POSSIBILITY OF A NEW METHOD FOR INVESTIGATING THE STATISTICAL BEHAVIOR OF LINEAR SYSTEMS

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A new approach is described for the investigation of the statistical behavior of linear systems. For the purpose of comparison with the kinetic equation method and the so-called "direct summation" method, some problems are considered which were not solved by these two methods.

1. For simplicity, let us first consider a system whose state is described by the single parameter x, which satisfies the equation of motion

$$dx/dt = V(x). \tag{1}$$

We shall also assume that the system experiences random bursts, distributed with respect to time by Poisson's law with intensity n. These bursts instantaneously change the parameter x by an amount ξ. The differential distribution of bursts relative to ξ is characterized by the function $\varphi(\xi)$ normalized to unity; this function is assumed to be known.

Owing to the action of the random bursts, the parameter x also becomes a random variable, described by the probability density f(x, t). As is well known, the function f(x, t) satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial (Vf)}{\partial x} + nf - n\int f(x-\xi, t)\,\varphi(\xi)\,d\xi = 0, \tag{2}$$

for whose solution it is necessary to have the initial distribution f(x, 0).

Different interpretations and methods of derivation of the kinetic equation are possible; they are all based on a comparison of the state of the system at some current moment of time t and at a subsequent instant of time t + dt. Inasmuch as the phenomena which take place upon starting the system are not considered, Eq. (2) obtained in such a fashion describes the behavior of a system for arbitrary initial conditions; the concrete form of the initial conditions is introduced only in the final stage, after obtaining a general solution of the kinetic equation.

In the case of linear systems for which the principle of superposition holds, a different consideration of the problem is possible. For purposes of illustration we apply it in the concrete case of an ionization chamber for which the collection time Δ is negligibly small in comparison with the characteristic time of the circuit of the chamber ($\Delta \ll RC$). The equation of motion in this case has the form

$$dx/dt + kx = 0, \tag{1'}$$

where k = 1/RC, and x has the meaning of the current which flows through the resistance R in the circuit of the chamber. The bursts are produced by the incidence of ionizing particles, and $\xi = Q/RC$, where Q is the charge released inside the chamber. The corresponding kinetic equation has the form

$$\frac{\partial f}{\partial t} - k\frac{\partial (xf)}{\partial x} + nf - n\int f(x-\xi, t)\,\varphi(\xi)\,d\xi = 0 \tag{2'}$$

and its properties are well known (see, for example, [1,2]).

Let the variation x = 0 at the initial moment, i.e., $f(x, 0) = \delta(x)$. It can be shown that for such initial conditions the function f(x, t) satisfies not only Eq. (2′), but also a certain other auxiliary equation. To obtain this equation, we shall be interested, as usual, in the value of the current x_t at a certain instant of time t. We shall consider an infinitesimally small time interval dt, elapsing after turning on the chamber at the time t = 0. If there is not a single burst during this time interval (the probability of such an event is equal to 1 − ndt), then x = 0 as before at the end of the interval dt. The process in the chamber is homogeneous in time, i.e., it depends only on the time difference; therefore, the value of x_t will coincide with the quantity x_{t-dt} in the given case.

We now assume that bursts take place in the interval dt (the probability of such a burst is equal to ndt). Then, the contribution to the variation x_t at the moment t is ξa, where ξ is the value of the given burst and the function a(t), which is important for the entire account following, has the meaning of the deviation of the system at the moment t, if x(0) = 1 and random bursts are "turned off." It follows from (1′) that, in the case of the ionization chamber,

$$a(t) = e^{-kt}. \tag{3}$$

It follows from the linearity of the system that under the conditions considered

$$x_t = \xi a + x_{t-dt}.$$

Thus the random quantity of interest to us $x_t = x_{t-dt}$ with probability $1-n\,dt$, and $x_t = \xi a(t) + x_{t-dt}$ with probability $n\,dt$. Hence it directly follows that

$$f(x, t) = (1 - ndt) f(x, t - dt)$$
$$+ ndt \int f(x - a\xi, t - dt)\, \varphi(\xi)\, d\xi,$$

which is equivalent to the equation

$$f(x, t + dt) = (1 - ndt) f(x, t) + ndt \int f(x - a\xi, t)\, \varphi(\xi)\, d\xi.$$

For small dt, we have $f(x, t+dt) = f(x,t) + (\partial f/\partial t)dt$, whence, after corresponding algebraic manipulations, the desired equation is found;

$$\frac{\partial f}{\partial t} = -nf + n \int f(x - a\xi, t)\, \varphi(\xi)\, d\xi. \tag{4}$$

Equation (4) naturally is of general value and refers to arbitrary linear systems; for the description of any concrete system, it is also necessary to use the function a(t) corresponding to it and characterizing the properties of the system under study.

It is again necessary to emphasize that in the derivation of Eq. (4) a zero initial condition was used in essential fashion. For another initial condition, the equation under consideration would have taken another form.[1)] Thus one cannot speak of reduction of Eq. (4) to Eq. (2′); these are completely different equations with different general solutions, having only a single common element—the partial solution corresponding to the zero initial conditions.[2)]

For illustration of this fact, we consider the quantity $\bar{x}$. Multiplying Eq. (2′) by x as usual, and integrating over x, we easily obtain the equation

$$d\bar{x} / dt + k\bar{x} = n\bar{\xi}. \tag{5}$$

Similarly, the following equation is derived from (4) in the case of the ionization chamber:

$$d\bar{x} / dt = n\bar{\xi}e^{-kt}. \tag{5′}$$

It is clear that Eqs. (5) and (5′) cannot reduce one to the other. The general solution of (5) has the form

$$\bar{x} = n\bar{\xi} / k + c'e^{-kt},$$

and that of Eq. (5′) is

$$\bar{x} = -(n\bar{\xi} / k)\, e^{-kt} + c''.$$

These are different sets of functions, but each contains the function

$$\bar{x} = (n\bar{\xi} / k)(1 - e^{-kt}), \tag{6}$$

which is a solution of the problem of interest to us for the case x(0) = 0.

2. The existence of Eq. (4), along with the usual kinetic equations, makes it possible first of all to compare them, which materially simplifies the solution of the corresponding statistical problems. In the case of an ionization chamber for example, we can subtract Eq. (4) from (2′) and obtain

$$k \frac{d(xf)}{dx} = n \int f(x - a\xi)\, \varphi(\xi)\, d\xi - n \int f(x - \xi)\, \varphi(\xi)\, d\xi. \tag{7}$$

It is evident that the partial derivative $\partial f/\partial t$ vanishes and all the dependence on time is associated with the single parameter a(t). This greatly simplifies the resultant considerations. In particular, by multiplying (7) by x^m and integrating with respect to x, we immediately obtain an algebraic equation connecting x^m with the moments of lower order:

$$n\overline{(x + a\xi)^m} - n\overline{(x + \xi)^m} + km\overline{x^m} = 0. \tag{8}$$

If we start out from only the single equation (2′), then it is convenient to use for the determination of the moments the set of coupled differential equations

$$d\overline{x^m} / dt = n\overline{(x + \xi)^m} - n\overline{x^m} - km\overline{x^m}. \tag{8′}$$

Systematic solution of the system (8′) is somewhat more complicated than solution of the set (8). A similar simplification takes place in the other cases.[3]

In a number of cases, Eq. (4), considered by itself, is more convenient for use than the kinetic equation. Usually this is the case when at the initial instant of time the behavior of the system is simpler than at an arbitrary instant. For example,

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1)Inasmuch as only linear systems are considered, use of zero initial conditions does not lead to any real limitation of generality.

2)The equations coincide if $a \equiv 1$. In this case Eq. (4) retains its form for any initial conditions, and the integral appearing in it is identical with the similar integral in Eq. (2). Inasmuch as $V \sim da/dt = 0$, the corresponding term in Eq. (2) also falls out.

we assume that the state of the system is described by simple interrelated parameters; if at the initial instant of time they are all equal to zero, and if the random bursts affect only one of them, then Eq. (4) can be much simpler than the corresponding kinetic equation.

By way of a first example, let us consider an ionization chamber with a finite collection time Δ. For a complete description of the state of the system, it is necessary to specify not only the current x but also the distribution of charges inside the working space of the chamber. In this connection, use of the kinetic equation encounters a number of technical difficulties, and, insofar as we know, the corresponding equations for the current have not yet been written out or solved.

At the same time, from the point of view that we have considered, the statistical behavior of the system is described as before by Eq. (4) for some other form of the function $a(t)$ which can easily be computed even in this case (see, for example, [2,4]). By multiplying (4) by x or by x^2 and integrating over x, we get the equations

$$d\bar{x}/dt = n\bar{\xi}a, \quad d\overline{x^2}/dt = n\overline{\xi^2}a^2 + 2n\bar{\xi}\bar{x}, \tag{9}$$

the solution of which has the following form for zero initial conditions:

$$\bar{x} = n\bar{\xi}\int_0^t a(\theta)\,d\theta, \quad \overline{x^2} = \bar{x}^2 + n\overline{\xi^2}\int_0^t a^2(\theta)\,d\theta. \tag{9'}$$

We now return to the problem of the multiple Coulomb scattering in the passage of a fast charged particle through matter, assuming as usual that the scattering angles are sufficiently small. For simplicity, we shall consider the projection of the trajectory of the particle on some plane parallel to the initial direction of motion. The state of the system is characterized by two parameters—the angle θ and the lateral deflection x, the value of the bursts ξ corresponds to a change in the angle during the elementary scattering act, the intensity of the bursts n is determined by the density of the medium and the effective scattering cross section. If the path followed by the particles is denoted by t, then the kinetic equation has the form

$$\theta\frac{\partial f}{\partial x} + nf - n\int f(x, \theta-\xi, t)\,\varphi(\xi)\,d\xi = -\frac{\partial f}{\partial t}, \quad f = f(x, \theta, t). \tag{10}$$

In Eq. (10) the variables x and θ are "intermixed," inasmuch as the rate of change of the parameter x depends upon the value of θ. To obtain any characteristics of the distribution over x, it is necessary, as is known, to consider the distribution over θ. Thus, for example, in the calculation of $\overline{x^2}$ it is necessary first to compute $\overline{\theta^2}$ and $\overline{\theta x}$; only then can one obtain an expression for $\overline{x^2}$.

Insofar as the approach presented here is concerned, it is easy to see that it leads to Eq. (4), in which $a(t) = t$. It is essential that in the resultant equation the deflection x enters alone and can be considered as unrelated to the angle θ, which is simply a certain parameter. The first two moments are determined as before, by the relations (9′). From symmetry considerations it is clear that $\bar{x} = 0$ and we quickly obtain the well-known expression $\overline{x^2} = n\overline{\xi^2}t^3/3$ for the quantity $\overline{x^2}$.

In the example under consideration, the random bursts affect the parameter θ which, in its turn, determines the rate of change of the parameter x that is of interest to us. Many problems of this type can be pointed out (Brownian motion, ionization chamber in the circuit in which a self-inductance is included, etc.). In all these cases, the methodology that we have set forth makes it possible to obtain the final results more rapidly. It should be noted that in the examples considered up to now, one could also use the methodology of "direct summation," which was proposed by Campbell [5] and exhaustively developed by Boonimovich. [6] It is also based on the introduction of the function $a(t)$ and in this connection it is close to the approach developed in the present work. Below we shall go to problems whose solution would be made more difficult with the use of "direct summation."

3. Let us consider a gas shower, the particles of which can diffuse in a transverse direction. We assume the progenitor of the shower to be one electron. The probability of multiplication in the distance ds is equal to αds, the number of electons at the depth s we denote by N_s, the lateral deviation of the i-th electron from the initial direction by x_s^i. We also assume that each electron undergoes jumps which change its lateral deflection by an amount ξ; the mean number of jumps per unit distance we denote by β.[3] For simplicity, the values of α and β, and also the laws of distribution of jumps ξ, are assumed to be independent of s.

We introduce the random variable

$$y_s = \sum_1^{N_s} x_s^i \tag{11}$$

and attempt to analyze its properties at a point lying a distance s + ds from the point of appearance of the shower. If there is neither a multipli-

[3]For simplicity we assume that the diffusion is brought about by jumps in the coordinates and not in the velocity. Such an assumption is valid if a sufficiently large number of jumps takes place in the time of interest to us.

cation nor a lateral jump in the initial interval ds, then the quantity y_{s+ds} will be equal to y_s. The probability of such an event is obviously $1-\alpha ds - \beta ds$. If a multiplication event takes place in the interval ds (probability αds), then two independent showers will be developed in the subsequent path s; the progenitors of these did not have any lateral deviations. One can show that in this case $y_{s+ds} = y_s + y'_s$, while the random variables y_s and y'_s are independent and have identical distributions. Finally, it is possible that a jump takes place in the interval ds which shifts the primary electron to one side by an amount ξ (the probability of such an event is equal to βds). We then have to deal with a shower which develops as usual but which is also shifted by the quantity ξ. We shall denote the quantity y_s that is of interest to us in this case by the symbol y_s^ξ.

Thus the behavior of the quantity y_{s+ds} is characterized by the following symbolic table:

$$y_{s+ds} = \begin{cases} y_s & \text{with probability } 1-\alpha\, ds-\beta\, ds \\ y_s+y'_s & \text{with probability } \alpha\, ds \\ y_s^\xi & \text{with probability } \beta\, ds \end{cases} \tag{12}$$

We assume that the distribution function of the jumps $\varphi(\xi)$ is even, i.e., $\bar{\xi} = 0$. In this case, diffusion takes place symmetrically and the equality

$$\overline{y_s} = 0 \tag{13}$$

is satisfied. To compute $\overline{y_s^2}$, we turn to (12), from which it follows that

$$\overline{y_{s+ds}^2} = (1-\alpha ds-\beta ds)\,\overline{y_s^2} + \alpha ds\,\overline{(y_s+y'_s)} + \beta ds\,\overline{(y_s^\xi)^2},$$

$$d\overline{y^2}/ds = -\alpha\overline{y^2} - \beta\overline{y^2} + \alpha\,\overline{(y+y')^2} + \beta\,\overline{(y^\xi)^2}. \tag{14}$$

From the independence of y and y′, we get the equality

$$\overline{(y+y')^2} = \overline{y^2} + \overline{y'^2} + 2\overline{y}\,\overline{y'} = 2\overline{y^2} + 2\overline{y^2} = 2\overline{y^2}.$$

Insofar as y_s^ξ is concerned, we have

$$y_s^\xi = \sum_1^{N_s}(x_s^i+\xi) = y_s + \xi N_s,$$

$$\overline{(y_s^\xi)^2} = \overline{y_s^2} + 2\bar{\xi}\,\overline{y_s N_s} + \overline{\xi^2}\,\overline{N_s^2}.$$

Inasmuch as it follows from the symmetry of the problem that

$$\overline{y_s N_s} = 0,$$

we finally obtain for $\overline{(y_s^\xi)^2}$

$$\overline{(y_s^\xi)^2} = \overline{y_s^2} + \overline{\xi^2}\,\overline{N_s^2}.$$

By making use of the results that we have obtained, we can transform (14) to the form

$$d\overline{y^2}/ds = \alpha\overline{y^2} + \beta\overline{\xi^2}\,\overline{N^2}, \tag{14'}$$

whence it is easy to show (see, for example, [2]) that

$$\overline{N^2} = 2e^{2\alpha s} - e^{\alpha s}. \tag{15}$$

With account of (15), the solution of Eq. (14′) (for the zero initial condition) yields

$$\overline{y^2} = (2\beta\overline{\xi^2}/\alpha)\,e^{\alpha s}(e^{\alpha s}-1) - \beta\overline{\xi^2}\,s e^{\alpha s}. \tag{16}$$

We now introduce the distribution function of lateral deviations f(x, s), which is identical for all electrons at the depth s. It follows immediately from (11) that

$$D_y = \bar{N} D_x + D_N \bar{x}^2. \tag{17}$$

Inasmuch as in the case of interest to us $\bar{x} = \bar{y} = 0$, Eq. (17) transforms to

$$\overline{y^2} = \bar{N}\,\overline{x^2}, \tag{17'}$$

Whence the quantity $\overline{x^2}$ can also be calculated. In this case, we get

$$\overline{x^2} = (2\beta\overline{\xi^2}/\alpha)(1-e^{-\alpha s}-\alpha s/2)\,\bar{N}. \tag{18}$$

If secondary ionization were important, then the mean square deviation of the primary particle after passage through a distance s would be $\beta\overline{\xi^2}s$, i.e., it would differ significantly from the value of $\overline{x^2}$.

It is probable that the investigation of this problem by means of the ordinary kinetic equations would be much more complicated. This evidently applies to all more or less complicated shower processes. [4)]

We shall consider one more example whose analysis by means of ordinary kinetic equations is at the least difficult. Some time back, apparatus was proposed and developed by Veksler and a number of other authors to measure the intensity of a radioactive source by the average current passing in the circuit of a proportional or Geiger counter (see, for example, [9–11]). In its simplest form the apparatus consists of a RC circuit in which the capacitance is replaced by the counter. If the dead time of the counter is equal to zero, then the system under investigation does not differ (from the statistical viewpoint) from an ionization chamber.

[4)]We found out that Gerasimova (see[7]) independently applied this method for calculation of the fluctuations of the number of particles in an electron-photon shower. The author thanks N. M. Gerasimova for reporting her calculations prior to publication. It should be noted that, on a purely mathematical plane, a similar method has already been applied in problems of such a type in the general study of so-called branching processes [see, for example,[8]).

The situation is different if the dead time $\tau \neq 0$. In this case, the development of the system is determined not only by its state at a given moment, but also by its history, in particular its state during the previous interval τ. By its nature, the ordinary kinetic equation is a local equation. Therefore, it cannot even be set up in the present case without the introduction of some additional parameters. So far as Eq. (4) is concerned, it can easily be obtained with appropriate changes. The difficulties associated with the effect of the prior history disappear here, because the prior history simply is absent after the system is turned on.

We shall consider the dead time of the so-called no-persistance type, assuming that each incidence of a particle on the counter that brings about its operation leads to a rapid jump in the current x by an amount ξ (in connection with the terminology used, see, for example,[12]). We shall also assume that at the instant of turning on the system, the counter is prepared for operation and $x(0) = 0$. Then arguing as before, we can write down the following for the case $t > \tau$:

$$x_{t+dt} = \begin{cases} x_t \text{ with probability } 1-n\,dt \\ x_{t-\tau} + \xi e^{-kt} \text{ with probability } n\,dt \end{cases}. \qquad (19)$$

If $t < \tau$, then

$$x_{t+dt} = \begin{cases} x_t \text{ with probability } 1-n\,dt \\ \xi e^{-kt} \text{ with probability } n\,dt \end{cases}. \qquad (19')$$

On the basis of (19) and (19′), we can immediately obtain the following generalization of Eq. (4). For purposes of economy of space, we limit ourselves to the results for $\bar{x}$ and $\overline{x^2}$. For $t > \tau$, we have

$$\bar{x}_{t+dt} = (1 - n\,dt)\,\bar{x}_t + n\,dt\,(\bar{x}_{t-\tau} + \xi e^{-kt}),$$

which gives

$$d\bar{x}/dt = -n\bar{x} + n\bar{x}_{t-\tau} + n\xi e^{-kt}, \quad t > \tau. \qquad (20)$$

Similarly, we immediately get

$$d\bar{x}/dt = -n\bar{x} + n\xi e^{-kt}, \quad t < \tau. \qquad (20')$$

For $\overline{x^2}$, we have, for $t > \tau$

$$\overline{x^2_{t+dt}} = (1 - n\,dt)\,\overline{x^2_t} + n\,dt\,\overline{(x_{t-\tau} + \xi e^{-kt})^2}$$

$$d\overline{x^2}/dt = -n\overline{x^2} + n\overline{x^2_{t-\tau}} + 2n\xi\bar{x}_{t-\tau}e^{-kt} + n\overline{\xi^2}e^{-2kt}, \quad t > \tau. \qquad (21)$$

Similarly, we get for $t < \tau$

$$d\overline{x^2}/dt = -n\overline{x^2} + n\overline{\xi^2}e^{-2kt}, \quad t < \tau. \qquad (21')$$

The quantities $\bar{x}$ and $\overline{x^2}$ can be found by direct solution of the resulting equations at the origin in the interval $(0, \tau)$, then for $(\tau, 2\tau)$ etc.

In a sufficiently long time after turning on the system, the stationary state is established in it, whose characteristics can be determined by a much simpler method. We begin with calculation of $\bar{x}$, for which we integrate Eq. (20′) from $t = 0$ to $t = \tau$, and Eq. (20) from τ to the running time t.

Collecting the results and taking it into account that $x(0) = 0$, we get

$$\bar{x}_t = -n\int_0^t \bar{x}_\theta\,d\theta + n\int_0^t \bar{x}_{\theta-\tau}\,d\theta + n\bar{\xi}\int_0^t e^{-k\theta}\,d\theta.$$

Making a change of variables in the second integral and computing the third integral, we transform this equation to the form

$$\bar{x}_t = -n\int_{t-\tau}^t \bar{x}_\theta\,d\theta + \frac{n\bar{\xi}}{k}(1 - e^{-kt}).$$

We can then write for the stationary state

$$\bar{x} = -n\tau\bar{x} + n\bar{\xi}/k$$

which yields

$$\bar{x} = n\bar{\xi}/k\,(1 + n\tau). \qquad (22)$$

Similarly, integrating over the corresponding limits Eq. (21) and (21′) and combining the results, we get the following equation for the stationary state:

$$\overline{x^2} = -n\tau\overline{x^2} + \frac{n\overline{\xi^2}}{2k} + 2n\bar{\xi}e^{-k\tau}\int_0^\infty x_\theta e^{-k\theta}\,d\theta.$$

For calculation of this integral, we multiply Eqs. (20) and (20′) by e^{-kt}, and integrate and combine the results. After integration by parts, the left hand side becomes

$$\bar{x}e^{-kt} + k\int_0^t \bar{x}_\theta e^{-k\theta}\,d\theta,$$

and the right side, after simple algebraic transformations, takes on the form

$$-n\int_0^t \bar{x}_\theta e^{-k\theta}\,d\theta + ne^{-k\tau}\int_0^{t-\tau} \bar{x}_\theta e^{-k\theta}\,d\theta + \frac{n\bar{\xi}}{2k}(1 - e^{-2kt}).$$

We then get the following equation for the stationary state:

$$k\int_0^\infty \bar{x}_\theta e^{-k\theta}\,d\theta = -n\int_0^\infty \bar{x}_\theta e^{-k\theta}\,d\theta + ne^{-k\tau}\int_0^\infty \bar{x}_\theta e^{-k\theta}\,d\theta + \frac{n\bar{\xi}}{2k},$$

which gives

$$\int_0^\infty \bar{x}_\theta e^{-k\theta}\,d\theta = \frac{n\bar{\xi}}{2k}\,\frac{1}{k + n(1 - e^{-k\tau})}.$$

Substituting this result in the expression for $\overline{x^2}$, we get

$$\overline{x^2} = \frac{n\overline{\xi^2}}{2k(1 + n\tau)} + \frac{n^2\bar{\xi}^2}{k + n(1 - e^{-k\tau})}\,\frac{e^{-k\tau}}{k(1 + n\tau)}. \qquad (23)$$

In the most interesting special case, in which the condition $k\tau \ll 1$ is satisfied,

$$\overline{x^2} = \frac{n\overline{\xi^2}}{2k(1+n\tau)} + \frac{n^2\overline{\xi}^2}{k^2(1+n\tau)^2} = \overline{x}^2 + \frac{n\overline{\xi^2}}{2k(1+n\tau)}.$$

In this case, the dispersion is

$$D_x = n\overline{\xi^2}/2k(1+n\tau). \qquad (24)$$

For very high intensities, we have

$$\overline{x} = \frac{\overline{\xi}}{k\tau}, \quad \overline{x^2} = \frac{\overline{\xi^2}}{2k\tau} + \frac{\overline{\xi}^2}{k\tau}\frac{e^{-k\tau}}{1-e^{-k\tau}}.$$

Thus, in contrast with the case in which the "dead time" is absent, the fluctuations remain finite.

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