QUANTUM ELECTRODYNAMICS IN TERMS OF ELECTROMAGNETIC FIELD INTENSITIES

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Submitted to JETP editor April 11, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1365-1370 (October, 1962)

A Lorentz invariant formulation of quantum electrodynamics is found which does not involve potentials, but only electromagnetic field intensities.

1. Unlike the equations of Maxwell's classical electrodynamics, the equations of quantum electrodynamics have so far not been successfully written in terms of electromagnetic field intensities alone. The quantum equations of motion ordinarily involve the four-vector potential A_μ.[1)] This fact leads to difficulties. For example, it is well known that the Maxwell equations for A_μ cannot be quantized. In fact, in principle they determine the A_μ only up to the four-gradient of an arbitrary function, and this means that there is a component in A_μ for which there is no equation of motion.[4] Therefore one cannot write for the A_μ commutation relations which are not in contradiction with the Maxwell equations.

The standard approach is that having fixed the gauge in some way or other one gets equations which determine the potentials A_μ more rigidly and therefore admit of quantization. The shortcomings of such formulations have been widely known for a long time. The clearest and most consistent of such formulations is that in the Coulomb gauge (Dirac). It is not explicitly covariant, however, and unlike the Maxwell theory it requires that one treat separately the interactions through the transverse field and the Coulomb field. In the Fermi formulation it is necessary to introduce the unphysical indefinite metric (and besides, this can still not be done in an explicitly covariant form).

Therefore many authors attempt[5-12] to overcome the difficulties of quantizing the Maxwell equations by prescribing commutation relations not for the A_μ but only for gauge-independent quantities, for example the field intensities. In such theories, however, either the vector potential is not completely eliminated and operating with it is difficult, or else explicit Lorentz covariance of the theory is lacking. Some authors have even been inclined to regard these difficulties as an indication that in quantum theory, unlike classical theory, the vector potential has an independent significance (see [13], and also [12]).

In the present paper it is shown that quantum electrodynamics can be constructed from beginning to end in terms of the electromagnetic field intensities and in explicitly covariant form. This formulation is based on our previous work.[4] The interaction of a charged field with photons here has an apparent nonlocal nature and can be written in many equivalent forms. The entire treatment is carried out in the interaction representation. Its purpose is not to replace the usual Feynman scheme of calculation, but to demonstrate that all calculations can be carried out in general without reference to the vector potential, and that in this sense there are no differences between the classical and quantum theories.

2. We assume that in the interaction representation the free-field operators obey the Maxwell and Dirac equations:

$$\partial F_{\mu\nu}/\partial x_\mu = 0, \tag{1}$$

$$\partial \check{F}_{\mu\nu}/\partial x_\mu = 0 \quad (\check{F}_{\mu\nu} = {}^1/_2 \varepsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}), \tag{2}$$

$$F_{\mu\nu} = - F_{\nu\mu}, \tag{3}$$

$$(\gamma_\mu\, \partial/\partial x_\mu + m)\psi = 0, \tag{4}$$

where $F_{\mu\nu}$ is the operator for the electromagnetic field intensities and ψ is the operator of the spinor field. There is consistency between these equations and the well known commutation relations:[2)]

$$[F_{\mu\nu}(x),\quad F_{\lambda\rho}(y)] = \Big[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\lambda}\delta_{\nu\rho} - \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\lambda}\delta_{\mu\rho} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\rho}\delta_{\nu\lambda} + \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\rho}\delta_{\mu\lambda}\Big]\, i\Delta(x-y), \tag{5}$$

1) At the same time it is well known that in a number of cases the results of quantum electrodynamics depend only on the intensities (cf., e.g., [1-3]).

2) In the introduction we already mentioned the well known fact that no commutation relations which are not in contradiction with the Maxwell equations can be written for the potentials.

$$\{\psi(x), \psi(y)\} = 0, \qquad \{\psi(x), \bar{\psi}(y)\} = -iS(x-y). \quad (6)$$

Equation (1) can be derived as the condition for a relative extremum of the Lagrangian $-\frac{1}{4}\int d^4x F_{\mu\nu}(x) F_{\mu\nu}(x)$ under the conditions (2) and (3). The commutation relations (5) can be obtained by the method of Peierls.[14] The Fourier expansion for $F_{\mu\nu}(x)$ and the commutation relations in the p representation are given in the Appendix.

3. As the interaction Lagrangian we take

$$L(x) = ie :\bar{\psi}(x)\gamma_\nu\psi(x)\Box^{-1}\frac{\partial}{\partial x_\mu}F_{\mu\nu}(x):. \quad (7)$$

By the operator symbol $\Box^{-1}$ we mean the operation of convolution with the Green's function of the d'Alembertian operator:

$$\Box^{-1}f(x) = \int dy\, G(x-y) f(y),$$

$$\Box G(x-y) = \delta(x-y). \quad (8)$$

We do not need to remove the arbitrariness in the choice of the Green's function. As we shall see, any choice leads to the same results.

We note also that from Eq. (1) there follows the vanishing of the quantity $\Box\Box^{-1}\partial F_{\mu\nu}/\partial x_\mu$, but not the vanishing of $\Box^{-1}\partial F_{\mu\nu}/\partial x_\mu$. We shall at the beginning stipulate the use of the operation $\Box^{-1}$.

We shall require of the S matrix only that it satisfy the conditions of Lorentz invariance, unitarity, and causality, in the spirit of the Stueckelberg–Bogolyubov approach.[15] We write it in the form

$$S = T^*\exp\left[i\int dx\, L(x)\right], \quad (9)$$

where the symbol T* means only that in the normal form of the S matrix one must take for the quantities $\Box^{-1}\partial F_{\mu\nu}/\partial x_\mu$ pairings of the form

$$\overline{\Box_x^{-1}\frac{\partial}{\partial x_\mu}F_{\mu\nu}(x)\,\Box_y^{-1}\frac{\partial}{\partial y_\lambda}F_{\lambda\rho}(y)}$$

$$= \left(\delta_{\nu\rho} - \Box_x^{-1}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\rho}\right)(-i)\Delta^c(x-y), \quad (10)$$

and for the spinor field one takes the usual pairings.[3)] The propagation functions (10) correspond to electrodynamics in the Landau-Khalatnikov gauge. The derivative term in Eq. (10) is unessential, since it drops out because of conservation of current and the vanishing of the equal-time commutator of two currents. Thus all of the coefficient functions are the same as in the standard form of electrodynamics. We note that the pairing (10) can be regarded as the result of the action of differential and integral operators on the pairing of the $F_{\mu\nu}$ defined by Hori[2]:

$$\overline{F_{\mu\nu}(x)F_{\lambda\rho}(y)} = \left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\lambda}\delta_{\nu\rho} - \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\lambda}\delta_{\mu\rho} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\rho}\delta_{\nu\lambda} + \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\rho}\delta_{\mu\lambda}\right)(-i)\Delta^c(x-y).$$

Instead of postulating Eq. (10) it would perhaps be more consistent to construct the S matrix for the nonlocal Lagrangian (7) by the method of Khirzhnits[16] with $\tilde{T}_g$-ordering.

4. In the normal form of the S matrix the only unusual element will be the N-products

$$:\Box^{-1}\frac{\partial}{\partial x^1_{\nu_1}}F_{\nu_1\mu_1}(x^1)\ldots\Box^{-1}\frac{\partial}{\partial x^n_{\nu_n}}F_{\nu_n\mu_n}(x^n):. \quad (11)$$

We shall show that the calculation of the matrix elements from such N-products causes no difficulties. To do this we first define the photon states as the result of the action of the negative-frequency part of the electromagnetic-field-intensity tensor on the vacuum state. Thus a one-particle state with definite momentum will be of the form[4)]

$$\frac{1}{q_0}\mathbf{f}^+_{\mu\nu}(\mathbf{q})\Psi_0 \quad (12)$$

with the orthogonality and normalization condition

$$\left(\frac{1}{p_0}\mathbf{f}^+_{\mu\nu}(\mathbf{p})\Psi_0, \frac{1}{q_0}\mathbf{f}^+_{\lambda\rho}(\mathbf{q})\Psi_0\right)$$

$$= \frac{1}{p_0^2}\{p_\mu p_\lambda\delta_{\nu\rho} - p_\mu p_\rho\delta_{\nu\lambda} - p_\nu p_\lambda\delta_{\mu\rho} + p_\nu p_\rho\delta_{\mu\lambda}\}\,\delta(\mathbf{p}-\mathbf{q}).$$

The state of a photon is uniquely determined by giving either its electric field intensity $E_m = iF_{m4}$, or its magnetic field intensity $H_m = i\check{F}_{m4}$, or[5)] $F_{\mu\nu} \pm \check{F}_{\mu\nu}$, i.e., $\mathbf{E} \pm i\mathbf{H}$. The first possibility (which we shall use) corresponds to the normalized state vector

$$\frac{-i}{q_0}\mathbf{f}^+_{\mu 4}(\mathbf{q})\Psi_0 = \frac{1}{q_0}\mathbf{E}^+_m(\mathbf{q})\Psi_0. \quad (13)$$

(The other possibilities can also be written out in an obvious way.) The fact that the state vector is normalized is expressed in the form

$$\left(\frac{1}{p_0}\mathbf{E}^+_m(\mathbf{p})\Psi_0, \frac{1}{q_0}\mathbf{E}^+_n(\mathbf{q})\Psi_0\right)$$

$$= \left(\delta_{mn} - \frac{p_m p_n}{p_0^2}\right)\delta(\mathbf{p}-\mathbf{q}). \quad (14)$$

Contraction on m and n (summation over the po-

[3)]It can be verified that with this choice of the pairings the conditions of Lorentz invariance, unitarity, and causality are satisfied in each order in the coupling constant.

[4)]The operators $\mathbf{f}_{\mu\nu}$ and their properties are defined in the Appendix.

[5)]These last quantities transform according to the irreducible representations (1, 0) or (0, 1) of the homogenous Lorentz group.

larizations) gives $2\delta(\mathbf{p}-\mathbf{q})$. The factor 2 corresponds to the number of independent spin states.

By means of the formula

$$\left[\Box^{-1}\frac{\partial}{\partial x_\mu}F_{\mu\nu}(x),\quad \mathfrak{f}^+_{\lambda\rho}(\mathbf{q})\right] = i(\delta_{\nu\lambda}q_\rho-\delta_{\nu\rho}q_\lambda)\frac{e^{iqx}}{\sqrt{(2\pi)^3 2q_0}}$$

[which follows from Eqs. (A.2) and (A.5)] we calculate the matrix element

$$\left(\Psi_0,\ \Box^{-1}\frac{\partial}{\partial x_\mu}F_{\mu\nu}(x)\frac{-i}{q_0}\mathfrak{f}_{\lambda 4}(\mathbf{q})\Psi_0\right)$$

$$=\frac{1}{q_0}(\delta_{\nu\lambda}q_4-\delta_{\nu 4}q_\lambda)\frac{e^{iqx}}{\sqrt{(2\pi)^3 2q_0}}. \tag{15}$$

Thus in the Feynman diagram the factor corresponding to a photon with polarization λ and coupled with the current j_ν is $q_0^{-1}(\delta_{\nu\lambda}q_4-\delta_{\nu 4}q_\lambda)$. With this the problem of calculating the matrix element from the normal product (11) is solved. After the matrix element has been squared the summation over photon polarizations gives

$$-q_0^{-2}(\delta_{\nu\nu'}q_4^2-\delta_{\nu 4}q_{\nu'}q_4-\delta_{\nu' 4}q_\nu q_4). \tag{16}$$

When we use the conservation of current and the (single-time) commutativity of currents only the first term remains. We thus arrive at the Feynman rule for summing over polarizations. If we form the photon states from the vacuum by means of the operators $c^+(\mathbf{p}s)$ (cf. Appendix), then we can write the matrix element of the S matrix for a process involving n photons in the following intuitive form:

$$\langle f|S|i\rangle \sim \bar{u}\ldots\gamma_{\mu_1}\ldots\gamma_{\mu_n}\ldots uF_{\mu_1 4}(\mathbf{q}_1 s_1)F_{\mu_2 4}(\mathbf{q}_2 s_2)\ldots F_{\mu_n 4}(\mathbf{q}_n s_n). \tag{17}$$

One can also obtain equivalent forms containing $\check{F}_{\mu\nu}$ or $F_{\mu\nu}\pm\check{F}_{\mu\nu}$ instead of $F_{\mu\nu}$.

5. Thus we have expounded a formulation of quantum electrodynamics in which the vector potential has never been mentioned. We have used only the intensities E and H, which are experimentally measurable and uniquely defined by the Maxwell equations.

This formulation is based on our previous work[4] in which we gave a decomposition of the vector potential into gauge-independent and gauge-dependent parts:

$$A_\mu=\left(A_\mu-\Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}A_\nu\right)+\Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}A_\nu$$

$$=\Box^{-1}\frac{\partial}{\partial x_\nu}F_{\nu\mu}+\Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}A_\nu.$$

The gauge-dependent part can be eliminated from the Heisenberg Lagrangian by the transformation $\psi\to\exp(i\Box^{-1}\partial A_\nu/\partial x_\nu)\psi$, and indeed the result of this transformation is that the interaction Lagrangian takes the form (7).[6)]

The potential A_μ is defined uniquely up to a four-gradient, and therefore is in general unmeasurable. Our definition of the gauge-independent part is also not without some arbitrariness. (One can take for the operator $\Box^{-1}$ an integral operator whose kernel is any Green's function of the d'Alembert equation.) This arbitrariness indeed remains in the formulation of quantum electrodynamics which we have given here. It does not relate, however, to the operator which describes the photons ($F_{\mu\nu}$), but relates to the choice of one of the equivalent forms for writing the S matrix in terms of these operators.

Recently we received a preprint of a paper by DeWitt[18] written in a spirit much like that of our own paper, and proposing a different method for eliminating the potentials. We take this occasion to express our sincere gratitude to Professor DeWitt.

APPENDIX

We write out the three-dimensional and four-dimensional expansions for the field-intensity tensor $F_{\mu\nu}(x)$:

$$F_{\mu\nu}(x)=\int\frac{d\mathbf{p}}{\sqrt{(2\pi)^3 2p_0}}\{\mathfrak{f}_{\mu\nu}(\mathbf{p})e^{ipx}+\mathfrak{f}^+_{\mu\nu}(\mathbf{p})e^{-ipx}\} \tag{A.1}$$

(here and in what follows $p_0=|\mathbf{p}|$) and

$$F_{\mu\nu}(x)=\int\frac{d^4p}{\sqrt{(2\pi)^3}}\{f_{\mu\nu}(p)e^{ipx}+f^+_{\mu\nu}(p)e^{-ipx}\}. \tag{A.2}$$

The operators $\mathfrak{f}_{\mu\nu}$ and $f_{\mu\nu}$ satisfy the Maxwell equations

$$p_\mu f_{\mu\nu}=0,\qquad p_\mu\check{f}_{\mu\nu}=0$$

and are connected by the relations

$$f_{\mu\nu}(p)=\delta(p^2)\theta(p_0)\sqrt{2p_0}\,\mathfrak{f}_{\mu\nu}(\mathbf{p}),$$

$$\mathfrak{f}_{\mu\nu}(\mathbf{p})=\int dp_0 f_{\mu\nu}(p)\sqrt{2p_0}.$$

The commutation relations for these quantities are

$$[\mathfrak{f}_{\mu\nu}(\mathbf{p}),\mathfrak{f}^+_{\lambda\rho}(\mathbf{q})]=\{p_\mu p_\lambda\delta_{\nu\rho}-p_\mu p_\rho\delta_{\nu\lambda}-p_\nu p_\lambda\delta_{\mu\rho}+p_\nu p_\rho\delta_{\mu\lambda}\}\delta(\mathbf{p}-\mathbf{q}),\quad p_0=|\mathbf{p}|; \tag{A.3}$$

$$[f_{\mu\nu}(p),f^+_{\lambda\rho}(q)]=\{p_\mu p_\lambda\delta_{\nu\rho}-p_\mu p_\rho\delta_{\nu\lambda}-p_\nu p_\lambda\delta_{\mu\rho}+p_\nu p_\rho\delta_{\mu\lambda}\}\delta(p-q)\theta(p_0)\delta(p^2); \tag{A.4}$$

6)The equations of motion in the Heisenberg representation are then

$$\partial_\mu F_{\mu\nu}=-j_\nu,\qquad \partial_\mu\check{F}_{\mu\nu}=0,\qquad (\gamma_\mu\partial_\mu+m)\psi=ie\gamma_\nu\psi\Box^{-1}\partial_\mu F_{\mu\nu}.$$

$$[f_{\mu\nu}(p), f^+_{\lambda\rho}(q)] = \{p_\mu p_\lambda \delta_{\nu\rho} - p_\mu p_\rho \delta_{\nu\lambda} - p_\nu p_\lambda \delta_{\mu\rho} + p_\nu p_\rho \delta_{\mu\lambda}\} \times \delta(\mathbf{p}-\mathbf{q})\,\theta(p_0)\,\delta(p^2)\sqrt{2p_0}. \quad (A.5)$$

The commutators not written are zero. In the use of the creation and annihilation operators $\mathbf{f}^+_{\mu\nu}(f^+_{\mu\nu})$ and $\mathbf{f}_{\mu\nu}(f_{\mu\nu})$ it is understood that the spin state of the photon is characterized by the pair of indices $(\mu\nu)$. We can go over to the expansion

$$f_{\mu\nu}(\mathbf{p}) = \sum_{s=1}^{2} \mathbf{c}(\mathbf{p}s)\, F_{\mu\nu}(\mathbf{p}s), \quad (A.6)$$

where s is the index of the spin state, $F_{\mu\nu}(\mathbf{p}s)$ are classical solutions of the Maxwell equations (wave functions of the photon[17]), and $\mathbf{c}(\mathbf{p}s)$ are annihilation operators with the commutation relations

$$[\mathbf{c}(\mathbf{p}s),\ \mathbf{c}^+(\mathbf{q}s')] = \delta_{ss'}\delta(\mathbf{p}-\mathbf{q}). \quad (A.7)$$

In terms of these quantities the Fourier expansion (7) can be written in the form

$$F_{\mu\nu}(x) = \int \frac{d\mathbf{p}}{\sqrt{(2\pi)^3\, 2p_0}} \sum_{s=1}^{2} \{\mathbf{c}(\mathbf{p}s)\, F_{\mu\nu}(\mathbf{p}s)\, e^{ipx} + \mathbf{c}^+(\mathbf{p}s)\, F^+_{\mu\nu}(\mathbf{p}s)\, e^{-ipx}\}. \quad (A.8)$$

We also give here a useful formula for the sum over spin states (unnormalized)

$$\sum_{s=1}^{2} F_{\mu\nu}(\mathbf{p}s)\, F^+_{\lambda\rho}(\mathbf{p}s) = p_\mu p_\lambda \delta_{\nu\rho} - p_\mu p_\rho \delta_{\nu\lambda} - p_\nu p_\lambda \delta_{\mu\rho} + p_\nu p_\rho \delta_{\mu\lambda}.$$

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Translated by W. H. Furry
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