THE PROBLEM OF "SURFACE" DIVERGENCES IN THE BOGOLYUBOV METHOD

A. D. SUKHANOV

Mathematics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 25, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1400-1410 (October, 1962)

A supplementary regularization procedure is proposed which allows us within the framework of the Bogolyubov method to obtain a matrix $S(\sigma, -\infty)$ not containing "surface" divergences (for fixed regularization masses M_i^2). The additional counterterms which then appear with arbitrary finite coefficients give a contribution only on the surface σ and can be separated out from the matrix $S(\sigma, -\infty)$ into a unitary operator which is without importance from the physical point of view. Some peculiar features of the derivation of the interaction Hamiltonian in the Bogolyubov method are also considered. The question of the passage to the limit $M_i^2 \to \infty$ in the matrix $S(\sigma, -\infty)$ remains an open one. Independent of its solution, however, the proposed procedure allows us in a consistent way to obtain the usual expression for the S matrix, starting from the Tomonaga-Schwinger equation.

1. INTRODUCTION

THE main shortcoming of the usual scheme of the Hamiltonian formalism [1] is the impossibility of obtaining a finite unitary Dyson matrix $S(\sigma, -\infty)$, even in perturbation theory, owing to the fact that in addition to the "ultraviolet" divergences there also appear peculiar "surface" divergences. [2] Because of this it has been proposed [3,4] to turn to the construction of the scheme of the Hamiltonian formalism on the basis of the axiomatic method of Bogolyubov, [5] which is the most consistent method in perturbation theory.

Following this path, and starting from a generalized Schrödinger variational equation, [5] it has been possible to obtain the Tomonaga-Schwinger equation and the interaction Hamiltonian in the expected form (for fixed regularization masses M_i^2). [4] Here also, however, a problem of "surface" divergences arose, and this time it appeared already in the derivation of the interaction Hamiltonian.

It was shown in a previous paper [4] that when proper account is taken of the symmetry properties of the quasi-local operators Λ_n introduced in [5] and the limiting processes are carried out more accurately, this problem occurs only in the calculation of the boson self-energy diagram. Since, however, a diagram of this kind is encountered in all renormalizable theories (except the Hearst-Thirring field), the solution of the problem of "surface divergences is as pressing a task as before.

A first attempt [6] in this direction was made on the basis of an examination of the usual Hamiltonian $H(\tau)$ in second-order perturbation theory for a theory with $L(x) = e:\varphi^4(x):$. Here there were revealed the physical and mathematical causes of the appearance of "surface" divergences, and it was shown that if one understands the passage from sufficiently smooth functions $g(x)$ [5] describing the "turning on and off of the interaction" to the limit of θ functions in an improper sense[1)] rather than in the usual sense, then one can carry out a "surface" regularization of the expression for the Hamiltonian $H(\tau)$, making it finite (for $M_i^2 = \text{const}$). The finite arbitrariness which thus arises is in principle of a "surface" nature, or in other words, the additional terms in the Hamiltonian must contribute only to the matrix $S(\sigma, -\infty)$, but not to the S matrix.

For a final solution of this problem, however, it is necessary to carry out a "surface" regularization of the interaction Hamiltonian in all orders of perturbation theory and obtain an expression for $S(\sigma, -\infty)$ by solving the Tomonaga-Schwinger equation by the method of successive approximations. A preliminary examination of the problem [7] showed that although this way of constructing the Dyson matrix in the Bogolyubov method is in principle possible, further difficulties appear in its practical application. These difficulties are connected both with satisfying the integrability conditions for the resulting finite Hamiltonian and

1)That is, as a definition of a generalized function on a class of functions on which it was not defined originally.

with the fact that the interaction Hamiltonian [4] in the Bogolyubov method contains terms of every order in the coupling constant. This last means that in solving the Tomonaga-Schwinger equation for $S(\sigma, -\infty)$ one gets an expression expanded in power series in $H(x; \sigma)$ which does not coincide in a given case with the expansion in powers of the coupling constant, so that there must be a further regrouping of terms in $S(\sigma, -\infty)$.

All of these difficulties lead one to think of approaching the matter from the other end, so to speak, i.e., trying first to get a finite Dyson matrix from the generalized scattering matrix $S(g)$ [5] by a formula of the type

$$S(\sigma, -\infty) = \lim_{g\to\theta_\sigma\equiv\theta(T_\sigma - x^0)} S(g) = \lim_{g\to\theta_\sigma} T\exp\left\{i\int L(x; g)\,dx\right\}, \tag{1}$$

where

$$L(x; g) = L(x)\,g(x) + \sum_{n=2} \frac{1}{n!}\int \Lambda_n(x, x_1 \ldots x_{n-1})\,g(x) \ldots g(x_{n-1})\,dx_1 \ldots dx_{n-1} \tag{2}$$

is the effective interaction Lagrangian, [5] and then getting from it (for fixed M_i^2) a finite expression for $H(x; \sigma)$ by the formula

$$H(x; \sigma) = i\,\frac{\delta S(\sigma, -\infty)}{\delta\sigma(x)}\,S^+(\sigma, -\infty). \tag{3}$$

Also the carrying out of the "surface" regularization directly in $S(g)$ is even more logical, since the matrix $S(g)$ is the fundamental quantity in the Bogolyubov method.

We shall realize this program here for the theory with $L(x) = e:\varphi^4(x):$ and shall show that the problem of "surface" divergences in the Bogolyubov method can be successfully solved in each order of perturbation theory.

2. "SURFACE" REGULARIZATION OF THE DYSON MATRIX

In obtaining the Dyson matrix by Eq. (1) we shall naturally be guided by the requirements of relativistic covariance and also by unitarity and causality, which are automatically assured in the Bogolyubov method, and by the requirements of finiteness and of correspondence with an S matrix given in the entire momentum space. The last requirement means that when we go to the limit $\lim_{\sigma\to\infty} S(\sigma, -\infty)$ or take the product $S(\infty, \sigma)S(\sigma, -\infty)$ we must get the usual S matrix. [5]

We emphasize that as in [3-7] the passage to the limit $g \to \theta_\sigma$ in Eq. (1) will be made with fixed M_i^2. In this case it is not hard to show that there is the following connection between the interaction Hamiltonian $H(x; \sigma)$ obtained in [4] and the $L(x; g)$ of the form (2):

$$\lim_{g\to\theta_\sigma}\int L(x; g)\,dx = -\int_{-\infty}^{\sigma} H(x'; \sigma')\,dx', \tag{4}$$

and now we are considering the theory for which

$$L(x; g) = eg(x)Z_4(g):\varphi^4(x): + \frac{1}{2}[Z_3(g) - 1]\left[-:\varphi\frac{\partial^2\varphi}{\partial x^2}: - m^2:\varphi^2(x):\right] - \delta m^2(g):\varphi^2(x): + \frac{1}{2}:\varphi^2(x):\frac{\partial g}{\partial x^k}\frac{\partial}{\partial x^k}\left[\frac{Z_3(g) - 1}{g(x)}\right], \tag{5}$$

where $Z_4(g)$, $Z_3(g)$ are constants which diverge logarithmically for $M_i^2 \to \infty$, [5,4] and $\delta m^2(g)$ is a constant which diverges quadratically. In particular,

$$Z_3(g) = 1 + \sum_{m=2} \frac{[eg(x)]^m}{m!} B_m. \tag{6}$$

Because of Eq. (4), in the passage to the limit $g \to \theta_\sigma$ in the argument of the exponential in Eq. (1) we encounter the same problem of a "surface" divergence for the boson proper-energy diagram that was discussed in detail in [4]. Namely, substituting Eq. (5) in Eq. (4), we get [for simplicity we take the surface σ to be a plane; $\theta_\tau = \theta(\tau - x^0)$]:

$$\lim_{g\to\theta_\tau}\int L(x; g)\,dx = \int_{-\infty}^{\tau} dx\,L(x; 1) + \frac{1}{2}\sum_{m=2}\frac{e^m}{m!}B_m \times \lim_{g\to\theta_\tau}\int dx:\varphi^2(x):[g^{m-1}(\tau - x^0)]'\,g'(\tau - x^0). \tag{7}$$

It is easy to see that direct passage to the limit in the second term in Eq. (7) leads to nonintegrable expressions of the type of products of δ functions of equal arguments. As has been pointed our earlier, [6] the physical reason for the appearance of such divergences is essentially the same as that for the appearance of "ultraviolet" divergences, namely the illegitimate use of the mathematical concept of a point in quantum field theory to describe physical processes. From the mathematical point of view the situation that arises is a reflection of the fact that the quasi-local operators Λ_n originally introduced in the $L(x; g)$ of the form (2) are generalized functions even for fixed M_i^2, and are not defined on a class of functions $g(x)$ which includes θ functions.

One can, however, give the limiting expression for $S(\sigma, -\infty)$ a quite definite meaning if we define these Λ_n in such a way that they are integrable generalized functions on a class of functions $g(x)$

including θ functions. In [6] it was shown for the example of the second-order diagram that it is hard to carry out such a redefinition of the Λ_n directly, since the Λ_n are rather rigidly fixed by the general requirements of relativistic covariance, unitarity, and causality.[5] But one can first separate out from the entire limiting expression in Eq. (7) the first term and make the transition $g \to \theta_\tau$ directly in it (as was done above), and then give a more detailed treatment to the second term.

Namely, following[6], we introduce the notation

$$f_m(x^0) = \frac{1}{2}\frac{e^m}{m!} B_m(m-1)\int dx : \varphi^2(x) : , \qquad (8)$$

where, as is well known, $f_m(x^0)$ is a sufficiently smooth function. Then the "surface" diverging term in Eq. (7) takes the form

$$\lim_{g\to\theta_\tau} \int f_m(x^0)\, g'(\tau - x^0)\, g'(\tau - x^0)\, g^{m-2}(\tau - x^0). \qquad (9)$$

To establish the procedure of "surface" regularization let us first consider for auxiliary purposes the problem of defining in Eq. (9) new generalized functions of a single variable

$$K_m(\tau - x^0) = \lim_{g\to\theta_\tau} g^{m-2}(\tau - x^0)\,[g'(\tau - x^0)]^2$$

on a corresponding class of functions $f_m(x^0)$, for which we take as an approximation to the function $\theta(\tau - x^0)$ the concrete expression

$$g(\tau - x^0) = \frac{1}{\pi}\operatorname{arc\,ctg}\frac{\tau - x^0}{\varepsilon}. \qquad (10)^*$$

In this case the passage to the limit $g \to \theta_\tau$ will correspond to the transition $\epsilon \to 0$. We shall further assume that the function $f_m(x^0)$ is n times differentiable and has an n-fold zero at the point $x^0 = \tau$, i.e.,

$$f_m(x^0) = (x^0 - \tau)^n f_m^n(x^0), \qquad (11)$$

where $f_m^n(x)$ is a continuous function. Our problem is to find out on what subclass of the functions of the form (11) the generalized function $K_m(\tau - x^0)$ is defined, and then to extend this generalized function, regarded as a linear continuous functional, onto the entire class of such functions, making explicit the arbitrariness that arises in this connection.

For this purpose we substitute Eq. (11) in Eq. (9); on making the interchange $(x^0 - \tau) \longleftrightarrow x^0$ we get

$$\lim_{\varepsilon\to 0}\frac{1}{\pi^2}\int dx^0\,(x^0)^n f_m^n(x^0 + \tau) \times\left[\frac{1}{2} + \frac{1}{\pi}\operatorname{arc\,ctg}\frac{x^0}{\varepsilon}\right]^{m-2}\frac{\varepsilon^2}{[(x^0)^2 + \varepsilon^2]^2}. \qquad (12)$$

*arc ctg = cot^{-1}

In Eq. (12) the only nonvanishing contribution is that from a range of integration $[-a, a]$ near the point $x^0 = 0$. It is not hard to show that the way of estimating integrals of the form (12) reduces to the examination of a number of simple integrals for small values of m and n, which can be calculated explicitly, but because of lack of space we shall not do this (cf. [7]).

The result is that in the solution of the problem of "surface" divergences in the higher orders of perturbation theory one encounters a total of two different cases. If m is even, the generalized function $K_{2k}(\tau - x^0)$ is integrable on the subclass of continuously differentiable functions $f_{2k}(x^0)$ having a first-order zero at the point $x^0 = \tau$, and for such functions the expression (9) is identically zero. By analogy with [6] we define it on the entire class of arbitrary continuously differentiable functions, choosing the special way of extending the functional which makes it identically zero. When we include the arbitrariness that arises here (cf. [6,8]), we have

$$K_{2k}(\tau - x^0) = \alpha_{02k}\delta(\tau - x^0), \qquad (13)$$

where α_{02k} are arbitrary finite constants. We note that in the case of spinor electrodynamics, for example, this settles everything.

If, on the other hand, m is odd, then the generalized function $K_{2k+1}(\tau - x^0)$ is integrable on the subclass of continuously twice differentiable functions $f_{2k+1}(x^0)$ which have a second-order zero at the point $x^0 = \tau$. We can define it to be identically zero on the entire class of arbitrary twice differentiable functions, and by general theorems of functional analysis (cf. e.g., [8]) the arbitrariness so introduced is of the form

$$K_{2k+1}(\tau - x^0) = \alpha_{02k+1}\delta(\tau - x^0) + \alpha_{1k}\delta'(\tau - x^0), \qquad (14)$$

where α_{02k+1} and α_{1k} are arbitrary finite constants.

This definition of the generalized functions $K_m(\tau - x^0)$ on corresponding classes of functions $f_m(x^0)$ allows us to construct a procedure for the "surface" regularization of the matrix $S(\sigma, -\infty)$, since it is clear that the expression for $S(\sigma, -\infty)$ will be finite in the "surface" sense if instead of Eq. (9) we consider the expression obtained by subtracting from it one (for even m) or two (for odd m) terms of its expansion in Maclaurin series centered at $x^0 = \tau$, and adding corresponding terms with arbitrary coefficients.

More explicitly, as the finite expression for $\lim_{g\to\theta_\tau}{}'' S(g)$, or in other words as the definition of the generalized functions Λ_n on a class of functions g(x) which includes the θ functions, it is most natural to take the expression

$$\tilde{S}(\tau, -\infty) = \lim_{g\to\theta_\tau} \tilde{S}(g) = T \exp\left\{i \lim_{g\to\theta_\tau} \int \tilde{L}(x; g)\, dx\right\}, \tag{15}$$

where

$$\lim_{g\to\theta_\tau} \int \tilde{L}(x; g)\, dx = \int_{-\infty}^{\tau} \tilde{L}(x; 1)\, dx = \int_{-\infty}^{\tau} \Big\{L(x; 1) + \frac{1}{2}\sum_{m=2} \frac{e^m}{m!} B_m \alpha_{0m} : \frac{\partial\varphi^2(x)}{\partial x_0} : + \frac{1}{2}\sum_{k=1} \frac{e^{2k+1}}{(2k+1)!} B_{2k+1}\alpha_{1k} : \frac{\partial^2\varphi^2(x)}{\partial x_0^2} : \Big\}\, dx, \tag{16}$$

and the second and third terms in the curly brackets in Eq. (16) embody the additional finite arbitrariness which arises in the introduction of the "surface" regularization.

The appearance of a finite arbitrariness in the $\tilde{S}(\tau, -\infty)$ of the form (15) should not especially surprise us, because it is a reflection of the additional arbitrariness in the coefficient functions of the Dyson matrix (for a fixed S matrix in the entire momentum space) which was to be expected (cf. [9]) because of the presence of an ambiguity in the definition of the Heisenberg field A(x). We shall hereafter analyze in detail the physical meaning of the finite "surface" counterterms which have appeared and of their contribution to the Dyson matrix.

It must also be pointed out that the expression $\tilde{S}(g)$ of the form (15), like the S(g) of the form (1), satisfies the requirements of relativistic covariance, unitarity, and causality, and also considerations of correspondence with the classical theory, but it cannot be used as a generalized scattering matrix for smooth functions g(x), because for $M_1^2 \to \infty$ it contains "ultraviolet" divergences. At the same time for g = 1

$$\tilde{S}(1) = S(1), \tag{17}$$

i.e., the S matrix is still the same.

3. GENERALIZED GAUGE TRANSFORMATION OF THE DYSON MATRIX

Passing now to the analysis of the role of finite "surface" counterterms in the Dyson matrix we first call attention to the following fact. The book of Bogolyubov and Shirkov [5] already expressed the idea that the vacuum "surface" counterterms are not important terms of the Hamiltonian H(τ) and can be removed from it by shifting the time phase by an infinite constant. It is true that we have shown [4] that just for the vacuum diagrams in H(τ) the problem of "surface" divergences does not arise, so that in this case there is no need of such a transformation. For boson self-energy diagrams also, however, the "surface" character of the terms added to L(x; 1) in the expression (16) leads to the idea that it may be possible to separate them out from $\tilde{S}(\tau, -\infty)$ into some unitary factor of no importance from the physical point of view, and thus that there may exist a generalized gauge transformation allowing us to remove such terms from the interaction Hamiltonian. In fact, the second and third terms in Eq. (16) give contributions to the Dyson matrix of the forms

$$\int_{-\infty}^{\tau} dx : \frac{\partial\varphi^2(x)}{\partial x_0} : = \int d\mathbf{x} : \varphi^2(x) : \Big|_{x^0=\tau}, \tag{18}$$

$$\int_{-\infty}^{\tau} dx : \frac{\partial^2\varphi^2(x)}{\partial x_0^2} : = \int d\mathbf{x} : \frac{\partial\varphi^2(x)}{\partial x_0} : \Big|_{x^0=\tau}, \tag{19}$$

that is, in other words, contributions only on the surface τ = const.

In this case the expression which follows from Eq. (15),

$$\tilde{S}(\tau, -\infty) = T \exp\Big\{i \int_{-\infty}^{\tau} L(x; 1)\, dx + i \int_{-\infty}^{\tau} dx : \frac{\partial\varphi^2(x)}{\partial x_0} : \frac{1}{2}\sum_{m=2} \frac{e^m}{m!} B_m\alpha_{0m}\Big\}, \tag{20}$$

can be transformed[2)] by representing the exponential as the product of two factors, and the factor that depends only on points of the surface τ = const can be brought out from under the sign of the T-product. An analogous result is also obtained if we first bring the second factor out from under the sign of the T-product and then integrate over x^0, because the necessary condition for this, $[L(\mathbf{x}, \tau; 1), \varphi^2(\mathbf{y}, \tau)] = 0$, is satisfied. Then

$$\tilde{S}(\tau, -\infty) = \exp\Big\{i \int d\mathbf{x} : \varphi^2(x) : \Big|_{x^0=\tau} \cdot \frac{1}{2}\sum_{m=2} \frac{e^m}{m!} B_m\alpha_{0m}\Big\} \times S(\tau, -\infty) = \exp\{iF(\tau)\}\, S(\tau, -\infty), \tag{21}$$

where

$$S(\tau, -\infty) = T \exp\Big\{i \int_{-\infty}^{\tau} L(x; 1)\, dx\Big\}. \tag{22}$$

Thus all of the finite "surface" counterterms of the matrix $S(\tau, -\infty)$ can be separated out into a unitary operator, which becomes unity for $\tau \to \infty$

---

2)Here and in what follows we shall for simplicity take all the $\alpha_{1k} = 0$, but the general case can also be treated.

[or drops out from the product $S(\infty, \tau)S(\tau, -\infty)$], i.e., does not change the value of the S matrix. The basic part of the expression for the Dyson matrix is the same for all theories (cf. [4]) and is of the form (22). Thus the arbitrariness associated with the "surface" counterterms is in fact just the arbitrariness which is possible in the Dyson matrix for a fixed S matrix.[3)]

We note, by the way, that if we had not introduced the apposite redefinition of the nonintegrable expressions which arose, it would have been by no means obvious whether or not the second term in Eq. (7) would vanish in the limit in the S matrix, because we should then have had to do with an expression of the form (18) but with an infinite coefficient. Thus the definition we have developed here of the operators Λ_n as generalized functions integrable on a class of functions $g(x)$ which includes θ functions is to some extent dictated by the requirements of internal consistency of the Bogolyubov method. Namely, it is necessary to recall that for fixed M_i^2 one must get the same S matrix from the matrix $S(g)$ both for $g \to 1$ and for $g \to \theta_\tau$ followed by $\tau \to \infty$.

It is natural that for the Hamiltonian $H(x; \sigma)$ of the form (3), starting from an $\widetilde{S}(\sigma, -\infty)$ of the form (21), we get an expression with "surface" counterterms which we shall give in the next section, but starting from an $S(\sigma, -\infty)$ of the form (22) we get the formula

$$H(x; \sigma) = -L(x; 1) \tag{23}$$

(on some supplementary assumptions which are explained below). In other words, one can always carry out a generalized gauge transformation of the state vectors,[10,11] $\Phi(\sigma) \to \exp\{-iF(\sigma)\}\Phi(\sigma)$, so as to remove the finite "surface" terms from the Dyson matrix and the interaction Hamiltonian (without changing the expression for the S matrix), and take as the effective Hamiltonian $H(x; \sigma)$ the expression (23) common to all theories.

Furthermore it is not hard to show how the presence of a finite "surface" arbitrariness in the Dyson matrix affects the connection between operators in the Heisenberg and interaction representations (beginning with the second order). For example, for the field operators we have

$$\begin{aligned} A(x) &= \widetilde{S}^+(x^0, -\infty)\,\varphi_{in}(x)\,\widetilde{S}(x^0, -\infty) \\ &= S^+(x^0, -\infty)\,\widetilde{\varphi}_{in}(x) S(x^0, -\infty), \end{aligned} \tag{24}$$

[3)]The fact that the Dyson matrix must be defined by putting conditions on it with accuracy up to a unitary operator which depends only on points of a surface σ has been established outside the framework of perturbation theory of B. V. Medvedev (private communication).

where

$$\widetilde{\varphi}_{in}(x) = \exp\{-iF(x^0)\}\,\varphi_{in}(x)\exp\{iF(x^0)\}. \tag{25}$$

Since $\widetilde{\varphi}_{in}(x)$ differs from $\varphi_{in}(x)$ only by a unitary transformation, this operator belongs to an equivalent representation of the canonical commutation relations.[12] Therefore from the physical point of view the descriptions of the system by means of the operators $\varphi_{in}(x)$, $\widetilde{S}(\sigma, -\infty)$ and $\widetilde{\varphi}_{in}(x)$, $S(\sigma, -\infty)$ are on precisely the same footing.

4. SOME FEATURES OF THE DERIVATION OF THE INTERACTION HAMILTONIAN

Turning to the derivation of $H(x; \sigma)$ according to Eq. (3) from the expression for $\widetilde{S}(\sigma, -\infty)$ of the form (21), we can easily get instead of Eq. (23) the formula

$$\begin{aligned} \widetilde{H}(x; \sigma) &= -\widetilde{L}(x; 1) + \frac{1}{2}\left[n_k \frac{\partial \widetilde{L}(x; 1)}{\partial(\partial\varphi/\partial x^k)}\right]^2 \\ &= \left\{-L(x; 1) - \frac{1}{2}\sum_{m=2} \frac{e^m}{m!} B_m \alpha_{0m} : \left(n_k \frac{\partial \varphi^2}{\partial x^k}\right):\right\} \\ &+ \frac{1}{2}\sum_{m=4}\sum_{k=2}^{m-2} \frac{e^m}{k!\,(m-k)!} B_k B_{m-k}\alpha_{0k}\alpha_{0m-k} : \varphi^2(x):, \end{aligned} \tag{26}$$

where n_k is the unit vector normal to the surface σ. The third term in $H(x; \sigma)$, which is necessary to satisfy the condition of integrability, appears automatically in the Bogolyubov method (just as in theories with derivative couplings [3]) because in expanding the expression $\left\langle T\left(\frac{\partial\varphi^2}{\partial x^\alpha}\frac{\partial\varphi^2}{\partial y^\beta}\right)\right\rangle_0$ one must use a definition of the chronological contraction (pairing) which contains a quasi-local term.

However, although the expressions (23) and (26) are admissible expressions (for fixed M_i^2) for the interaction Hamiltonian, or in other words, although by using them in the solution of the Tomonaga-Schwinger equation one can obtain the expressions $S(\sigma, -\infty)$ and $\widetilde{S}(\sigma, -\infty)$ in the indicated form, these expressions (23) and (26) are still not the most general. Furthermore the problem that now arises is associated not with the problem of "surface" regularization, but with that of satisfying the conditions of integrability for the main term of the Hamiltonian of the form $-L(x; 1)$.

The point is that unlike the cases considered in [3], an $L(x; 1)$ of the form (5) contains not the first derivative, but the second derivative of the field functions. For such a Lagrangian one can no longer directly obtain the Hamiltonian by the method of Matthews,[11,3] which in its usual form is designed only for the case of theories with Lagrangians containing the first derivatives of the fields. As for the conditions of integrability, they are formally satisfied for the term of the form $-L(x; 1)$ in the Hamiltonian (cf. [3]).

When, however, we derive $H(x; \sigma)$ by the formula (3), it turns out that besides contributions from the quasi-local operators $\Lambda_n(x_1 \ldots x_n)$ themselves there are contributions from expressions of the forms $T(\Lambda_n(x_1 \ldots x_n)\Lambda_m(y_1 \ldots y_m))$, $T(\Lambda_n(x_1 \ldots x_n)\Lambda_m(y_1 \ldots y_m)\Lambda_k(z_1 \ldots z_k))$, and so on, where the indices n, m, k, . . . run through values from 1 to ∞ and $\Lambda_1(x) = L(x)$. This occurs for the same reasons that lead [3] to the appearance in $H(x; \sigma)$ of terms depending quadratically on the normals—that is, because in the Bogolyubov method one uses definitions of the chronological pairings of the form

$$i\left\langle T\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\right)\right\rangle_0 = \theta(x^0 - y^0)\frac{\partial^4 D^-(x-y)}{\partial x^2\,\partial y^2} - \theta(y^0 - x^0)\frac{\partial^4 D^+(x-y)}{\partial x^2\,\partial y^2} + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta(x-y) - m^2\delta(x-y). \quad (27)$$

When, now, we go to derive $H(x; \sigma)$, only the first two terms of the expression (27) vanish, and, as was indeed to be expected, the quasi-local operators in them contribute to $H(x; \sigma)$.

Furthermore, if the question of the passage to the limit $g \to \theta_\sigma$ in the part of $S(\sigma, -\infty)$ corresponding to the terms of the expression (27) which simply contain the δ function requires no special explanation, still the treatment of the limiting process in the part of $S(\sigma, -\infty)$ corresponding to the expression $\frac{1}{2}(\partial^2/\partial x^2 + \partial^2/\partial y^2)\delta(x-y)$ in Eq. (27) also leads to a problem of "surface" divergences. Thus it turns out that the quantities not defined on a class of functions $g(x)$ which includes θ functions, and hence still in need of regularization, include not only the operators Λ_n but also the expression for the pairing $\left\langle T\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\right)\right\rangle_0$ in the form (27), which follows directly from [5]. The "surface" regularization of such expressions can be carried out in analogy with the treatment given above, and the finite "surface" terms that then arise are of the nature already indicated and can also be separated out from the Dyson matrix into a unitary multiplying factor.

Besides these additional "surface" terms, however, the presence of quasi-local operators in expressions of the type (27) leads to the appearance in $H(x; \sigma)$ of additional terms of the same operator structure as that of the terms of $-L(x; 1)$, but with different coefficients, which also depend on M_i^2. As an example we write out the additional terms of the final Hamiltonian $H(x; \sigma)$ that are obtained from $S(\sigma, -\infty)$ of the form (22) and contribute to the fourth-order expression for $H(x; \sigma)$. These terms are

$$\Delta H(x; \sigma) = -\left\{:\frac{\partial L(x; 1)}{\partial\varphi(x)}\frac{\partial L(x; 1)}{\partial(\partial^2\varphi/\partial x^2)}: - \frac{1}{2}:\frac{\partial L(x; 1)}{\partial(\partial^2\varphi/\partial x^2)}\frac{\partial^2}{\partial x^2}\left[\frac{\partial L(x; 1)}{\partial(\partial^2\varphi/\partial x^2)}\right]: + \frac{m^2}{2}\left[\frac{\partial L(x; 1)}{\partial(\partial^2\varphi/\partial x^2)}\right]^2\right\}. \quad (28)$$

In the higher orders in $H(x; \sigma)$ there will be contributions not only from the terms of Eqs. (23) and (28), but also from more complicated combinations of larger numbers of factors consisting of derivatives (including higher derivatives) of $L(x; 1)$ with respect to $\varphi(x)$ and $\partial^2\varphi/\partial x^2$, which there is no use writing out, since it is easy to establish the law of their occurrence, by using for the development of the T-product formulas of the type

$$T(L(x)L(y)) = :L(x)L(y): + \frac{1}{i}D^c(x-y):\frac{\partial L(x)}{\partial\varphi(x)}\frac{\partial L(y)}{\partial\varphi(y)}: + \ldots. \quad (29)$$

Thus we find that even without inclusion of the "surface" counterterms $H(x; \sigma) \neq -L(x; 1)$ in the Bogolyubov method.

In order to understand what is involved here, we recall (cf. [13,3]) that since the integrability condition contains first variational derivatives of $H(x; \sigma)$ it determines $H(x; \sigma)$ only up to an arbitrary function of the field operators which does not depend on the shape of the surface σ. Usually this function is set equal to zero by considerations of simplicity.[13]

Naturally, in the given case, although $L(x; 1)$ satisfies the integrability condition we can add to $-L(x; 1)$ a term $\Delta L(x)$ satisfying the conditions

$$[\Delta L(x), L(y; 1)] = [\Delta L(x), \Delta L(y)] = 0 \quad \text{for } x \sim y \quad \text{or } x = y. \quad (30)$$

It is not hard to see that the expression (28) actually satisfies the conditions (30). Thus from the point of view of satisfying the integrability condition the addition of terms of the form (28) to $H(x; \sigma)$ has no effect, and it is necessary to bring in other arguments to make a correct choice of $H(x; \sigma)$, because there is no justification in this case for using considerations of simplicity.

If we start from considerations of correspondence with the finite (for $M_i^2 \to \infty$) S matrix obtained in the Bogolyubov method,[5] it is necessary to include additional terms of the form (28) in

$H(x; \sigma)$ so that when we derive $S(\sigma, -\infty)$ (or the S matrix) by solving the Tomonaga-Schwinger equation we may be able to proceed to the definition of pairings of the form (27) adopted in [5] without changing the expression for $S(\sigma, -\infty)$. But for fixed M_i^2, as assumed everywhere in this paper, we can also use the expression (23) to obtain $S(\sigma, -\infty)$. Thus the problem of satisfying the integrability condition for the effective Lagrangian $L(x; 1)$ is of a narrower character in the Bogolyubov method than might at first glance be expected. [3] In particular, although the boson proper-energy counterterms contain derivatives of the fields they actually do not lead to terms in $H(x; \sigma)$ depending quadratically on the normals.

5. DISCUSSION

Thus within the framework of the Bogolyubov method one can carry out in a reasonable way a "surface" regularization of the Dyson matrix (for fixed M_i^2), and in perturbation theory one can get a finite expression for this matrix for any renormalizable theory. Furthermore, as long as the M_i^2 are finite, we can with equal success give the name of Dyson matrix either to the expression $\widetilde{S}(\sigma, -\infty)$ of the form of Eq. (21) or to the $S(\sigma, -\infty)$ of Eq. (22).

Meanwhile a question of great interest is whether one can carry out the transition $M_i^2 \to \infty$ in the resulting expression for the Dyson matrix. It is easy to see that the factor $\exp\{iF(\sigma)\}$ itself diverges for $M_i^2 \to \infty$, though one can separate out from it a finite part of the same operator structure. As for the expression for $S(\sigma, -\infty)$, our preliminary investigation shows that in the case of the Hearst-Thirring field and for all diagrams of other theories that do not contain boson proper-energy insertions the transition $M_i^2 \to \infty$ in $S(\sigma, -\infty)$ in all probability is possible and leads in every order of perturbation theory to an expression which is finite in the ordinary and "surface" senses and satisfies the requirements of correspondence with the S matrix.

When we consider the transition $M_i^2 \to \infty$ in the Dyson matrix for a theory with $L(x; g)$ of the form (5) our attention is caught by the fact that the finite arbitrary constants α_{0m} have the dimensions of mass and that in principle one can even set $\alpha_{0m} \sim M_i$. Thus there is a possibility that for this theory also either $S(\sigma, -\infty)$ or $\widetilde{S}(\sigma, -\infty)$ will be finite in every order for $M_i^2 \to \infty$ (for a suitable choice of α_{0m}). Further studies are needed, however, for a final solution of this problem. Still, apart from this, the results of this paper (together with [3,4]) show that within the framework of the Bogolyubov method one can construct a scheme of the Hamiltonian formalism which allows a consistent derivation of the usual expression for the S matrix starting from the Tomonaga-Schwinger equation (if we go to the limit $M_i^2 \to \infty$ already in the S matrix).

Another interesting and still unsettled question is that of obtaining the Dyson matrix according to Eq. (1) with a different order of the passages to limits, i.e., first letting $M_i^2 \to \infty$, and then $g \to \theta_\sigma$. A preliminary examination shows that for the Hearst-Thirring field the expression for $S(\sigma, -\infty)$ does not depend on the order of the passages to the limits, but in a theory with $L(x; g)$ of the form (5) new difficulties arise, which have not yet been overcome. Another open question is whether it is possible to construct a finite Dyson matrix outside the framework of perturbation theory. Worthy of attention in this connection are attempts [14] to approach the derivation of the Dyson matrix and the interaction Hamiltonian in such a way that no divergent expressions appear at any stage.

Finally, let us touch on a problem posed by Haag [12] outside the framework of perturbation theory—that there may not exist any finite Dyson matrix, because it must connect fields in the Heisenberg and interaction representations [cf. Eq. (24)] which belong to nonequivalent representations of the canonical commutation relations. It is easy to see that by Eq. (24) the value of the commutator

$$\langle [\dot{A}(\mathbf{x}, \tau),\ A(\mathbf{y}, \tau)] \rangle_0 = -iZ^{-1}\delta(\mathbf{x} - \mathbf{y}), \qquad (31)$$

where $Z^{-1} > 1$, does not depend on whether we use the matrix $\widetilde{S}(\sigma, -\infty)$ or $S(\sigma, -\infty)$, so that the "surface" arbitrariness we have discussed has no effect on the quantity Z^{-1} and leaves the question open as to the existence of the Dyson matrix in the general case.

On the other hand, in perturbation theory one may be able to construct an expression for the Dyson matrix which is finite even for $M_i^2 \to \infty$. If this is so only for the Hearst-Thirring field, then in all probability this case can be regarded as an exception, for which Haag's problem does not arise. If, however, one manages to construct an expression for $S(\sigma, -\infty)$ which is finite for $M_i^2 \to \infty$ in every renormalizable theory, then it is quite possible that this problem, which arises in the framework of the axiomatic method, has no meaning in any order of perturbation theory, but is connected with problems of the summability of the perturbation-theory series.

In conclusion I express my sincere gratitude to N. N. Bogolyubov and V. S. Vladimirov, who in conversations have helped greatly to clear up for the writer a number of questions of principle. I also express my gratitude to D. V. Shirkov, B. V. Medvedev, and D. A. Slavnov for a helpful discussion.

[1] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[2] E. C. G. Stueckelberg, Phys. Rev. **81**, 130 (1951).

[3] A. D. Sukhanov, JETP **41**, 1915 (1961), Soviet Phys. **14**, 1361 (1962).

[4] A. D. Sukhanov, JETP **43**, 932 (1962), Soviet Phys. **16**, 660 (1963).

[5] N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh polei (Introduction to the Theory of Quantized Fields), Gostekhizdat, 1957.

[6] A. D. Sukhanov, DAN SSSR **145**, 1042 (1962), Soviet Phys. Doklady **7**, in press.

[7] A. D. Sukhanov, Candidate's Dissertation, V. A. Steklov Mathematics Institute, Academy of Sciences, U.S.S.R., 1962.

[8] I. M. Gel'fand and G. E. Shilov, Prostranstva osnovnykh i obobshchennykh funktsiĭ (Spaces of Basic and Generalized Functions), Fizmatgiz, 1958.

[9] D. A. Slavnov and A. D. Sukhanov, JETP **41**, 1940 (1961), Soviet Phys. JETP **14**, 1379 (1962).

[10] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[11] P. T. Matthews, Phys. Rev. **76**, 1657 (1949).

[12] R. Haag, Kgl. Danske, Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 12 (1955).

[13] S. Kanesawa and Z. Koba, Progr. Theoret. Phys. **4**, 297 (1949).

[14] D. A. Slavnov, DAN SSSR **143**, 570 (1962), Soviet Phys. Doklady **7**, 220 (1962).

Translated by W. H. Furry
242