THE THREE-BODY PROBLEM FOR SHORT RANGE FORCES IN THE LINEAR APPROXIMATION IN THE FORCE RANGE

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Equations for the scattering amplitude for a system of three particles are derived in the linear approximation in $E^{1/2}r_0$, assuming short-range resonance-type forces between the particles (E is the energy of the system, r_0 is the range of the forces). In this approximation the scattering amplitude can be expressed in terms of the zero-energy two-particle amplitudes, the energy of the three-particle bound state, and the range of the forces.

INTRODUCTION

An equation for the wave function of a system of three identical particles interacting via short-range resonance-type forces has been obtained by Skornyakov and Ter-Martirosyan (STM) [1] in the limit $r_0 \to 0$, where r_0 is the range of the forces. This equation has been investigated in a paper of the author. [2] It was shown in [2] that this equation has a non-unique solution. In choosing the solution, one must require that the wave functions for different energies be orthogonal to one another. The amplitude for the scattering of one of the particles on the two remaining particles can be expressed in terms of the two-particle amplitudes and a single three-particle parameter, for example, the energy of the three-particle bound state. The non-uniqueness of the solution of the STM equation is connected with the fact that a three-particle system collapses into the center in the limit $r_0 \to 0$. [3,4]

As a result of the collapse into the center the STM equation gives rise to an infinite number of levels for the system of three particles interacting via resonance-type forces. However, we must consider only those levels whose energies satisfy the inequality $|ME/\hbar|^{1/2}r_0 \ll 1$, since the other levels lie outside the range of applicability of the theory. For realistic values of r_0 and a_0, only one level lies within the range of validity of the theory. The energy of the next level is, according to Minlos and Faddeev, [4] about 1000 times larger in absolute value than the energy of the former. Minlos and Faddeev [4,5] express the opinion that the presence of an infinite number of levels casts some doubt on the validity of the STM equation. Actually, the presence of levels so remote cannot affect the validity of the STM equation for real systems, but simply reflects the fact that three nucleons could form two bound states if the range of the forces were $1/50$ of the actual range.

Minlos and Faddeev [5] have proposed a set of equations for the description of a three-particle system with short range forces which is different from the STM equation. However, as will be shown in the present paper, their equations are valid only in the presence of three-body forces with a range $R_0 \gg r_0$. We shall not consider such forces, since they are not in agreement with our present ideas of the interaction mechanism between particles.

In Sec. 2 of the present paper we shall derive an equation for the determination of the scattering amplitude for a system of three particles with an accuracy up to and including terms $\sim |ME/\hbar|^{1/2}r_0$. In this approximation the three-particle amplitude is expressed in terms of the two-particle amplitudes at zero energy, the range of the two-body forces r_0, and the energy of the three-particle bound state. The method of deriving this equation is illustrated in Sec. 1 on the example of the two-body problem. For simplicity we shall assume that all particles are identical and spinless. The generalization to other cases is obvious.

1. TWO-PARTICLE WAVE FUNCTION ACCURATE TO TERMS $\sim kr_0$

In order to illustrate the method to be used later for the derivation of the three-particle wave function, we consider first the scattering for a system of two identical spinless particles. The equation for the wave function $\psi(\mathbf{r})$ of this system is

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}} + \int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} v(r')\psi(\mathbf{r}')\,d^3r'. \quad (1)$$

Here k is the wave vector in the center of mass system (c.m.s.) and the quantity v(r) is related to the potential V(r) by

$$v(r) = -MV(r) / 4\pi\hbar^2, \tag{2}$$

where M is the mass of the particle.

For $kr_0 \ll 1$, $r \lesssim r_0$, where r_0 is the range of the forces, Eq. (1) can be written as

$$\psi(\mathbf{r}) = 1 + \int \frac{1}{|\mathbf{r}-\mathbf{r}'|} v(r') \psi(\mathbf{r}') d^3r' + \left(ik - \frac{k^2 r_0}{2}\right) \int v(r') \psi(\mathbf{r}') d^3r' + \Delta_1(\mathbf{r}) + \Delta_2(\mathbf{r}); \tag{3}$$

$$\Delta_1(\mathbf{r}) = -\frac{k^2}{2} \int (|\mathbf{r}-\mathbf{r}'| - r_0) v(r') \psi(\mathbf{r}') d^3r', \tag{4}$$

where r_0 is some parameter of the order of magnitude of the range of the forces. The parameter r_0 will be defined more precisely below. The quantity $\Delta_2(\mathbf{r})$ contains terms $\sim \mathbf{kr}$ and $k^3 r_0^2 a_0$ arising from the expansion of the exponential in (1).

Let us introduce the function $\tilde{\psi}(\mathbf{r})$ defined by

$$\psi(\mathbf{r}) = A\tilde{\psi}(\mathbf{r}), \tag{5}$$

$$A = 1 + \left(ik - \frac{k^2 r_0}{2}\right) \int v(r') \psi(\mathbf{r}') d^3r'. \tag{6}$$

The function $\tilde{\psi}(\mathbf{r})$ satisfies the equation

$$\tilde{\psi}(\mathbf{r}) = 1 + \int \frac{1}{|\mathbf{r}-\mathbf{r}'|} v(r') \tilde{\psi}(\mathbf{r}') d^3r' + \tilde{\Delta}_1(\mathbf{r}) + \tilde{\Delta}_2(\mathbf{r});$$

$$\tilde{\Delta}_1(\mathbf{r}) = \Delta_1(\mathbf{r})/A, \quad \tilde{\Delta}_2(\mathbf{r}) = \Delta_2(\mathbf{r})/A. \tag{7}$$

We solve Eq. (7) by regarding $\tilde{\Delta}_1(\mathbf{r})$ and $\tilde{\Delta}_2(\mathbf{r})$ as inhomogeneous terms. For this purpose we introduce the eigenfunctions $\varphi_n(\mathbf{r})$ of the equation

$$\varphi_n(\mathbf{r}) = \lambda_n \int \frac{1}{|\mathbf{r}-\mathbf{r}'|} v(r') \varphi_n(\mathbf{r}') d^3r'. \tag{8}$$

It is easily seen that the functions $\varphi_n(\mathbf{r})$ corresponding to different values of λ_n satisfy the condition

$$\int \varphi_n(\mathbf{r}) v(r) \varphi_m(\mathbf{r}) d^3r = 0 \text{ for } n \neq m.$$

We can therefore assume that the functions $\varphi_n(\mathbf{r})$ are orthogonal to one another with weight v(r) and are normalized to unity:

$$\int \varphi_n(\mathbf{r}) v(r) \varphi_m(\mathbf{r}) d^3r = \delta_{nm}. \tag{9}$$

With this normalization, the function $|\mathbf{r}-\mathbf{r}'|^{-1}$ is equal to [1]

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=1}^{\infty} \frac{\varphi_n(\mathbf{r}) \varphi_n(\mathbf{r}')}{\lambda_n}. \tag{10}$$

It is seen at once that $\tilde{\psi}(\mathbf{r})$ is given in terms of the functions $\varphi_n(\mathbf{r})$ according to the formula

$$\tilde{\psi}(\mathbf{r}) = \psi_0(r) + \tilde{\Delta}_1(\mathbf{r}) + \tilde{\Delta}_2(\mathbf{r}) + \sum_{n=1}^{\infty} \frac{\varphi_n(\mathbf{r})}{\lambda_n - 1} \int \varphi_n(\mathbf{r}') v(r') [\tilde{\Delta}_1(\mathbf{r}') + \tilde{\Delta}_2(\mathbf{r}')] d^3r', \tag{11}$$

where

$$\psi_0(r) = 1 + \sum_{n=1}^{\infty} \frac{\varphi_n(\mathbf{r})}{\lambda_n - 1} \int \varphi_n(\mathbf{r}') v(r') d^3r' \tag{12}$$

is the wave function of the system at zero energy, which satisfies (7) without the last two terms. It follows from (12) that the zero energy amplitude a_0 is equal to

$$a_0 = \int v(r) \psi_0(r) d^3r = \int v(r) d^3r + \sum_{n=1}^{\infty} \frac{[\int \varphi_n(\mathbf{r}) v(r) d^3r]^2}{\lambda_n - 1}. \tag{13}$$

If the system has no real or virtual level for energies $E = E_0 \ll r_0^{-2}\hbar^2 M^{-1}$, then $a_0 \lesssim r_0$. We shall be interested in the case when there is such a level. Then $a_0 \sim |ME_0/\hbar^2|^{1/2} \gg r_0$. The ground state wave function $\varphi_d(\mathbf{r})$ then satisfies the equation

$$\varphi_d(r) = \int \frac{1}{|\mathbf{r}-\mathbf{r}'|} v(r') \varphi_d(r') d^3r' \tag{14}$$

up to terms of order $\sim r_0/a_0$ relative to the main term. Comparing this equation with (8), we see that in this case one of the λ_n, which we shall denote by λ, is close to unity. The function $\varphi_n(\mathbf{r})$ with $\lambda_n = \lambda$ will be denoted by $\varphi(\mathbf{r})$. Evidently, $\varphi(\mathbf{r})$ is spherically symmetric, since it coincides with the bound state wave function $\varphi_d(\mathbf{r})$ up to terms $\sim r_0/a_0$.

The integrals $\tilde{r}_0 = \int v(r) d^3r$ and $r_n = \left[\int \varphi_n(\mathbf{r}) v(r) d^3r\right]^2$ remain finite as $|\lambda - 1| \to 0$ and are small compared to a_0 for sufficiently small $|\lambda - 1|$. These quantities are in general of the order of magnitude of the range of the forces. Indeed, it follows from (11) that for $kr_n \sim 1$ not all the terms in the sum over n are small, and the determination of the function $\tilde{\psi}(\mathbf{r})$ requires the knowledge of the functions $\varphi_n(\mathbf{r})$, which are in turn determined by the shape of the potential. The condition $k\tilde{r}_0 \ll 1$ has been used by us in obtaining Eq. (3). Thus the parameters $\tilde{r}_0$ and r_n determine the effective range of the interaction. We shall therefore regard $k\tilde{r}_0$ and kr_n as small parameters and deter-

[1] Formula (10) follows from the general theory of integral equations[6] if the potential v(r) has no zeros anywhere. It can be shown that (10) is valid for all potentials for which the right-hand side of (10) converges. However, as will be shown below, the results of this section are also valid when the right-hand side of (10) has no meaning.

mine the function $\widetilde{\psi}(\mathbf{r})$ with an accuracy up to terms $\sim kr_0$ and kr_n inclusive.

The terms $\widetilde{\Delta}_1(r)$ and $\widetilde{\Delta}_2(r)$ on the right-hand side of (11) and all terms in the sum over n except the large one proportional to $(\lambda-1)^{-1}$ will be $\sim k^2 r_n r_0$ compared to $\psi_0(r)$ and can be neglected. We choose the parameter r_0 such that the term proportional to $(\lambda-1)^{-1}$ vanishes with the required accuracy. For this we must require

$$\int \varphi(r)\, v(r)\, \widetilde{\Delta}_1(\mathbf{r})\, d^3r = 0, \tag{15}$$

which implies

$$r_0 = \frac{\int \psi_0(r)\, v(r)\, |\mathbf{r}-\mathbf{r}'|\, v(r')\, \varphi(r')\, d^3r\, d^3r'}{\int \psi_0(r)\, v(r)\, d^3r \int \varphi(r')\, v(r')\, d^3r'}$$
$$\approx \frac{1}{a_0^2} \int \psi_0(r)\, v(r)\, |\mathbf{r}-\mathbf{r}'|\, \psi_0(r')\, v(r')\, d^3r d^3r'. \tag{16}$$

With r_0 chosen in this way, it follows from (11) that

$$\widetilde{\psi}(\mathbf{r}) = \psi_0(r) \tag{17}$$

up to terms $\sim k^2 r_0^2$ as compared to the main term. Therefore the wave function $\psi(\mathbf{r})$ for $r \lesssim r_0$ is, with the same accuracy,

$$\psi(r) = A\psi_0(r), \tag{18}$$

where A is given by (6).[2] Substituting (18) in the right-hand side of (6), we obtain an equation for A which leads to

$$A = (1 - ika_0 + k^2 a_0 r_0/2)^{-1}. \tag{19}$$

Up to and including terms $\sim kr_0$, formula (19) can be written as

$$A = \frac{\alpha}{\alpha + ik}\left(1 - \frac{ikr_0}{2}\right), \tag{20}$$

where $\hbar^2\alpha^2/M = E_0$ is the energy of the bound state.

The amplitude a_0 is related to α by

$$a_0 = -\alpha^{-1}(1 + {}^1/{}_2 r_0 \alpha). \tag{21}$$

[2]If the expansion (10) diverges, it follows from the theory of integral equations that the function $\widetilde{\psi}(\mathbf{r})$ can, as before, be written in a form analogous to (11) by separating the large term containing $(\lambda - 1)^{-1}$ from the sum over n and writing the rest of the sum in the form of the integral

$$\int R(\mathbf{r}, \mathbf{r}')\, v(r')\, [\widetilde{\Delta}_1(r') + \widetilde{\Delta}_2(r')]\, d^3r'.$$

The form of the function $R(\mathbf{r}, \mathbf{r}')$ is given by the form of the potential. The integral containing $R(\mathbf{r}, \mathbf{r}')$ can be neglected for the same reason as the sum over n in (11). This applies also to the formulas obtained below in Sec. 2. The results of the present paper are therefore valid independently of any assumptions about the convergence of the expansion (10).

The amplitude $a(k)$ for the energy $E = \hbar^2 k^2/M$ is equal to

$$a(k) = -\frac{1}{\alpha + ik}\left(1 + \frac{1}{2} r_0 \alpha - \frac{ikr_0}{2}\right). \tag{22}$$

The assertion that, for $r \lesssim r_0$, the wave functions for different energies are proportional to one another with an accuracy up to and including terms $\sim kr_0$ applies also to the bound state function $\varphi_d(r)$. The function $\varphi_d(r)$ satisfies Eq. (1) without the free term. The kernel of this equation can be expanded in powers of $\alpha|\mathbf{r}-\mathbf{r}'|$ (see above). It is then evident that

$$\varphi_d(r) = N\widetilde{\psi}(r) = N\psi_0(r). \tag{23}$$

The factor N is determined by the normalization condition

$$\int \varphi_d^2(r)\, d^3r = 1, \tag{24}$$

substituting for $\varphi_d(r)$ in (24) the expression

$$\int \frac{e^{-\alpha|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, v(r')\, \varphi_d(r')\, d^3r'$$
$$= N \int \frac{e^{-\alpha|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, v(r')\, \psi_0(r')\, d^3r'.$$

The calculation shows that the factor N has no terms $\sim \alpha r_0$, so that

$$N^2 = \alpha^3/2\pi,\; N = -\alpha\sqrt{\alpha/2\pi}. \tag{25}$$

With such a choice of N, the function $\varphi_d(r)$ is for $r \gg r_0$ equal to

$$\varphi_d(r) = \sqrt{\frac{\alpha}{2\pi}}\left(1 + \frac{\alpha r_0}{2}\right)\frac{e^{-\alpha r}}{r}.$$

2. EQUATIONS FOR THE WAVE FUNCTION OF A SYSTEM OF THREE IDENTICAL PARTICLES

Let us now apply the method developed above to the problem of three particles (three nucleons) and obtain equations for the amplitudes for the scattering of a particle on the bound state of the two other particles (scattering of a neutron on a deuteron) with an accuracy up to and including terms $\sim \alpha r_0$ and $|ME/\hbar^2|^{1/2} r_0$, where $\hbar^2\alpha^2/M$ is the binding energy of the deuteron and E the energy of the system. We shall assume for simplicity that all particles are identical and spinless. The exact equation for the wave function of such a system is ($\hbar = M = 1$, M is the mass of the particle)

$$F(\mathbf{r},\mathbf{k}) = \frac{8\pi\cos[(\mathbf{k}/2+\mathbf{k}_0)\mathbf{r}]\int\cos[(\mathbf{k}+\mathbf{k}_0/2)\mathbf{r}']v(r')\varphi_d(r')d^3r'}{k^2+k_0^2+\mathbf{k}\mathbf{k}_0-E}$$
$$+\int\frac{e^{-\gamma_k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}v(r')F(\mathbf{r}',\mathbf{k})d^3r'$$
$$+\frac{8\pi}{(2\pi)^3}\int\frac{\cos[(\mathbf{k}/2+\mathbf{k}')\mathbf{r}]}{k^2+k'^2+\mathbf{k}\mathbf{k}'-E-i\delta}$$
$$\times\cos\left[\left(\mathbf{k}+\frac{1}{2}\mathbf{k}'\right)\mathbf{r}'\right]v(r')F(\mathbf{r}',\mathbf{k}')d^3r'd^3k'$$
$$-\frac{1}{4\pi}\int\frac{e^{-\gamma_k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}',\boldsymbol{\rho}')e^{ik\rho'}\psi(\mathbf{r}',\boldsymbol{\rho}')d^3r'd^3\rho'. \qquad (26)$$

In contrast to formula (8) of STM, Eq. (26) contains an additional term involving the three-body potential $V(\mathbf{r},\boldsymbol{\rho})$, where $\mathbf{r}\equiv\mathbf{r}_{23}$, $\boldsymbol{\rho}\equiv\boldsymbol{\rho}_1$ are Jacobian coordinates.

We assume that the range of the three-body forces $R_0\lesssim r_0$. The function $\psi(\mathbf{r},\boldsymbol{\rho})$ in the last term of (26) is the wave function in the coordinate representation. The function $F(\mathbf{r},\mathbf{k})$ is equal to

$$F(\mathbf{r},\mathbf{k})=\int e^{-i\mathbf{k}\boldsymbol{\rho}}\psi(\mathbf{r},\boldsymbol{\rho})\,d^3\rho-(2\pi)^3\varphi_d(r)\,\delta(\mathbf{k}-\mathbf{k}_0),$$

where $\mathbf{k}_0$ is the wave vector of the incident particle and $\varphi_d(r)$ is the wave function of the deuteron.[3)] The quantity γ_k is equal to

$$\gamma_k^2={}^3/_4\,k^2-E,\qquad \gamma_k=-i|\gamma_k|,\ \text{if}\ \gamma_k^2<0.$$

Let us consider Eq. (26) for $|E|^{1/2}r_0$, $kr_0\ll1$, and $r\lesssim r_0$, and determine the function $F(\mathbf{r},\mathbf{k})$ in this region with an accuracy up to and including terms $\sim\alpha r_0$.[4)] For $kr_0\ll1$, the cosines in the first term of (26) can be replaced by unity. The kernel in the second term can be expanded in powers of $\gamma_k|\mathbf{r}-\mathbf{r}'|$, as we did before in solving Eq. (1). If we want to determine $F(\mathbf{r},\mathbf{k})$ with an accuracy up to and including terms of order $\sim\alpha r_0$, we cannot replace the cosines under the integral in the third term of (26) by unity because of the weak fall-off of the function $F(\mathbf{r},\mathbf{k})$ for

$$k\gg|E|^{1/2},\quad k\gg\alpha,\quad kr_0\ll1.$$

Indeed, in this region the function $F(\mathbf{r},\mathbf{k})$ has the asymptotic form [2]

$$F(\mathbf{r},\mathbf{k})\sim\varphi_d(r)\frac{1}{k^2\alpha}\left[A\left(\frac{k}{\alpha}\right)^{is_0}+B\left(\frac{k}{\alpha}\right)^{-is_0}\right],\qquad(27)$$

where $s_0\approx1$, and A and B are certain coefficients. Therefore, the terms neglected in the expansion of the cosines make a contribution of the order

$$\alpha^{1/2}\int_0^{r_0^{-1}}\frac{k^2r_0^2}{k^2}\frac{k^2dk}{k^2\alpha}\sim(\alpha r_0)\,\alpha^{-3/2}$$

to the integral over k'.

It is usually assumed that the contribution from the integrals (26) (including the last term) over the region $r\lesssim r_0$, $k'\lesssim r_0^{-1}$ is of the order $\sim r_0$ (larger $k'\lesssim r_0^{-1}$ evidently correspond to $\rho\lesssim r_0$). The validity of this assumption is, however, not obvious in the presence of a strong three-body interaction. For example, it can be shown [7] that the three-body forces, and hence the integrals over the region $r\lesssim r_0$, $k\gg r_0^{-1}$, can play an essential role if there are no two-body interactions so that the potential $V(\mathbf{r},\boldsymbol{\rho})$ is different from zero only for $R\lesssim R_0$, where $R^2=r^2+{}^4/_3\rho^2$. Nevertheless, it can be shown that in the presence of resonance-type two-body interactions the usual assumptions about the magnitude of the integrals (26) over the region $r\lesssim r_0$, $\rho\lesssim r_0$ are valid for arbitrary three-body forces with a range $R_0\lesssim r_0$. We shall therefore assume that the integrals (26) over the region $r\lesssim r_0$, $\rho\lesssim r_0$ contribute a term $\sim(\alpha r_0)\alpha^{-3/2}$.

Within the required accuracy, Eq. (26) can therefore be written in the form

$$F(\mathbf{r},\mathbf{k})=f(\mathbf{k})+Q(\mathbf{k})+\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}v(r')F(\mathbf{r}',\mathbf{k})d^3r'$$
$$-\left(\gamma_k-\frac{\gamma_k^2r_0}{2}\right)\int v(r')F(\mathbf{r}',\mathbf{k})d^3r'+B(\mathbf{r},\mathbf{k}),\qquad(28)$$

where r_0 is given by (16), and

$$f(\mathbf{k})=\frac{8\pi}{k^2+k_0^2+\mathbf{k}\mathbf{k}_0-E}\int\varphi_d(r)v(r)d^3r,\qquad(29)$$
$$Q(\mathbf{k})=\frac{8\pi}{(2\pi)^3}\int\frac{v(r')F(\mathbf{r}',\mathbf{k}')d^3r'd^3k'}{k^2+k'^2+\mathbf{k}\mathbf{k}'-E-i\delta},\qquad(30)$$
$$B(\mathbf{r},\mathbf{k})=\frac{8\pi}{(2\pi)^3}\int\frac{[\cos((\mathbf{k}/2+\mathbf{k}')\mathbf{r})\cos((\mathbf{k}+\mathbf{k}'/2)\mathbf{r}')-1]}{k^2+k'^2+\mathbf{k}\mathbf{k}'-E-i\delta}$$
$$\times v(r')F(\mathbf{r}',\mathbf{k}')d^3r'd^3k'$$
$$-\frac{1}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}',\boldsymbol{\rho})\psi(\mathbf{r}',\boldsymbol{\rho})d^3r'd^3\rho.\qquad(31)$$

Since only values of $k'\gtrsim r_0^{-1}$ are important in the first integral in (31), the function $B(\mathbf{r},\mathbf{k})$ does not depend on k for $kr_0\ll1$:

$$B(\mathbf{r},\mathbf{k})\equiv B(r)\sim(\alpha r_0)\,\alpha^{-3/2}.$$

Equation (28) has the same structure as (3). If $Q(\mathbf{k})$ and $B(\mathbf{r})$ are regarded as inhomogeneous terms, the solution of (28) can be written as an

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3)In STM, $F(\mathbf{r},\mathbf{k})$ denotes the function

$$F(\mathbf{r},\mathbf{k})=\int e^{-i\mathbf{k}\boldsymbol{\rho}}\psi(\mathbf{r},\boldsymbol{\rho})\,d^3\rho.$$

4)If not indicated otherwise, the statement that one of the parameters $|E|^{1/2}r_0$, αr_0, and kr_0 is small is meant, here and in what follows, to imply that the other two parameters are also small.

expansion in the functions $\varphi_n(\mathbf{r})$ in analogy to the expansion (11):

$$F(\mathbf{r}, \mathbf{k}) = A(\mathbf{k})\psi_0(r) + B_1(\mathbf{r}), \qquad (32)$$

where $\psi_0(\mathbf{r})$ is the wave function of a system of two particles at zero energy given in terms of the $\varphi_n(\mathbf{r})$ by formula (12),

$$B_1(\mathbf{r}) = B(\mathbf{r}) + \sum_{n=1}^{\infty} \frac{\varphi_n(\mathbf{r})}{\lambda_n - 1} \int \varphi_n(\mathbf{r}') v(r') B(\mathbf{r}') d^3r', \qquad (33)$$

and the function $A(\mathbf{k})$ is equal to

$$A(\mathbf{k}) = f(\mathbf{k}) + Q(\mathbf{k}) - \left(\gamma_k - \frac{\gamma_k^2 r_0}{2}\right) \int v(r') F(\mathbf{r}', \mathbf{k}) d^3r'. \qquad (34)$$

The sum in (33) contains a large term proportional to $(\lambda - 1)^{-1}$. The analogous term in the sum in (11) vanishes by virtue of the choice of the parameter r_0 according to (16). In the sum in (33) the term proportional to $(\lambda - 1)^{-1}$ is in general not equal to zero and must be considered separately. In order to single out this term, we write $B_1(\mathbf{r})$ in the form

$$B_1(\mathbf{r}) = C(E)\psi_0(r) + B_2(\mathbf{r}), \qquad (35)$$

where $C(E)$ is equal to

$$C(E) = \frac{1}{a_0} \int v(r) B_1(\mathbf{r}) d^3r \sim (\alpha r_0)\alpha^{-3/2}, \qquad (36)$$

here a_0 is the two-body amplitude at zero energy. Then the quantity $B_2(\mathbf{r})$ has the property

$$\int v(r) B_2(\mathbf{r}) d^3r = 0. \qquad (37)$$

It follows from (12), (33), (35), and (36) that $B_2(\mathbf{r})$ does not contain terms proportional to $(\lambda - 1)^{-1}$ and can therefore be neglected in the approximation under consideration. Then

$$F(\mathbf{r}, \mathbf{k}) = \chi(\mathbf{k})\psi_0(r), \qquad (38)$$

$$\chi(\mathbf{k}) = A(\mathbf{k}) + C(E). \qquad (39)$$

If (38) is substituted in the integral of (34) and $A(\mathbf{k})$ expressed through $\chi(\mathbf{k})$ according to (39), one finds that up to and including terms $\sim \alpha r_0$

$$\chi(\mathbf{k}) = -\frac{\alpha}{\gamma_k - \alpha}[f(\mathbf{k}) + Q(\mathbf{k}) + C(E)] - \frac{\alpha r_0}{2} \frac{\gamma_k}{\gamma_k - \alpha}[f(\mathbf{k}) + Q(\mathbf{k})]. \qquad (40)$$

We show now how the STM equation is obtained from (40). In this approximation, terms $\sim \alpha r_0$ must be neglected. Then (40) reduces to

$$\chi(\mathbf{k}) = -\frac{\alpha}{\gamma_k - \alpha}[f(\mathbf{k}) + Q(\mathbf{k})]. \qquad (41)$$

The function $F(\mathbf{r}, \mathbf{k})$ under the integral (30) for $Q(\mathbf{k})$ can in this approximation be replaced by the expression (38), since the integrals over the region $k' \gtrsim r_0^{-1}$ give a contribution $\sim \alpha r_0$ (see above). We then obtain the STM equation for $\chi(\mathbf{k})$.[5)]

This is easily seen by writing $f(\mathbf{k})$ and $Q(\mathbf{k})$ out explicitly. Let us denote $\chi(\mathbf{k})$ in the STM approximation by $\chi_0(\mathbf{k})$ and rewrite (40) in the form

$$\chi(\mathbf{k}) = -\frac{\alpha}{\gamma_k - \alpha}[f(\mathbf{k}) + Q(\mathbf{k}) + C(E)] + \frac{r_0\gamma_k}{2}\chi_0(\mathbf{k}). \qquad (42)$$

Equation (42) could be regarded as an equation for the determination of $\chi(\mathbf{k})$, if we could substitute the expression (38) in the integral (30) for $Q(\mathbf{k})$. However, this we are not allowed to do, since the integrals over the region $k' \gtrsim r_0^{-1}$ [where (38) is not valid] make a contribution $\sim \alpha r_0$. Therefore we write $Q(\mathbf{k})$ in the form

$$Q(\mathbf{k}) = \int [R(\mathbf{k}, \mathbf{k}') - R(0, k')] v(r') F(\mathbf{r}', \mathbf{k}') d^3r' d^3k' + \int R(0, k') v(r') F(\mathbf{r}', \mathbf{k}') d^3r' d^3k',$$

$$R(\mathbf{k}, \mathbf{k}') = 8\pi(2\pi)^{-3}(k^2 + k'^2 + \mathbf{k}\mathbf{k}' - E - i\delta)^{-1}. \qquad (43)$$

In view of the fast convergence of the integral we can neglect the contribution from the region $k' \gtrsim r_0^{-1}$ in the first term of (43) and substitute (38) for $F(\mathbf{r}, \mathbf{k})$. The second term is independent of k. Let us denote it by $C_1(E)$:

$$C_1(E) = \int R(0, k') v(r') F(\mathbf{r}', \mathbf{k}') d^3r' d^3k'. \qquad (44)$$

Then we obtain the following equation for $\chi(\mathbf{k})$:

$$\chi(\mathbf{k}) = -\frac{\alpha}{\gamma_k - \alpha}\Big[f(\mathbf{k}) + \int [R(\mathbf{k}, \mathbf{k}') - R(0, k')] v(r')\psi_0(r')\chi(\mathbf{k}') d^3r' d^3k' + C_1(E) + C(E)\Big] + \frac{r_0\gamma_k}{2}\chi_0(\mathbf{k}). \qquad (45)$$

Since $\chi_0(\mathbf{k})$ has the asymptotic form (27) for $k \to \infty$

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5)The equations proposed by Minlos and Faddeev[5] differ from Eq. (41) by the term $-\alpha(\gamma_k - \alpha)^{-1}C(E)$, where C is a function of $\mathbf{k}$ and $\chi(\mathbf{k})$:

$$C = \int L(\mathbf{k}, \mathbf{k}')\chi(\mathbf{k}') d^3k'.$$

The function $L(\mathbf{k}, \mathbf{k}')$ is subject only to very general restrictions and arbitrary otherwise. Minlos and Faddeev propose the assumption that C is constant for $k \ll R_0^{-1}$ and a function of k for $k \sim R_0^{-1}$, where R_0 is some parameter which evidently has the meaning of the range of the three-body forces. Here we must, of course, assume that $R_0 \gg r_0$, since otherwise we would go beyond the accuracy of our approximation in taking account of the dependence of C on k. If we assume that C does not depend at all on k for $kr_0 \ll 1$, it can be shown that $\psi(\mathbf{k})$ must satisfy the STM equation and the constant C must be of the order $\sim(\alpha r_0)\alpha^{-3/2}$, as assumed in the present work.

and the last term in (45) behaves like $k\chi_0(\mathbf{k})$ for large k, we may expect that for $k \to \infty$

$$\chi(k) \sim k^{-1}[d_1 k^{is_0} + d_2 k^{-is_0}].$$

Therefore the integral $\int R(\mathbf{k}, \mathbf{k}')\chi(\mathbf{k}')d^3k'$ diverges [although the integral $\int [R(\mathbf{k}, \mathbf{k}') - R(0, k')]\chi(\mathbf{k}')d^3k'$ exists]. However, this integral can be given a meaning by introducing a convergence factor under the integral, for example, $k^{-\sigma}$, with $\sigma > 0$, where $\sigma \to 0$ after the evaluation of the integral. In the following, all integrals of this type are to be understood in this sense.

Equation (45) can be treated with the method of [2]. The calculations are completely analogous to those given in the appendix of this reference. Equation (45) has a unique solution which has the property

$$\int R(0, k')v(r')\psi_0(r')\chi(\mathbf{k}')d^3r'd^3k' = C_1(E) + C(E). \tag{46}$$

This equation is a consequence of the fact that the asymptotic form of $\chi(\mathbf{k})$ for large k has no term independent of k.

If we now consider instead of (45) the equation

$$\chi(\mathbf{k}) = -\frac{\alpha}{\gamma_k - \alpha}[f(\mathbf{k}) + a_0\int R(\mathbf{k}, \mathbf{k}')\chi(\mathbf{k}')d^3k'] + \frac{r_0\gamma_k}{2}\chi_0(\mathbf{k}), \tag{47}$$

where $a_0 = \int v(r)\psi_0(r)d^3r$ is the two-body amplitude at zero energy, we see that this equation differs from the STM equation only by the last term and will therefore also have a nonunique solution, just like the STM equation. One of the solutions of (47) will coincide with the solution of (45), since the latter satisfies (47) identically owing to (46). Equation (47) can thus be regarded as an equation for the determination of $\chi(\mathbf{k})$ if we choose that solution which satisfies the condition (46).

If we write out $C_1(E)$ explicitly, (47) takes on the form

$$\int R(0, k')v(r')[\psi_0(r')\chi(\mathbf{k}') - F(\mathbf{r}', \mathbf{k}')]d^3r'd^3k' = C(E). \tag{48}$$

Only large values of $k' \lesssim r_0^{-1}$ are important in the integral on the left-hand side of (48) and in the integrals defining $C(E)$, since $F(\mathbf{r}', \mathbf{k}') = \psi_0(\mathbf{r}')\chi(\mathbf{k}')$ for $r \lesssim r_0$ and $k'r_0 \ll 1$. But in the region $r \lesssim r_0$, $\rho \lesssim r_0$ (i.e., $k \gtrsim r_0^{-1}$) the wave functions for different energies are proportional to each other. Hence Eq. (46) can be satisfied only if, in the region of large k, the functions $\chi(\mathbf{k})$ for different energies are proportional to each other with an accuracy up to and including terms $\sim \alpha r_0$. This condition on $\chi(\mathbf{k})$ allows us to remove the arbitrariness in the solution of (47).

Let us write the function $\chi(\mathbf{k})$ in the form

$$\chi(\mathbf{k}) = \chi_0(\mathbf{k}) + \chi'(\mathbf{k}), \tag{49}$$

where $\chi'(\mathbf{k})$ is of the order $\sim(\alpha r_0)\alpha^{-3/2}$. Following STM, [1] we write $\chi_0(\mathbf{k})$ in the form [6)]

$$\chi_0(\mathbf{k}) = -\alpha\sqrt{\frac{\alpha}{2\pi}}\frac{4\pi a(\mathbf{k}, \mathbf{k}_0)}{k^2 - k_0^2 - i\delta},$$

$$a(\mathbf{k}, \mathbf{k}_0) = \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)a_l(k, k_0). \tag{50}$$

The function $\chi'(k)$ is written in a similar form:

$$\chi'(\mathbf{k}) = -\alpha\sqrt{\frac{\alpha}{2\pi}}\frac{4\pi\Delta(\mathbf{k}, \mathbf{k}_0)}{k^2 - k_0^2 - i\delta},$$

$$\Delta(\mathbf{k}, \mathbf{k}_0) = \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\Delta_l(k, k_0). \tag{51}$$

The function $\Delta(\mathbf{k}, \mathbf{k}_0)$ satisfies the equation

$$\frac{\gamma_k - \alpha}{k^2 - k_0^2}\Delta(\mathbf{k}, \mathbf{k}_0) = \frac{3}{8}r_0 a(\mathbf{k}, \mathbf{k}_0) + \frac{1}{\pi^2}\int\frac{\Delta(\mathbf{k}', \mathbf{k}_0)}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - E - i\delta}\frac{d^3k'}{k'^2 - k_0^2 - i\delta}. \tag{52}$$

In deriving (52) we must take into account that $a(\mathbf{k}, \mathbf{k}_0)$ satisfies the STM equation.

Using (51), we can obtain equations for $\Delta_l(k, k_0)$. For $l \neq 0$, these equations have, according to [2], a unique solution. The equation for $\Delta_0(k, k_0)$ has a nonunique solution and must be considered in more detail. It has the form

$$\frac{\gamma_k - \alpha}{k^2 - k_0^2}\Delta_0(k, k_0) = \frac{3}{8}r_0 a_0(k, k_0) + \frac{2}{\pi}\int_0^{\infty}\ln\frac{k^2 + k'^2 + kk' - E - i\delta}{k^2 + k'^2 - kk' - E - i\delta}\frac{\Delta_0(k', k_0)}{k'^2 - k_0^2 - i\delta}\frac{k'^2dk'}{kk'}. \tag{53}$$

Equation (53) can be investigated using the method of [2]. In particular, we can determine the asymptotic form of $\Delta_0(k, k_0)$ for $k \gg (|E|^{1/2}, \alpha)$:

$$\Delta_0(k, k_0) \sim A_E(s_0)[\zeta(s_0, k) + d_E(s_0)(k/\alpha)^{is_0}] + A_E(-s_0)[\zeta(-s_0, k) + d_E(-s_0)(k/\alpha)^{-is_0}], \tag{54}$$

$$\zeta(s_0, k) = (\alpha r_0)\delta_1(s_0)(k/\alpha)^{is_0+1} + (\alpha r_0)(k/\alpha)^{is_0}\delta_2(s_0)\ln(k/\alpha), \tag{55}$$

where $A_E(s_0)$ and $A_E(-s_0)$ are coefficients in the asymptotic form of $a_0(k, k_0)$: [2]

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6)The function $a(\mathbf{k}, \mathbf{k}_0)$ defined by (50) coincides with the function $a(\mathbf{k}, \mathbf{k}_0)$ defined by (50) coincides with the function $a(\mathbf{k}, \mathbf{k}_0)$ of STM.[1]

$$a_0(k, k_0) \sim A_E(s_0)(k/\alpha)^{is_0} + A_E(-s_0)(k/\alpha)^{-is_0}, \; s_0 \approx 1. \tag{56}$$

The quantities $\delta_1(s_0)$ and $\delta_2(s_0)$ in (55) are numerical coefficients which are independent of the energy. We shall not write down the formulas for $\delta_1(s_0)$ and $\delta_2(s_0)$ in view of their complexity, especially since their explicit form is not important for us.

Since the solution of (53) without the free term has the asymptotic form (56) and an arbitrary multiple of the solution of the homogeneous equation can be added to the solution of the inhomogeneous equation, which has the asymptotic form (54), one of the coefficients $d_E(s_0) \sim \alpha r_0$ and $d_E(-s_0) \sim \alpha r_0$, for example, $d_E(s_0)$, remains arbitrary. Then $d_E(-s_0)$ is determined by this arbitrary parameter through the condition of joining with $\Delta_0(k, k_0)$ at small k. Since large values of $k' \sim k$ are important in the integral (52) if k is large, the asymptotic form (54) can be verified immediately by substituting it into (53).

The arbitrary constant $d_E(s_0)$ can be determined in principle from the requirement that for large k the functions $\chi(\mathbf{k})$ belonging to different energies must be proportional to each other. This implies, according to the definitions (49) to (51), that the quantities $a_0(k, k_0) + \Delta_0(k, k_0)$ for different energies must be proportional to each other in the region of large k. According to (53) and (55) we have for $kr_0 \ll 1$, $k\alpha^{-1} \gg 1$, $k|E|^{-1/2} \gg 1$

$$a_0(k, k_0) + \Delta_0(k, k_0)$$

$$\sim A_E(s_0)\left[1 + \frac{d_E(s_0)}{A_E(s_0)}\right]\left[\zeta(s_0, k) + \left(\frac{k}{\alpha}\right)^{is_0}\right] + A_E(-s_0) \times \left[1 + \frac{d_E(s_0)}{A_E(-s_0)}\right]\left[\zeta(-s_0, k) + \left(\frac{k}{\alpha}\right)^{-is_0}\right]. \tag{57}$$

In deriving (57) we must take into account that $d_E(s_0)\,\zeta(s_0, k)$ and $d_E(s_0)\,\zeta(-s_0, k)$ are second order quantities. In order that the expressions (57) for different energies be proportional to each other with an accuracy up to and including terms $\sim \alpha r_0$, we must require that the relation

$$A_E(s_0)\,[1 + d_E(s_0)/A_E(s_0)]\,/\,A_E(-s_0) \times [1 + d_E(-s_0)/A_E(-s_0)] = \beta \tag{58}$$

is independent of the energy. Within our limitations of accuracy, we can write (58) in the form

$$d_E(s_0)/A_E(s_0) - d_E(-s_0)/A_E(-s_0) = \beta/\beta_0, \tag{59}$$

where $\beta_0 = A_E(s_0)/A_E(-s_0)$. β_0 is independent of E according to [2].

In order to determine the constant β we must fix one of the three-particle amplitudes or the energy of the bound state of the three particles. Let us fix, for example, the energy of the bound state, E_0. The function $\chi(k)$ for the bound state (triton) satisfies (47) with $f(k) = 0$. Hence the function $\Delta_0(k, k_0)$ for the bound state satisfies (52) with $E = E_0$, $k_0^2 = \frac{4}{3}(E_0^2 + \alpha^2)$, whereas the function $a_0(k, k_0)$ for triton satisfies Eq. (52) without the free term. Thus the arbitrariness in the solution of (53) due to the addition of the solution of the homogeneous equation reduces in this case to the addition of the term $\lambda a_0(k, k_0)$, where λ is an arbitrary constant $\sim \alpha r_0$. This implies the replacement of the function

$$\chi(k) \sim a_0(k, k_0) + \Delta_0(k, k_0)$$

by

$$(1 + \lambda)\,[a_0(k, k_0) + \Delta_0(k, k_0)]$$

(since the quantity $\lambda\Delta_0 \sim r_0^2$). The arbitrariness in the solution of (52) for the bound state is therefore no serious difficulty at all and amounts only to an arbitrariness in the normalization of $\chi(k, k_0)$.

This arbitrariness does not affect the ratio of the coefficients in the asymptotic form. Hence the value of β is also unaffected. In solving (52) for triton we can, for definiteness, require, e.g., that the tritium wave function be real and normalized to unity like the wave function in the first approximation (which we also regard as real). These requirements allow us, in principle, to make a unique choice of the solution of (53) for the determination of the corrections to the triton wave function. In order to determine all desired quantities we must, therefore, find the triton wave function according to the recipe discussed above, determine the value of β, and remove the last arbitrariness in the amplitude with an accuracy up to and including terms $\sim \alpha r_0$ with the help of (59).

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