CALCULATING THE INVARIANT PHASE VOLUME OF N PARTICLES

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Submitted to JETP editor April 28, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1448-1455 (October, 1962)

A representation of an invariant phase volume for N particles is obtained in the form of a single contour integral. The phase volume is expanded in infinite series, depending on the relationship between the masses of particles and the total energy of the system. A simple formula is obtained by the saddle-point method.

IN the study of the decay and multiple production of particles, it becomes necessary to calculate the invariant phase volume Ω_N for particles with arbitrary masses $(M, m_1, \ldots, m_N)$:

$$\Omega_N(M, m_1, \ldots, m_N) = \int_{-\infty}^{+\infty} \delta^{(4)}\Big(Q - \sum_{i=1}^{N} P_i\Big) \prod_{j=1}^{N} E_j^{-1} d^3p_j, \quad (1)$$

where N is the number of particles at the end of the reaction, $P_j \equiv (E_j, p_j)$ is the four-momentum of the j-th particle with mass m_j, Q is the total four-momentum of the system, and $M = \sqrt{Q^2}$ is the mass of the system or decaying particle.

In many papers, the expression discussed differs from (1) (nonrelativistic phase volume):

$$\omega_N = \int_{-\infty}^{+\infty} \delta^{(4)}\Big(Q - \sum_{i=1}^{N} P_i\Big) \prod_{j=1}^{N} d^3p_j. \quad (2)$$

Fermi [1] advanced the idea that when the number of particles N is increased, the factor ω_N determines the entire character of the multiple production process. In later papers, the tensor ω_N was replaced by the scalar Ω_N, which coincides with ω_N in the nonrelativistic limit. In the present paper we calculate Ω_N in detail, although the method is applicable to ω_N.

The value of ω_N was calculated in [1] for several particular cases without account of the three-momentum conservation law. Lepore and Stuart [2] reduced the phase volume to a double integral, with account of the momentum conservation law, and calculated the first term of the expansion in the extremely relativistic and nonrelativistic cases. This double integral was estimated by Fialho [3] by the saddle-point method and by Rozental' and Maksimenko [4] by series expansion of the interand. In the present paper, the double integral that represents the phase volume, is reduced to a single integral [formula (8)]. The latter is calculated both by series expansion of the integrand, in a form different from that given in [4], for the relativistic case (13) and for the nonrelativistic case (11), as well as by the saddle-point method (22). We note that the formulas obtained are much simpler than those in [3,4].

The phase volume Ω_N can be readily reduced by the method detailed in [2,4] to the four-dimensional integral

$$\Omega_N = (2\pi)^{-4} \int_{-\infty}^{+\infty} \exp(iQx)\, d^4x \prod_{j=1}^{N} 2\pi^2\, im_j \varkappa^{-1} H_1^{(2)}(\varkappa m_j), \quad (3)$$

where $\kappa = \sqrt{x^2} = \sqrt{t^2 - r^2}$. The integrand depends on κ in a rather complicated form, but has a clear and simple dependence on (Qx), so that the four-dimensional integral (3) reduces to a one-dimensional one. Let us prove that

$$\int_{-\infty}^{+\infty} d^4x f(\varkappa) \exp(iQx) = \frac{4\pi^2}{iM} \int_C z^2 f(z) J_1(Mz)\, dz, \quad (4)$$

if $f(z) \to 0$ when $z \to i\infty$. The contour C begins at $-\infty$, circuits the origin from below, and goes to $+\infty$; $J_1(Mz)$ is the first-order Bessel function.

We shall calculate (4) in a coordinate system in which Q has only one component $Q_0 = M$:

$$\int_{-\infty}^{+\infty} d^4x f(\varkappa) \exp(iQx) = 2\pi \int_{-\infty}^{+\infty} r^2 dr \int_{C_1} dt f(\varkappa) \exp(iMt). \quad (5)$$

The integral with respect to the variable t must be considered in the complex plane along the contour C_1 (Fig. 1, continuous line). Since $f(i\infty) = 0$, the contour C_1 can be deformed into C (dashed line). We replace the integration variable t by ξ using the formula $t = \sqrt{r^2 + \xi^2}$. The contour C goes over into L (Fig. 2). The contour L begins at $-\infty$ on the first sheet of the Riemann surface of the root $\sqrt{r^2 + \xi^2}$ and goes to the point A (continuous line). At the point A it rises to the second sheet (dash-dot line) and goes to $+\infty$, then to $-\infty$ along an arc of radius $R \to \infty$, continuing to B

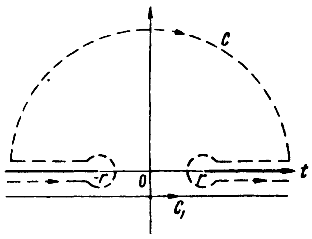

FIG. 1

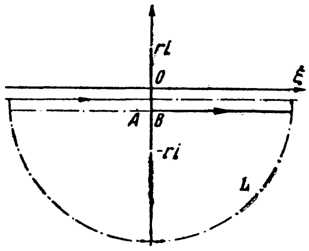

FIG. 2

where it again drops to the first sheet. The integral is transformed to the form

$$\int_{-\infty}^{+\infty} r^2 dr \int_L \frac{f(\xi)\, d\xi}{\sqrt{\xi^2+r^2}} \exp(iM\sqrt{\xi^2+r^2}). \qquad (6)$$

If we change the order of summation, then the integral with respect to r reduces to the Bessel function $J_1(M\xi)$ [7].

According to (4), the relativistic phase volume can be written in terms of a one-dimensional contour integral in the form

$$\Omega_N = \frac{(2\pi^2 iQ^2)^N}{4\pi^2 iQ^4} \int_{S_1} \frac{dz}{z^{N-2}} J_1(z) \prod_{j=1}^{N} \mu_j H_1^{(2)}(z\mu_j), \qquad (7)$$

where $\mu_j = m_j/M$.

The contour S_1 in Fig. 3 is shown by the dashed line. If we replace J_1 by the half sum of $H_1^{(1)}$ and $H_1^{(2)}$, then Ω_N is expressed in the form

$$\Omega_N = \frac{(2\pi^2 iQ^2)^N}{8\pi^2 iQ^4} \int_{S} \frac{dz}{z^{N-2}} H_1^{(1)}(z) \prod_{j=1}^{N} \mu_j H_1^{(2)}(z\mu_j), \qquad (8)$$

and the contour S (solid line) is shown in Fig. 3.

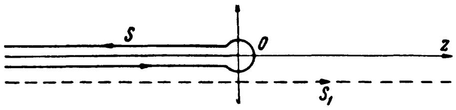

FIG. 3

The integral (8) cannot be expressed in its general form in terms of known tabulated functions, so that we must expand in a series. Depending on the relation between the particle masses and the total energy of the system, let us consider the following cases: nonrelativistic, when $m_j > M - \Sigma m$ for all particles; relativistic, when $m_j \ll M - \Sigma m$ also for all particles; and mixed, when some particles satisfy the relativistic conditions and others the nonrelativistic conditions.

We start with the nonrelativistic case. We deform the contour S into S_R (circle of radius R) and then let R tend to infinity. On S_R we have $|z| = R \gg 1$ and we can therefore use the expansion of the functions $H_1^{(1)}$ and $H_1^{(2)}$ for large absolute values of the argument

$$H_1^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left(iz - \frac{3\pi i}{4}\right) \sum_{m=0}^{\infty} c(m)(2iz)^{-m},$$

$$H_1^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left(-iz + \frac{3\pi i}{4}\right) \sum_{m=0}^{\infty} c(m)(-2iz)^{-m}, \qquad (9)$$

where

$$c(m) = \frac{\Gamma(m+3/2)\,\Gamma(m-1/2)}{\Gamma(3/2)\,\Gamma(-1/2)\,\Gamma(1+m)}.$$

As a result of the integration we obtain for the phase volume Ω_N an expansion in the form

$$\Omega_N = (2\pi)^{3(N-1)/2} Q^{2N-4} \left(1 - \sum_{j=1}^{N} \mu_i^{(3N-5)/2} \left(\prod \sqrt{\mu}\right)\right.$$
$$\times \sum_{m=0}^{\infty} \frac{c(m_0)\, x_0^{m_0} c(m_1)\, x_1^{m_1} \ldots c(m_N)\, x_N^{m_N}}{\Gamma((3N-3)/2 + m_0 + m_1 + \ldots + m_N)}, \qquad (10)$$

where

$$x_0 = \frac{1 - \sum \mu}{2}, \qquad x_k = \frac{\sum \mu - 1}{2\mu_k}, \qquad \prod \sqrt{\mu} = \sqrt{\mu_1 \ldots \mu_N}.$$

It is obvious that the series (10) diverges when $x_k < -1$.

It is easy to note that $c(m_0)\ldots c(m_N) \le \Gamma((3N-3)/2 + m_0 + \ldots + m_N)$, and therefore the multidimensional series (10) is majored by the series

$$\sum_{m=0}^{\infty} x_0^{m_0} \ldots x_N^{m_N}.$$

Consequently, the region of convergence of the series (10) is the hypercube $-1 < x_k < 0$. This condition means that the mass of the k-th particle exceeds the kinetic energy of the system, i.e., the particle is nonrelativistic. However, it may turn out that some or all the quantities $x_k < -1$. In this case the series (10) with respect to these variables can be continued analytically, if one uses the properties of the hypergeometric functions and the relation

$$\sum_{m_k=0}^{\infty} \frac{c(m_k)\, x_k^{m_k}}{\Gamma((3N-3)/2 + m_0 + \ldots + m_N)}$$
$$= \frac{F(-1/2, 3/2, (3N-3)/2 + m_0 + \ldots m_{k-1} + m_{k+1} + \ldots m_N, x_k)}{\Gamma((3N-3)/2 + m_0 + \ldots + m_{k-1} + m_{k+1} + \ldots + m_N)}. \qquad (11)$$

Direct continuation of the hypergeometric functions is quite cumbersome, so that we choose a different path.

In the relativistic case, i.e., when $x_k < -1$, we use the expansion of $H_1^{(2)}(z)$ in the vicinity of the point $z = 0$. This expansion is more complicated:

$$H_1^{(2)}(z) = \frac{2i}{\pi z}\left[1 + \sum_{n=0}^{\infty} \frac{\partial}{\partial n} \frac{(iz/2)^{2+2n}}{\Gamma(n+1)\Gamma(n+2)}\right]. \tag{12}$$

We substitute formula (12) in (8) and integrate:

$$\frac{\Omega_N}{2\pi(\pi Q^2)^{N-2}} = \sum_{k=0}^{N} P_k^N \sum_{n=0}^{\infty} \frac{\partial^k}{\partial n_1 \ldots \partial n_k} \frac{\prod_{j=1}^{k} \mu_j^{2+2n_j} \Gamma^{-1}(1+n_j)\Gamma^{-1}(2+n_j)}{\Gamma(N-k-\sum n)\Gamma(N-1-k-\sum n)}, \tag{13}$$

where P_k^N is an operator which makes up all possible products of k factors among $\mu_1, \ldots \mu_N$.

Let us examine the behavior of some of the terms. The term of the series with k = 1 has a finite number of members:

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial n} \frac{\mu^{2+2n}}{\Gamma(n+1)\Gamma(n+2)\Gamma(N-1-n)\Gamma(N-2-n)} = \sum_{n=0}^{N-2} \frac{2\ln\mu + \psi(N-1-n) + \psi(N-2-n) - \psi(1+n) - \psi(2+n)}{\Gamma(1+n)\Gamma(2+n)\Gamma(N-1-n)\Gamma(N-2-n)} \mu^{2+2n}$$

and therefore the phase volume of N particles, of which only one has a mass, is expressed in final form as

$$\Omega_N(M, m, 0) = 2\pi(\pi Q^2)^{N-2}\left[\frac{1-\mu^{2N-2}}{\Gamma(N)\Gamma(N-1)} + \sum_{n=0}^{N-3} \frac{2\ln\mu + \psi(N-1-n) + \psi(N-2-n) - \psi(1+n) - \psi(2+n)}{\Gamma(1+n)\Gamma(2+n)\Gamma(N-1-n)\Gamma(N-2-n)} \mu^{2+2n}\right].$$

$$\Omega_2 = 2\pi(1-\mu^2), \qquad \Omega_3 = \pi^2 Q^2 (1 - \mu^4 + 4\mu^2 \ln\mu),$$

$$\Omega_5 = \frac{\pi^4 Q^6}{72}[1 + 28\mu^2 - 28\mu^6 - \mu^8 + 24\mu^2(1+3\mu^2+\mu^4)\ln\mu]. \tag{14}$$

The next term (k = 2) is much more complicated.

$$\sum_{n_1, n_2=0}^{\infty} \frac{\partial^2}{\partial n_1 \partial n_2} \frac{\mu_1^{2+2n_1}\mu_2^{2+2n_2}}{\Gamma(1+n_1)\Gamma(2+n_1)\Gamma(1+n_2)\Gamma(2+n_2)\Gamma(N-2-n_1-n_2)\Gamma(N-3-n_1-n_2)}. \tag{15}$$

We sum the series (15) only for N = 2 and 3. This result is obtained from an examination of the phase volume for two particles with mass and (N = 2) without mass. In fact, the phase volume $\Omega_N(M, m_1, m_2, 0)$ is

$$\Omega_N(M, m_1, m_2, 0)$$
$$= \int d^4q \int \delta^{(4)}\left(Q - q - \sum_3^N P\right) \prod_3^N \frac{d^3p}{E} \int \delta^{(4)}(q - P_1 - P_2)$$
$$\times \frac{d^3p_1 d^3p_2}{E_1 E_2} = \int d^4q \frac{2\pi[\pi(Q-q)^2]^{N-4}}{\Gamma(N-2)\Gamma(N-3)} 2\pi$$
$$\times \left\{\frac{[q^2-(m_1+m_2)^2][q^2-(m_1-m_2)^2]}{q^4}\right\}^{1/2}.$$

With the aid of this procedure we can obtain the more general relation

$$\Omega_{n+k}(\sqrt{Q^2}, m_1, \ldots, m_k, \mu_1 \ldots \mu_k)$$
$$= \int d^4q \Omega_k(\sqrt{q^2}, \mu_1, \ldots, \mu_k)\Omega_n(\sqrt{(Q-q)^2}, m_1 \ldots m_n).$$

In the coordinate system where the vector Q has one component, we can take relatively simply the integral over all the angle variables. The latter integration with respect to q^2 will be of the type

$$\int dq^2 \left\{[q^2-(m_1+m_2)^2][q^2-(m_1-m_2)^2]\right\}^{1/2}\left[R_1(q)\sqrt{1-q^2} + R_2(q)\ln\frac{1+\sqrt{1-q^2}}{q}\right],$$

where $R_1(q)$ and $R_2(q)$ are rational-fraction expressions in q. The term with R_1 reduces to elliptic functions, while the one with R_2 is not expressed in terms of any known functions. For example, the phase volumes $\Omega_4(M, m_1, m_2, 0)$ and Ω_5 have the form

$$\Omega_4(M, m_1, m_2, 0) = 4\pi^3 M^4 \int_a^1 dq^2 \sqrt{(q^2-a^2)(q^2-b^2)}$$
$$\times \left[\frac{\sqrt{1-q^2}}{q^2} - \ln\frac{1+\sqrt{1-q^2}}{q}\right],$$

$$\Omega_5(M, m_1, m_2, 0)$$
$$= 2\pi^4 M^6 \int_a^1 dq^2 \sqrt{(q^2-a^2)(q^2-b^2)}\left[\frac{(7q^2-1)\sqrt{1-q^2}}{3q^2} - (1+q^2)\ln\frac{1+\sqrt{1-q^2}}{q}\right],$$

where

$$a = (m_1 + m_3)/M, \qquad b = (m_1 - m_2)/M.$$

Let us proceed to the mixed case, when there are H nonrelativistic particles and P relativistic particles. In the integral (8), the functions $H_1^{(2)}(z\mu)$ of the nonrelativistic particles will be expanded like (9), those of the relativistic particles will be expanded like (12) and we integrate:

$$\Omega_N = (4\pi Q^4)^{-1}\ (8\pi^2 Q^2)^{H+P}\left(\frac{2}{\pi}\right)^{(1+P+N)/2}$$
$$\times\left(\prod^{H}\sqrt{\mu}\right)\left(1-\sum^{H}\mu\right)^{(3H-5)/2+2P}\sum_{k=0}^{P}\sum_{n,\,m=0}^{\infty}P_k^P\frac{\partial^k}{\partial n_1\ldots\partial n_k}$$
$$\times\frac{c(m_0)\,x_0^{m_0}\ldots c(m_H)x_H^{m_H}}{\Gamma(N-2+(H+1)/2+P+\sum m-2k-2\sum n)}$$
$$\times\prod_{j=1}^{k}\frac{y_j^{1+n_j}}{\Gamma(1+n_j)\,\Gamma(2+n_j)},$$

where

$$x_0=\left(1-\sum^{H}\mu\right)/2,\quad x_k=\left(\sum^{H}\mu-1\right)/2\mu_k,$$
$$y_k=\left[\mu_k/2\left(1-\sum^{H}\mu\right)\right]^2;$$

$\sum^{H}$ denotes summation over the nonrelativistic particles only.

Formulas (10) and (13) for the phase volume work quite well only when certain relations exist between the particle masses and the total energy. It is natural to attempt to find a formula which is valid over the entire mass interval. If we examine the behavior of the integrand $z^{2-N}H_1^{(1)}(z)\,\Pi\mu H_1^{(2)}(\mu z) = F(z)$, then we can readily establish that it has one saddle point in the lower half plane of z. Indeed, in the vicinity of the zero F is proportional to z^{1-2N}, while at infinity it is proportional to $\exp[iz(1-\Sigma\mu)]$. The saddle point z_0 is determined from the equation $F'(z_0) = 0$. Its exact solution cannot be found, but an approximate one can. To this end we write the integrand as a product of rapidly and slowly varying factors

$$F(z) = z^{-\rho}R(z)\exp(i\alpha z), \qquad (16)$$

where $\alpha = 1-\Sigma\mu$, ρ is a certain positive number of order N, R(z) a smooth function of z, and therefore we neglect $R'(z)$ (see the appendix). From the equation

$$\frac{F'}{F} = i\alpha-\frac{\rho}{z}+\frac{R'}{R} = i\alpha\left(1-\frac{iR'}{\alpha R}\right)-\frac{\rho}{z}=0 \qquad (17)$$

we obtain $z_0 \approx z_1 = \rho/i\alpha$. We now expand R(z) in the integral

$$\int_C F(z)\,dz = \int_C e^{i\alpha z}z^{-\rho}R(z)\,dz$$

in the vicinity of the point z_1: $R(z) \approx R(z_1) + (z-z_1)R'(z_1)$. The integral of these two terms is then easy to calculate:

$$\int_C F(z)\,dz \approx \frac{2\pi i\,(i\alpha)^{\rho-1}z_1^{\rho}}{\Gamma(\rho)\exp(i\alpha z_1)}F(z_1)\left[1+\frac{iR'(z_1)}{\alpha R(z_1)}+\ldots\right] \qquad (18)$$

For $\Gamma(\rho)$ we can use the Stirling expansion, since $\rho \sim N \gg 1$ (see the appendix). This introduces into the result an error of the order $1/12\rho$, which, as will be shown below, is smaller than in other approximations. Consequently, confining ourselves only to the first term in the square brackets, expression (18) can be rewritten in the form

$$\int_C F(z)\,dz \approx \frac{\sqrt{2\pi\rho}}{\alpha}F(z_1). \qquad (19)$$

We now establish the interval of variation of ρ. If all $\mu \to 0$, then (8) is proportional to the following integral:

$$\int z^{2-2N}H_1^{(1)}(z)\,dz \approx \int z^{1-2N}\exp(iz)\,dz,$$

i.e., the maximum value of ρ is $2N-1$. The minimum value can be determined from the condition when $\Sigma\mu \to 1$. Expanding the Hankel function in integral (8) in accordance with formulas (9), we obtain the lower bound for ρ, which is equal to $(3N-3)/2$.

To determine ρ for arbitrary μ we use the linear interpolation

$$\rho(\mu) = (3N-3+P)/2. \qquad (20)$$

Starting from (19), we can readily obtain an approximate formula for the phase volume

$$\Omega_N = \frac{(2\pi^2 iQ^2)^N}{8\pi^2 iQ^4}\frac{\sqrt{2\pi\rho}}{\alpha}\frac{H_1^{(1)}(z_1)\,\Pi\mu H_1^{(2)}(\mu z_1)}{z_1^{N-2}} \qquad (21)$$

The functions $H_1^{(2)}(\mu z_1) = H_1^{(2)}(\rho\mu/i\alpha)$ are simply expressed in terms of the tabulated values of the functions K_1:

$$H_1^{(2)}\left(\frac{\rho\mu}{i\alpha}\right) = -\frac{2}{\pi}K_1\left(\frac{\rho\mu}{\alpha}\right).$$

We represent the function $H_1^{(1)}(z_1)$ approximately in terms of $I_1(\rho/\alpha)$. Substituting in (21), we obtain

$$\Omega_N = 4(4\pi Q^2\alpha/\rho)^{N-2}\,I_1(\rho/\alpha)\,\alpha^{-1}\sqrt{2\pi\rho}\prod_{j=1}^{N}\mu_j K_1\left(\frac{\rho\mu_j}{\alpha}\right). \qquad (22)$$

Recognizing that $\rho/\alpha \gg 1$, we can expand $I_1(\rho/\alpha)$ with accuracy $\sim \alpha/\rho$:

$$\Omega_N = 4\,(4\pi Q^2\alpha/\rho)^{N-2}\alpha^{-1/2}\exp(\rho/\alpha)\prod_{j=1}^{N}\mu_j K_1\left(\frac{\rho\mu_j}{\alpha}\right). \qquad (22')$$

When $\rho\mu/\alpha \to 0$ and $\rho\mu/\alpha \to \infty$ formula (22) goes over into (10) and (13), respectively. Figure 4

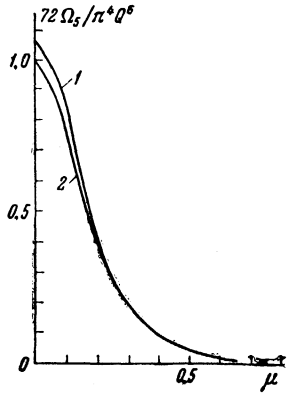

FIG. 4

shows the behavior of the exact formula $\Omega_5(\mathrm{Mm}, 0)$ (14) (curve 2) and the approximate formula (22) (curve 1). The deviation of curve 1 from 2 in the region of small μ is due to the small number of particles in the phase volume.

To conclude the paper we make a few remarks concerning the calculation of the quantity ω_N. First, ω_N is not a scalar function of Q and m, but is a tensor of rank N, since d^3p is the zeroth component of the four-vector dual to P. In the coordinate system where Q has only one component, this tensor also has only one. But it is easy to see that the scalar function

$$\varphi_N = \int \delta^{(4)}(Q - \Sigma P) \prod_{j=1}^{N} d^3p_j E_j^{-1}(P_j Q)\, Q^{-2}$$

is exactly proportional to this component. The calculation of φ_N is analogous to that of Ω_N. The difference will be quite insignificant; for example, in place of (5) we obtain

$$\int \frac{d^3p}{E} \frac{PQ}{Q^2} \exp(-ipx) = -\frac{2\pi^3 m^3 (Qx)}{Q^2 \varkappa^3} \frac{\partial}{\partial m} \frac{H_1^{(2)}(m\varkappa)}{m}.$$

The integral over d^4x can again be reduced to (4) by differentiating with respect to Q. The remaining calculations are similar to those developed above.

The author is deeply grateful to L. B. Okun' and I. Yu. Kobzarev for discussions and to G. I. Kopylov for the valuable hint that the integral (8) can be calculated by the saddle-point method.

APPENDIX

In the derivation of (22) we assumed that the function R(z) varies little in the vicinity of the saddle point. It is easy to see from (17) that this condition leads to the inequality

$$|R'/\alpha R| < 1, \qquad \text{(A.1)}$$

the correctness of which we now desire to prove. Let us take the logarithmic derivatives of the right and left halves of (16):

$$\frac{F'}{F} = \frac{2-N}{z} + \frac{d \ln H_1^{(1)}(z)}{dz} + \sum \mu \frac{\partial \ln H_1^{(2)}(\mu z)}{\partial \mu z}$$

$$= i\alpha - \frac{\rho}{z} + \frac{R'}{R}. \qquad \text{(A.2)}$$

We consider the expression (A.2) at the point $z_1 = \rho/i\alpha$. Using the expansion of $H_1^{(1)}(z)$ and $H_1^{(2)}(z)$ for $|z| > 1$ and $|z| < 1$, we simplify (A.2):

$$\frac{R'}{R} \approx \frac{3 - 3N - P + 2\rho}{2z} + \frac{3i}{8z^2}\left(\sum^{H} \mu^{-1} - 1\right). \qquad \text{(A.3)}$$

If we choose ρ equal to $(3N - 3 + P)/2$, then it follows from (A.3) that

$$\left|\frac{R'}{\alpha R}\right| \approx \frac{3\alpha}{8\rho^2}\left(\sum^{H} \mu^{-1} - 1\right) \ll 1.$$

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Translated by J. G. Adashko
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