THEORY OF QUANTUM CYCLOTRON RESONANCE IN METALS[1)]

I. M. LIFSHITZ, M. Ya. AZBEL', and A. A. SLUTSKIN

Khar'kov State University and the Physico-technical Institute, Academy of Sciences, Ukrainian SSR

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A theory of quantum cyclotron resonance in metals is developed. The total surface impedance of the metal is found for this case. It is shown that for sufficiently low temperatures and long relaxation times, "giant" oscillations of the high frequency characteristics appear which are periodic in the reciprocal of the magnetic field. These oscillations do not coincide in period with those of the de Haas−van Alphen type, and are a manifestation of resonance at discrete frequencies resulting from the quantization of the effective mass in a strong magnetic field (for metals with a complex dispersion law). The special features of this quantum cyclotron resonance for different sections of the Fermi surface are investigated. A discussion is given of the possibility of observing these effects and of the additional information about the features of the energy spectrum which can be derived from experiments on quantum cyclotron resonance.

1. INTRODUCTION

CYCLOTRON resonance in metals is observed when one applies a high frequency electromagnetic field and a dc magnetic field parallel to the surface of the metal. In a dc magnetic field H_0, an electron moves so that its energy ϵ and the projection p_z of its momentum on the direction of H_0 are conserved. If the trajectory of the electron in momentum space is closed, it will traverse this trajectory in a periodic motion with a frequency Ω, which for a complex dispersion law depends on ϵ and p_z, i.e., $\Omega = \Omega(\epsilon, p_z)$. At low temperatures, the main contribution to all high frequency characteristics comes from those electrons which are in the neighborhood of the limiting Fermi surface. Thus the frequencies actually depend on only the one continuous parameter p_z: $\Omega = \Omega(\zeta, p_z)$, where ζ is the Fermi energy. In this case, resonance is observed only at certain selected frequencies corresponding to singular points in the continuous spectrum of $\Omega = \Omega(\zeta, p_z)$. In particular, special "resonance" sections of the Fermi surface are determined by those values of p_z for which the density of states at a given frequency goes to infinity. Near these sections, which are given by the condition $(\partial\Omega/\partial p_z)_{\epsilon=\zeta} = 0$, the number of electrons simultaneously taking part in the resonance is relatively large.

Under certain conditions which will be formulated later, the consideration of quantum effects causes a considerable change in the picture of resonance. We know that in a dc magnetic field there is a quantization of the energy levels of the electron. In the quasiclassical approximation the quantized energy levels are given by the equation

$$S(\epsilon, p_z) = ehHn/c, \tag{1.1}$$

where $S(\epsilon, p_z)$ is the area of the section of the constant-energy surface $\epsilon = \text{const}$ by the plane $p_z = \text{const}$; n is an integer. From Eq. (1.1) it follows that the electron energy depends on two quantum numbers: the discrete n and the continuous p_z, i.e., $\epsilon = \epsilon(n, p_z)$. On the other hand, to each energy ϵ there corresponds the set of states determined by the discrete values of the quasimomentum p_{zn},

$$S(\epsilon, p_{zn}) = ehHn/c.$$

To these quantum states there corresponds a discrete frequency spectrum

$$\Omega_n = \mu_n H_0, \qquad \mu_n = \mu(\epsilon, p_{zn}) = e/m^* c$$

(where m^* is the effective mass of the electron[2]).

In the limit as the mean free time goes to infinity, a very sharp resonance can be observed at each of these frequencies. A reduction of the mean free time causes a reduction in the height of the resonance. It is easy to see that the quantum splitting of the frequencies is important as long as the distance between neighboring frequencies is larger

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[1)]The first report on quantum cyclotron resonance was given in [1].

than, or of the order of, the collision frequency, i.e., $\Delta\Omega \gtrsim 1/\tau$ (where τ is the relaxation time, $\Delta\Omega_n = \Omega(n+1) - \Omega(n)$ is the distance between neighboring frequencies). Since we are considering the case of resonance, $\Omega_n \sim \omega$, where ω is the frequency of the external electromagnetic field and $\Delta\Omega \sim \hbar\Omega^2/\zeta \sim \hbar\omega/\zeta$. Thus the condition for observing quantum cyclotron resonance can be put in the form

$$\gamma = \frac{\zeta}{\hbar\omega}\Big/\omega\tau \lesssim 1. \qquad (1.2)$$

When this condition is satisfied, the amplitude of the quantum cyclotron oscillations of any macroscopic quantity is of the same order as the quantity itself. If $\gamma \gg 1$, then we actually have a "classical" resonance in the continuous spectrum, which is observed at the extremal frequencies selected by the equation $(\partial\Omega/\partial p_z)_{\epsilon=\zeta} = 0$. In this case the amplitude of the resonant oscillations goes to zero like $\exp(-\pi\gamma)$.

The treatment given applied to electrons with a single energy. Since the field H_0 is parallel to the metal surface, the high frequency electromagnetic field with frequency $\omega = \Omega_n$ causes only transitions in which the quantum number n changes by unity. The energy then changes by $\hbar\Omega_n$; it then follows that at absolute zero the contribution to the current comes from the electrons with energies lying in the narrow range $|\epsilon - \zeta| \leq \hbar\Omega$. Because of the restrictions on this range, the resonance, as usual, is logarithmic.

Quantum effects in cyclotron resonance vanish with increasing temperature. From general arguments it follows that the temperatures at which one can still observe quantum oscillations satisfy the condition $kT \leq \hbar\omega$. A more exact treatment (cf. below) gives the estimate $kT \leq 2\pi^2\hbar\omega$.

Thus, at sufficiently low temperatures and sufficiently long relaxation times a logarithmic quantum cyclotron resonance occurs at a discrete spectrum of frequencies.

2. CALCULATION OF CURRENT AND IMPEDANCE IN GENERAL FORM

Assuming that the metal surface is a plane, we introduce a coordinate system in which the y axis is directed into the metal, perpendicular to its surface, while the z axis is along the magnetic field H_0. Then the differential equation relating the electric field intensity $\mathbf{E}$ in the metal to the current density $\mathbf{j}$ has the form

$$E_{x,z}''(y) + 4\pi i\omega c^{-2}j_{x,z}(y) = 0, \qquad j_y = 0. \qquad (2.1)$$

The connection of the statistical operator $\hat{\rho}$ with the current density is given in the quasiclassical approximation by the relation

$$\mathbf{j}(\mathbf{R}) = e\mathrm{Sp}\,\hat{\rho}\hat{\mathbf{v}}\delta(\mathbf{R} - \hat{\mathbf{r}}), \qquad (2.2)$$

where $\hat{\mathbf{v}}$ is the operator for the particle velocity. The radius vector $\mathbf{R}$, which gives the point in space at which the current density is calculated, is a c-number in this expression.

The change of the statistical operator with time is given by the kinetic equation

$$\dot{\hat{\rho}} + i\,[\hat{\mathcal{H}}\hat{\rho}]/\hbar + \hat{\rho}/\tau = 0, \qquad (2.3)$$

where τ is the relaxation time[2] and $\hat{\mathcal{H}}$ is the Hamiltonian of the system.

Equations (2.1)—(2.3) form a complete system whose solution enables one to find all the high frequency characteristics of the metal. The system (2.1)—(2.3) was treated in a paper of one of the authors,[3] who calculated the quantum corrections to the current density, proportional to $(\hbar\omega/\zeta)^{1/2} \ll 1$, and investigated the small nonresonant oscillations of the high frequency characteristics. In our treatment this corresponds to the case of $\gamma \gg 1$. It is clear that the results of this paper[3] cannot give effects associated with resonance in the discrete frequency spectrum. In the case of quantum cyclotron resonance which we have investigated ($\gamma \lesssim 1$), the influence of quantum effects is no longer small, and in deriving the formulas one can stop at the zeroth approximation in the quasiclassical parameter $\hbar\omega/\zeta$.

To calculate other quantities for the case of cyclotron resonance in metals, one must consider the boundary conditions. The fact that the dc magnetic field is parallel to the surface permits a separation of the particles into those which do and those which do not collide with the metal surface. Thus the expression for the current density can be written in the form $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, where $\mathbf{j}_1$ and $\mathbf{j}_2$ are determined, respectively, by those electrons which experience collisions with the metal surface and those which do not. The electrons contributing to $\mathbf{j}_1$ are nonresonant. This means that the quantum expression for $\mathbf{j}_1$ differs from the classical one by a quantity of first order in $\hbar\omega/\zeta$. The quantum effects may be important in calculating $\mathbf{j}_2$, to which the resonant electrons also contribute. Taking account of boundary conditions even in the classical case means that the field in the metal is determined by an integral equation whose solution cannot be written in closed form. These difficulties remain in the quantum case. But we shall

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[2] As was shown earlier, [3] for the anomalous skin-effect one can always introduce a mean free time.

show in Appendix 2 that finding the various high frequency characteristics in the case of quantum cyclotron resonance reduces to solving a classical problem. In this section, for a qualitative study of the dependence of the surface impedance on magnetic field, we make simplifying assumptions (cf. below) which do not spoil the physical picture of the phenomenon but at the same time permit us to get formulas suitable for study.

Let us consider the derivation of the quantum expression for j_2. Calculations completely analogous to those in [3] give the following result, valid for any values of γ:

$$j_2 = \frac{2e^3 H_0}{\hbar^2 c} \sum_{n,\, l} \int_{-\infty}^{\infty} dp_z \, \frac{f_0(\varepsilon_{n+l,\, p_z}) - f_0(\varepsilon_{n,\, p_z})}{\varepsilon_{n+l,\, p_z} - \varepsilon_{n,\, p_z}}$$

$$\times \, \frac{A_l(y, p_z)}{T^2(-i\omega + il\Omega + 1/\tau)},$$

$$A_l(y, p_z) = \int_0^T dt \, \mathbf{v}(t) \, u(y - r(t) - r_0) e^{-il\Omega t}$$

$$\times \int_0^T dt' e^{il\Omega t'} \mathbf{v}(t') \, \mathbf{E}\left(y - \int_{t'}^{t} v_y dt''\right). \qquad (2.4)$$

Here f_0 is the Fermi distribution function, $T = 2\pi/\Omega(n, p_z)$, $u(x) = 0$ for $x < 0$ and $u(x) = 1$ for $x > 0$, $\mathbf{v}$ is the particle velocity; the variable of integration is the time of revolution of the electron in its orbit in the dc magnetic field, $r(t)$ is the y coordinate of the particle measured from the center of the orbit, and r_0 is half the diameter of the electron orbit in coordinate space.

In proceeding with the solution of the problem, we assume that the temperature is equal to zero. Considering the quantity $f_0(\epsilon_{n+l, p_z}) - f_0(\epsilon_{n, p_z})$ for fixed n, it is easy to see that this difference is different from zero only for p_z in the interval $[p_z(n), p_z(n+l)]$, where $p_z(n)$ is found from the equation $\epsilon(n, p_z) = \zeta$. Thus the integration in (2.4) goes over the small interval

$$\Delta p_z(n, l) = |p_z(n + l) - p_z(n)| \sim p_0 l\hbar\omega/\zeta$$

(where p_0 is the characteristic momentum at the Fermi surface).

To be specific, we shall treat the resonance for $l = 1$. Performing some simple algebraic transformations and using formulas (B_1) and (D_1) of Appendix 1, we get for j_2 the following expression:

$$j_2 = j_2^{cl}(y)$$

$$+ \frac{e^3 H_0 \omega A(y, \omega/H_0)}{2i\hbar^2 c\pi^2 (\partial\Omega/\partial p_z)_S}\left(\ln\frac{\sin \pi n_2}{\sin \pi n_1} - \pi i\varkappa\, \text{sign}\left(\frac{\partial S}{\partial\Omega}\right)_{\varepsilon=\zeta}\right),$$

$$n_2 = n_1 - \varkappa, \quad n_1 = \frac{cS(\omega/H)}{ehH} - \frac{ic(\partial S/\partial\Omega)_{\varepsilon=\zeta}}{ehH\tau}, \quad \varkappa = \frac{(\partial\Omega/\partial p_z)_S}{(\partial\Omega/\partial p_z)_\varepsilon}.$$

$$(2.5)$$

In the second term of formula (2.5) (which we shall call j^{qu}), all quantities are taken on the resonant section of the Fermi surface: $\mu(\zeta, p_z)H_0 = \omega$. The quantity j^{qu} appears as a result of the inclusion of the quantum resonance effects.[3] The logarithm in (2.5) oscillates rapidly with changing ω and H_0 and has an amplitude of the order of unity. Its oscillations are quasiperiodic in the reciprocal of the magnetic field. From the definition of n_1 and n_2 it follows that this period $\Delta(1/H_0)$ is equal to $eh/c \, (S + \mu \, \partial S/\partial\mu)$. The difference in the brackets in (2.5) is greater than or of the order of unity if

$$\text{Im } n_1 = \text{Im } n_2 = \frac{c}{ehH}\left(\frac{\partial S}{\partial\Omega}\right)_{\varepsilon=\zeta}/\tau,$$

which corresponds to condition (1.2). For $\gamma \to \infty$, this quantity tends to zero like $\exp(-\pi\gamma)$. (A detailed investigation of the special cases where $(\partial S/\partial\Omega)_{\varepsilon=\zeta}$ goes either to zero or to infinity will be carried out in Sec. 4.)

Now we proceed to calculate the surface impedance Z_{ik}, which is given by the relation

$$E_i(0) = \sum_{k=1}^{2} Z_{ik} E_k'(0)$$

[where $\mathbf{E}'(0)$ is the derivative of the electric field at the surface of the metal]. First we note that condition (1.2) is also the condition for the existence of "spikes" of the current and field (cf. [4]) at distances which are multiples of d_0, the diameter of the electron orbit in the dc magnetic field. This statement follows from the following estimates. In order to get "spikes," it is necessary that the condition $\gamma_1 = (\omega\tau)^{-1} d_0/\delta \gtrsim 1$ (where δ is the skin depth) be satisfied; since $\delta \sim c/\omega_0$ (where ω_0 is the plasma frequency),

$$\gamma_1 = \frac{v}{c}\frac{\zeta}{\hbar\omega}\Big/\omega\tau = \frac{v}{c}\gamma,$$

i.e., $\gamma_1 \ll \gamma$. Thus, in finding the various high-frequency characteristics one must include those terms in the conductivity tensor which determine the "spikes." However, it can be shown that the presence of "spikes" does not spoil the qualitative picture of quantum cyclotron resonance; to simplify the calculations, we shall therefore omit the terms containing the "spikes" from now on.

In the case of the anomalous skin effect, those electrons are resonant which do not collide with

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[3]We have obtained j_2^{cl} — the classical expression for j_2, by replacing the summation of the nonresonant terms by an integration, i.e., in the zeroth approximation in $(\hbar\omega/\zeta)^{1/2}$. Including first order terms in $(\hbar\omega/\zeta)^{1/2}$ (cf. [3]) allows one to determine the nonresonant oscillations of the high frequency characteristics.

the surface and traverse a path of order $\sqrt{r\delta}$ in the skin layer, which is close to the maximum possible. Thus the picture of quantum cyclotron resonance is not essentially changed if we assume that the electric field is given over all space in such a way that $E(y) = E(-y)$. Mathematically this means that we replace by unity the function $u(y - r(t) - r_0)$ in the expression for A for electrons not striking the surface.

With these assumptions, the relations between the Fourier components of the current and electric field have the following form:[4]

$$j_m^{cl}(k) = \frac{1}{k} \sum_{n=1}^{2} A_{mn} \mathcal{E}_n(k), \qquad j_m^{qu}(k) = \frac{\lambda}{k} \sum_{n=1}^{2} B_{mn} \mathcal{E}_n(k),$$

$$\lambda = \ln \frac{\sin \pi n_2}{\sin \pi n_1} - \pi i \varkappa \, \text{sign}\left(\frac{\partial S}{\partial \Omega}\right)_{\varepsilon=\zeta}, \qquad (2.6)$$

where the subscripts m and n denote transverse components of the current and field. The matrices $\hat{A}$ and $\hat{B}$ are independent of k and are smooth functions of ω and H_0:

$$A_{mn} = \frac{2\pi e^3 H_0}{h^3 c} \sum_{\nu} \int \frac{v_m(\eta_\nu) v_n(\eta_\nu)}{|v_y'(\eta_\nu)|}$$

$$\times \left[1 - \exp\left(-2\pi i \frac{\omega}{\Omega} - \frac{1}{\Omega\tau}\right)\right]^{-1} dp_z,$$

$$B_{mn} = \frac{8e^3 H_0 \pi^2 \omega}{i h^3 c (\partial\Omega/\partial p_z)_S} \sum_{\nu} \frac{v_m(\eta_\nu) v_n(\eta_\nu)}{|v_y'(\eta_\nu)|}.$$

Here η_ν are those points on the section of the Fermi surface for which $v_y = 0$, and the prime denotes differentiation with respect to the time of revolution in the magnetic field. In the matrix $\hat{B}$, all quantities are taken on the resonant section of the Fermi surface.

In expression (2.6) not only is λ an oscillating function of H_0 but so also is $\mathcal{E}(k)$ which determines the change in field with depth. We get an algebraic equation for $\mathcal{E}(k)$ by substituting (2.6) in Eq. (2.1), written in Fourier components:

$$-k^2 \mathcal{E}_m(k) - 2E_m'(0) = -\frac{4\pi i \omega}{c^2} (A_{mn} + \lambda B_{mn}) \frac{\mathcal{E}_n(k)}{k}.$$

It is then easy to find the surface impedance, which turns out to be a quite complicated function of λ:

$$Z_{mn} = -\frac{16\pi^2 \omega i}{3\sqrt{3} c^2} \sum_{r=1}^{2} \xi_r^{-1/3}(\lambda) e_m^r(\lambda) e_n^r(\lambda).$$

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[4]In [1], where electrons all having energies $\varepsilon = \zeta$ were treated, the oscillations of the high frequency characteristics were determined by the function $\cot \pi n_1$. The quantity $\ln(\sin \pi n_2/\sin \pi n_1)$ in the expression for j^{qu} appears as the result of integrating over an energy interval of order $\hbar\omega$ which comes from transitions in the high frequency field.

Here ξ_r and e^r are, respectively, the eigenvalues and unit eigenvectors of the matrix $\hat{Q} = \hat{A} + \lambda\hat{B}$ (in general the e_n^r are complex numbers), i.e., ξ_r are the roots of the characteristic equation $\xi^2 - \xi \, \text{Tr} \, \hat{Q} - \Delta_Q = 0$ (where Δ_Q is the determinant of $\hat{Q}$).

From the definition of ξ_r and e^r it follows that when $\lambda \gg 1$, the expression for the impedance has the form

$$Z_{mn} = Z_{mn}^B \left(\ln \frac{\sin \pi n_2}{\sin \pi n_1}\right)^{-1/3},$$

$$Z_{mn}^B = -\frac{16\pi^2 \omega i}{3\sqrt{3} c^2} \sum_{r=1}^{2} \xi_{rB}^{-1/3} e_{mB}^r e_{nB}^r,$$

where ξ_{rB} and e_B^r satisfy the equation $\hat{B} e_B = \xi_B e_B$. When $\lambda \to 0$ we get the classical expression for the impedance, where $\xi(0)$ and $e(0)$ are given by the equation $\hat{A} e(0) = \xi(0) e(0)$.

3. TEMPERATURE DEPENDENCE

At temperatures different from zero, electrons from a "belt" of energies of order kT around the Fermi surface contribute to the density. From formula (2.5) and formula (B_1) of the appendix, it follows that the dependence of the amplitude of the quantum resonance oscillations on temperature is given by the integral*

$$J = \frac{1}{i\hbar\omega} \int_{-\infty}^{\infty} [f_0(\varepsilon + \hbar\omega)$$

$$- f_0(\varepsilon)] \left[\text{ctg} \, \pi \left(\frac{cS}{ehH} + i\gamma\right) + i \, \text{sign} \, \gamma\right] d\varepsilon,$$

which goes to zero in the classical case. The function $S = S(\zeta, \omega/H) + (\partial S/\partial\varepsilon)_\Omega (\varepsilon - \zeta) + \ldots$ is expressed in terms of the variables ε and Ω, where $\Omega = \omega$. The integral can be written as a sum of residues; we have

$$J = -\frac{2\pi kT}{\hbar\omega} \sum_{m=1}^{\infty} \left[\text{ctg} \, \pi\left(\frac{S_0}{\sigma} + 2\pi i kT \left(\frac{\partial S}{\partial\varepsilon}\right)_\Omega \frac{m}{\sigma} + i\gamma\right)\right.$$

$$\left. - \text{ctg} \, \pi\left(\frac{S_0}{\sigma} + 2\pi i kT\left(\frac{\partial S}{\partial\varepsilon}\right)_\Omega \frac{m}{\sigma} - 2\pi i \hbar\omega\left(\frac{\partial S}{\partial\varepsilon}\right)_\Omega \frac{1}{\sigma} + i\gamma\right)\right].$$

Here $S_0 = S(\zeta, \omega/H)$, $\sigma = ehH/c$.

From the expression given it follows that the parameter which characterizes the damping of the oscillations with increasing temperature is $\beta_0 = 2\pi^2 kT(\partial S/\partial\varepsilon)_\Omega/\sigma$. If $\beta_0 \gg 1$ the oscillations are proportional to $\exp(-\beta_0)$.

It is convenient to write the temperature parameter in the form $\beta_0 = 2\pi^2 kT/\hbar\Omega_1$ where $\Omega_1 = eH/m_1^* c$, $m_1^* = (2\pi)^{-1}(\partial S/\partial\varepsilon)_{\Omega=\omega}$. For most

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*ctg = cot.

sections of the Fermi surface, $\Omega_1 \sim \omega$, and the condition for occurrence of quantum resonance oscillations has the form

$$\beta_0 \sim 2\pi^2 \, kT/\hbar\omega \lesssim 1. \tag{3.1}$$

4. SPECIAL SECTIONS

As we see from the preceding treatment, the amplitude of the resonance oscillations is determined by the parameters

$$\beta_0 = 2\pi^2 \, kT/\hbar\Omega_1,$$

$$\gamma = \frac{\pi c}{ehH} \left(\frac{\partial S}{\partial \Omega}\right)_{\epsilon=\zeta} \Big/ \tau \qquad (S = S\,(\zeta, \omega/H)).$$

For most sections of the Fermi surface, $\partial S/\partial \Omega \sim S_0/\omega$ and $\partial S/\partial \epsilon \sim m_0^*$ (where S_0 and m_0^* are typical values), and the condition for observing quantum cyclotron resonance for these sections has the form (1.2) and (3.1).

For certain frequencies Ω, the derivative $(\partial S/\partial \Omega)_{\epsilon=\zeta}$ goes to zero. Near these special sections the separation of neighboring frequencies of the discrete spectrum, $\Delta\Omega_n = \Omega(n+1) - \Omega(n)$, increases markedly, which makes it much easier for quantum resonance oscillations to occur. Condition (1.2) must then be replaced by a much more stringent condition. Another peculiarity in the behavior of the quantum resonance oscillations occurs when ω coincides with those frequencies of the Fermi surface at which classical cyclotron resonance is observed. As already mentioned, for these sections, which are given by the condition $(\partial \Omega/\partial p_z)_{\epsilon=\zeta} = 0$, the density of states with a given frequency goes to infinity. Thus for ω close to one of the extremal frequencies, we may expect the amplitude of the quantum oscillations to increase.

A. Let us consider a section of the first type, given by the equation $(\partial S/\partial \Omega)_{\epsilon=\zeta} = 0$. Since

$$(\partial S/\partial \Omega)_\epsilon = (\partial S/\partial p_z)_\epsilon/(\partial \Omega/\partial p_z)_\epsilon,$$

there correspond to these special sections extremals of the area of the section of the Fermi surface and sections of the "figure-8" type $[(\partial \Omega/\partial p_z)_{\epsilon=\zeta} \to \infty]$.

1) Near a section with an extremal area, the dependence of S and Ω on p_z is given by the expansions

$$S = S_0 + S''\,(p_z - p_{z0})^2/2, \quad \Omega = \Omega_0 + \Omega'\,(p_z - p_{z0});$$

S_0, Ω_0, and p_{z0} are quantities referring to the special section; from now on the prime denotes a derivative with respect to p_z at constant energy. Using these equations and the quantization condi-

tion (1.1), we get a discrete spectrum of frequencies lying close to Ω_0,

$$\Omega_n = \Omega_0 \pm \Omega'\,(2\sigma/S'')^{1/2}(n - z_0) \qquad (S_0 = \sigma z_0).$$

In the present case, the condition for observing quantum cyclotron resonance has the form

$$(\zeta/\hbar\omega)^{1/2}/\omega\tau \lesssim 1. \tag{4.1}$$

The derivation of the expression for the current density for the case where $\Delta\Omega \gtrsim \omega - \Omega_0$ is somewhat more complicated. The reason for this is that the section on which $(\partial S/\partial \Omega)_{\epsilon=\zeta} = 0$ cuts out two regions on the constant energy surface with different dependences of Ω^ν on n ($\nu = 1, 2$); Ω_0 is the limiting frequency for each of these branches of the spectrum. It then follows that, in calculating $j_\nu^{qu}(y)$, one must use equation (C$_1$) of the appendix. After quite lengthy transformations we get the following formula for j^{qu}:

$$j^{qu}(y) = \frac{e^3 H_0 A \omega}{2\pi^2 ich^3\,(\partial\Omega/\partial p_z)_S} \left\{ \varphi\,(a,\, N - z_0) \right.$$
$$\left. - \left[\ln \frac{\sin \pi n_2}{\sin \pi n_1} - \pi i(n_2 - n_1)\,\text{sign}\,\text{Im}\,n_1 \right] \right\}. \tag{4.2}$$

Here a is a numerical parameter, equal to $(\omega - \Omega_0 - i/\tau)/\Omega'\sqrt{2\sigma/S''}$; the number N gives the level $p_z(N)$ which is closest to the special section. The function $\varphi(a, x)$ has the following form

$$\varphi\,(a,\, x) = \int_{-\infty}^{\infty} \frac{\text{sign}\,y}{e^{2\pi|y|} - 1}\,(\sqrt{iy + x} - \sqrt{x})\,\Delta\psi\,(iy + x, a)\,dy$$
$$+ \int_{x}^{\infty} \left(\frac{d}{dy} + \Delta\right) \psi\,(a, y)\,(\sqrt{y} - \sqrt{x})\,dy,$$

$$\Delta\psi = \psi\,(x - 1) - \psi\,(x),$$

$$\psi\,(a,\, x) = (\sqrt{x + 1} - \sqrt{x})^{-1} \ln\left(\frac{a + \sqrt{x+1}}{a - \sqrt{x+1}}\,\frac{a - \sqrt{x}}{a + \sqrt{x}}\right).$$

The dc magnetic field H_0 appears in the first term of the expression for j^{qu} in the form of the difference $N - z_0$. (To terms of order $(\hbar\omega/\zeta)^{1/2}$, we can neglect the dependence of the parameter a on H_0.) The quantity $N - z_0$ is a periodic function of $1/H_0$ with period $\Delta(1/H_0) = eh/cS_0$. Thus the first term oscillates with this same period when we change the dc magnetic field. The amplitude of oscillation of φ is determined by the parameter a: if $a \lesssim 1$, $\varphi \sim 1$; for $a \gg 1$, we have $\varphi \sim a^{-3}$. If $\omega - \Omega_0 \gg \Omega(\hbar\omega/\zeta)^{1/2} \sim \Delta\Omega$, then $a \sim (\zeta/\hbar\omega)^{1/2} \gg 1$ and the amplitude of oscillation of φ becomes of order $(\hbar\omega/\zeta)^{3/2}$. Then formula (4.2) goes over into formula (2.5) for an ordinary section. For the case of $\omega - \Omega_0 \sim \Delta\Omega$ the parameter $a \sim 1 - i(\hbar\omega/\zeta)^{1/2}/\omega\tau$, so that when condition

(4.1) is satisfied, $\varphi \sim 1$. It is interesting to note that even for $\tau \to \infty$, the amplitude of the oscillations of φ remains of order unity. From the treatment given it follows that in the limiting case of $a \gg 1$ the first term in the expression for j^{qu} determines the nonresonant oscillations of the high frequency characteristics, where were treated in [3].

The quantities n_1 and n_2 are given by the equations $\omega - 1/\tau = \Omega(\zeta, n_1) = \Omega(\zeta + \hbar\omega, n_2)$. On sections with an extremal area, $(\partial\Omega/\partial p_z)_S = (\partial\Omega/\partial p_z)_\epsilon$; thus, for $\omega = \Omega_0$ the difference $n_2^\nu - n_1^\nu = \pm 1$ and the expression in square brackets in (4.2) vanishes. Thus for frequencies $\omega - \Omega_0 \sim \Delta\Omega$, the main contribution to j^{qu} comes from the function $\varphi(a, N - z_0)$.

2) The curve $\epsilon = $ const, $p_z = $ const is self-intersecting (i.e., it has the form of a "figure 8"), when the plane $p_z = $ const is tangent at a saddle point of the constant energy surface. The loops of the "figure 8" divide the constant energy surface into several regions, to which there correspond different branches of the spectrum. Near the special section, the dependence of Ω and S on ϵ and p_z has the form [2]

$$\Omega^\nu = \Omega_\nu^0 \Big/ \ln \frac{p_1^\nu}{p_z - p_0(\epsilon)},$$

$$S = S_0^\nu(\epsilon) + p_2^\nu(p_z - p_0(\epsilon)) \ln \frac{p_3^\nu}{p_z - p_0(\epsilon)}.$$

Here $p_0(\epsilon)$ is the projection along the direction of the dc magnetic field of the momentum at the point of self-intersection, $S_0^\nu(\epsilon)$ is the area of the loop of the "figure 8": the quantities p_1^ν, p_2^ν, $p_3^\nu \sim p_0$, the characteristic momentum at the Fermi surface. When $p_z - p_0(\epsilon) \to 0$, the frequency Ω^ν of revolution of the particle goes to zero logarithmically. The distance between neighboring frequencies of the discrete spectrum depends essentially on n, and increases sharply near the special section:

$$\Delta\Omega_n \sim \Omega_0 \frac{\Delta p_n}{(p_{zn} - p_0) \ln^2(\zeta/\hbar\Omega_0)} \qquad \left(\Delta p_n \sim p_0 \frac{\hbar\omega}{\zeta} \Big/ \ln \frac{\zeta}{\hbar\omega}\right).$$

If the frequency $\omega \sim \Omega_0/\ln(\zeta\hbar/\Omega_0)$, the condition for observing quantum cyclotron resonance looks as follows:

$$\gamma = \frac{\zeta}{\hbar\omega} \exp\left(-\frac{\Omega_0}{\omega}\right) \ln^3\left(\frac{\zeta}{\hbar\omega}\right) \Big/ \omega\tau \lesssim 1. \qquad (4.3)$$

In deriving the equation for the current one should take account of the fact that for $\Omega_0/\omega \sim \ln(\zeta/\hbar\Omega_0)$, the frequency ω lies near the edge of the spectrum. Using formula (C_1) of the appendix and neglecting terms $\sim 1/\ln^2(\zeta/\hbar\omega)$, we get the following result:

$$j_\nu^{qu} = \frac{e^3 H_0 A\omega}{2\pi^2 ich^3} \left\{ \frac{p_3\alpha\Omega_0}{|\ln\alpha|\omega^2} \varphi(a, N - z_0) \right.$$
$$\left. - \left(\frac{\partial p_z}{\partial\Omega}\right)_S \left[\ln \frac{\sin \pi n_2}{\sin \pi n_1} + \pi i(n_2 - n_1) \text{sign Im } n_1\right] \right\}. \qquad (4.4)$$

The index ν has been omitted on the right side of (4.4). The quantum number N_ν gives the level which is closest to the special section; $\alpha^\nu = \sigma/p_2^\nu p_3^\nu \sim \hbar\Omega/\zeta$; the numerical parameter $a = \ln[p_1|\ln\alpha| \exp(-\Omega_0/\omega)/\alpha p_3]$. The function φ is given by the equation

$$\varphi(a, x) = \int_{-\infty}^\infty \frac{|y|}{e^{2\pi|y|} - 1} \Delta\psi(x + iy, a)\, dy$$

$$+ \int_0^\infty y\left(\frac{d}{dy} + \Delta\right)\psi(y + x, a)\, dy;$$

$$\psi(x, a) = \int_0^1 \frac{du}{a + \ln(x + u)}.$$

Near the special section, i.e., for $\omega \sim \Omega(N) \sim \Omega_0/\ln(\zeta/\hbar\Omega_0)$, the quantities n_1 and n_2 have the form

$$\sigma n_1 = S_0 + p_1 p_2 \exp\left(-\frac{\Omega_0}{\omega - i/\tau}\right)\left(\ln \frac{p_1}{p_3} - \frac{\Omega_0}{\omega - i/\tau}\right);$$

$$\sigma n_2 = S_0 + p_2 p_3\left[\frac{p_1}{p_3} \exp\left(-\frac{\Omega_0}{\omega - i/\tau}\right) + \frac{\alpha}{\ln\alpha}\right]$$
$$\times \ln\left[\frac{p_1}{p_3} \exp\left(-\frac{\Omega_0}{\omega - i\tau}\right) + \frac{\alpha}{\ln\alpha}\right]. \qquad (4.5)$$

From Eq. (4.5) we see that Im n_1 and Im n_2, which determine the damping of the quantum resonance oscillations, are of the order

$$\frac{\zeta}{\hbar\omega} \exp\left(-\frac{\Omega_0}{\omega}\right) \frac{\Omega_0}{\omega}\left(\left|\sin \frac{\Omega_0}{\omega^2\tau}\right| + \frac{1}{\omega\tau}\right),$$

which is in agreement with condition (4.3).

The expression for j_ν^{qu} is a sum of two terms which are periodic in $1/H_0$. The function φ, which depends on the difference $N - z_0$, has the period $\Delta(1/H_0) = eh/cS_0^\nu$; the amplitude of its oscillations is determined by the parameter a: $\varphi \sim 1$ for $a \lesssim 1$, $\varphi \sim a^{-2}$ for $a \gg 1$. Just as in the case of a section with an extremal area, for $a \gg 1$ the first term determines the nonresonant oscillations. These oscillations are the result of edge effects caused by the special section. For $\omega \sim \Omega_0$, the parameter $a \sim \ln(\hbar\omega/\zeta)$ and $\varphi \sim 1/\ln^2(\zeta/\hbar\omega)$. For $\omega \sim \Omega(N)$, this parameter is of order $1 + i\Omega_0/\omega^2\tau$, i.e., in this case the condition $a \lesssim 1$ does not coincide with condition (4.3). The second term in the expression for j^{qu} corresponds to logarithmic quantum cyclotron resonance. From formula (4.5) we see that for $\omega \sim \Omega(N)$, the oscillation period $\ln(\sin \pi n_2/\sin \pi n_1)$ coincides with the period of the function φ.

From formula (4.4) it follows that even in the case where condition (4.3) is satisfied and $\varphi \sim 1$, the ratio

$$\frac{|j^{qu}|}{|j^{cl}|} \sim \exp\left(-\frac{\Omega_0}{\omega}\right) \ln^2 \frac{\zeta}{\hbar\omega} \ll 1.$$

The reason for the decrease in the amplitude of quantum resonance oscillations for $\omega \sim \Omega(N)$ is that near the "figure 8" the density of states with a given frequency, which is determined by the derivative $\partial p_z / \partial\Omega$, is relatively low—for a level with quantum number N,

$$\frac{\partial p_z}{\partial\Omega} \sim \exp\left(-\frac{\Omega_0}{\Omega(N)}\right) \ln^2 \frac{\zeta}{\hbar\Omega_0}.$$

B. Let us consider quantum cyclotron resonance when ω is close to one of the extremal frequencies of the Fermi surface, $(\partial\Omega/\partial p_z)_{\epsilon=\zeta} = 0$. It turns out that the picture of quantum cyclotron resonance is different for the central section and for the other sections having extremal frequencies. The difference arises because, for noncentral sections with extremal frequencies, the derivative $(\partial\Omega/\partial S)_\epsilon$ vanishes ($\Omega' = 0$, but $S' \neq 0$). The distance between neighboring frequencies of the spectrum is then very small: $\Delta\Omega \sim (\hbar\omega/\zeta)^2 \Omega_0$, and the amplitude of quantum resonance oscillations near such sections is proportional to $\exp[-(\zeta/\hbar\omega)^2/\omega\tau]$, which is practically always much less than unity.

Near the central section,

$$\Omega = \Omega_0 + \Omega'' p_z^2/2, \qquad S = S_0 + S'' p_z^2/2,$$

and the frequency spectrum has the form

$$\Omega_n = \Omega_\theta + \Omega'_S \sigma \, (n - z_0),$$

so that $\Delta\Omega \sim \hbar\omega\Omega_0/\zeta$, i.e., of the same order as for the "ordinary" sections of the Fermi surface. The frequency Ω_0 is the limiting frequency of the spectrum. Therefore, just as in the cases treated before, in deriving the expression for the current one must start from formula (C$_1$) of the appendix. Using the fact that for $\omega - \Omega_0 \ll \Omega_0$ the derivative $(\partial\Omega/\partial p_z)_{S=\sigma n} = [2\dot{\Omega}(\omega - \Omega(\sigma n, 0) - i/\tau]^{1/2}$ (where the dot denotes differentiation with respect to p_z at constant S), we get for j^{qu} the following expression:

$$j^{qu} = \frac{e^3 H_0 A\omega}{2\pi i c h^3 [\sigma\dot{\Omega}(\partial\Omega/\partial S)_{p_z}]^{1/2}} \left\{ \varphi\,(a, \ N - z_0) - \int_0^a \frac{\text{ctg}\,\pi\,(n_1 - y)\,dy}{\sqrt{ab + y}} \right.$$

$$\left. + \pi i \sqrt{b}\,(\sqrt{a+1} - \sqrt{a})\,\text{sign}\left(\frac{\partial S}{\partial\Omega}\right)_{\epsilon=\zeta} \right\}. \tag{4.6}$$

Here N is the quantum number giving the level nearest to the central section; $a = (\omega - \Omega_0 - i/\tau)/\sigma(\partial\Omega/\partial S)_{p_z}$; the parameter b does not depend on the magnetic field: $b = m^*/(\ddot{m}^*)'' \sim 1$;

$$n_1 = S\left(\frac{\omega}{H}\right) \bigg/ \sigma - \frac{i\,(\partial S/\partial\Omega)_{\epsilon=\zeta}}{\sigma\tau}\,.$$

We shall not give the specific form of φ because it is too lengthy.

The physical meaning of the function φ was considered earlier in studying the analogous formulas for the special sections with extremal area and those of the "figure 8" type. Just as before, the parameter a determines the damping of the oscillations of the first term: $\varphi \sim 1$ for a $\lesssim 1$, and when a $\gg 1$, $\varphi \sim a^{-2}$. The second term in (4.6) corresponds to logarithmic cyclotron resonance: if $\gamma \to 0$, this term increases like $\ln|\gamma|$. From the expression for j^{qu} it follows that for a $\sim \gamma \lesssim 1$ and $\omega - \Omega_0 \sim \Delta\Omega$ the amplitude of the quantum resonance oscillations of the current density is $(\zeta/\hbar\omega)^{1/2}$ times greater than the corresponding amplitude for the "ordinary" sections. This peculiarity in the behavior of j^{qu} as a function of H_0 and ω is related to the fact that the quantum resonance oscillations are superposed on the picture of classical cyclotron resonance corresponding to the extremal frequency Ω_0. When $\gamma \lesssim 1$ and a $\lesssim 1$, the dependence on $1/H_0$ enters into j^{qu} not just in the form of the periodic functions $N - z_0$ and $\cot \pi\,(n_1 + y)$. Now one must also consider the dependence of the parameter a on the magnetic field: a changes by an amount of the order of unity when $1/H_0$ changes by eh/cS_0. Thus for $\omega - \Omega_0 \sim \Delta\Omega$ the oscillations become aperiodic.

If $\omega \sim \Omega_0$, the parameter a $\sim \zeta/\hbar\omega$. It is easy to see that in this case the expression for j^{qu} is given by (2.5) correct to terms of order $(\hbar\omega/\zeta)^{3/2}$.

C. There may be still another type of section of the Fermi surface which gives a peculiar behavior of the quantum cyclotron resonance with temperature. The parameter β_0 which gives the T dependence, contains the quantity $\Omega_1 = eH/m_1^*c$, where

$$m_1^* = (2\pi)^{-1}\,(\partial S/\partial\epsilon)_\Omega.$$

One can show that

$$\overset{\bullet}{m_1^*} = m^*\,(\partial m^*/\partial p_z)_S/(\partial m^*/\partial p_z)_\epsilon.$$

On sections for which $(\partial m^*/\partial p_z)_S = 0$ and $(\partial m^*/\partial p_z)_\epsilon \neq 0$, the parameter β_0 vanishes. For these sections the temperature condition for observing quantum cyclotron resonance is $(kT)^2/\hbar\omega\zeta \lesssim 1$, which is practically always satisfied for metals. The temperature dependence for sections with β_0 equal to zero is determined only by the fact that the relaxation time decreases with increasing T.

5. EXPERIMENTAL POSSIBILITIES FOR OBSERVING QUANTUM CYCLOTRON RESONANCE

Now let us see what values of the dc magnetic field H_0 and the frequency ω are needed so that the conditions for observing quantum cyclotron resonance are satisfied: $\gamma \lesssim 1$, $\beta_0 \lesssim 1$. First let us consider any sections of the Fermi surface for which $\gamma \sim \zeta/\hbar\omega^2\tau$ and $\beta_0 \sim (2\pi)^2 kT/\hbar\omega$. The condition $\beta_0 \lesssim 1$, which is related to the temperature dependence of the quantum cyclotron resonance, is most restrictive. If the frequency of the external electromagnetic field is in the millimeter range, satisfying this condition requires temperatures of the order of 0.1°K; at liquid helium temperatures, ω must be of the order of 10^{12} sec^{-1}. The parameter γ contains the relaxation time τ, which is determined both by the scattering of electrons at impurities and by the electron-electron and electron-phonon interaction, i.e., $1/\tau = 1/\tau_0 + 1/\tau_1 + 1/\tau_2$ (where τ_0 is the impurity relaxation time, and τ_1 and τ_2 are respectively the electron-electron and electron-phonon relaxation times). The latter two types of interaction give a strong dependence of τ on ω:

$$\frac{1}{\tau_1} \sim \omega_0 \left(\frac{\hbar\omega + kT}{\hbar\omega_0}\right)^2, \qquad \frac{1}{\tau_2} \sim \frac{k\Theta}{\hbar}\left(\frac{\hbar\omega + kT}{k\Theta}\right)^3$$

($\omega_0 = \zeta/\hbar$, and Θ is the Debye temperature).

If we assume that the condition $\beta_0 \lesssim 1$ is satisfied, then for the ordinary sections

$$\gamma \sim \frac{\omega_0}{\omega^2}\left(\frac{1}{\tau_0} + \omega_0\left(\frac{\omega}{\omega_0}\right)^2 + \omega_\Theta\left(\frac{\omega}{\omega_\Theta}\right)^3\right),$$

where $\omega_\Theta = k\Theta/\hbar$. This expression has a minimum at the optimum frequencies $\omega_{opt} \sim \omega_\Theta^{2/3}\tau_0^{-1/3}$. The value of γ_{opt} corresponding to these frequencies is $\gamma_{opt} \sim \omega_0\omega_\Theta^{-4/3}\tau_0^{-1/3} + \gamma_0$, $\gamma_0 \sim 1$; γ_{opt} and ω_{opt} are independent of the magnetic field and are determined by the fundamental parameters of the metal. Assuming $\omega_\Theta \sim 10^{13}$ sec^{-1} and $\tau_0 \sim 10^{-9}$ sec, we get $\omega_{opt} \sim 5 \times 10^{11}$ sec^{-1}. In order for the condition $\gamma_{opt} \lesssim 1$ to be satisfied, we would need $\omega_0 \lesssim 10^{14}$ sec^{-1}. In most metals such values of ω_0 correspond to "anomalous" electron groups with long-period oscillations in the de Haas–van Alphen effect.[5] For the ordinary electron groups the situation $\gamma \lesssim 1$ can be reached if $\tau_0 \sim 10^{-8}$ sec. Such long mean free paths can be obtained in metals of very high purity.

The possibility of observing quantum cyclotron resonance is improved near the special sections

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[5]In bismuth, with an electron density $\sim 10^{18}$ cm^{-3}, the frequencies $\omega_0 \sim 10^{14}$ sec^{-1} correspond to the main electron groups.

considered above. As was shown in the preceding section, for sections with an extremal area, the parameter γ is $(\zeta/\hbar\omega)^{1/2}\omega\tau$; for the "figure 8's", γ is given by formula (4.3). The condition for observing quantum cyclotron resonance in the first case is well satisfied even for the main electron groups when $\tau_0 \sim 10^{-9}$ sec, $\omega \sim 10^{11}$ sec^{-1}, $H_0 \sim 10^4$ Oe. In the second case, for the main electron groups we need $H_0 \sim 10^5$ Oe with $\tau_0 \sim 10^{-9}$ sec and $\omega \sim 10^{11}$ sec^{-1}. The temperature condition for the occurrence of quantum resonance oscillations for these two types of sections is the same as for "ordinary" sections of the Fermi surface. On sections where $(\partial m^*/\partial p_z)_S = 0$ and $(\partial m^*/\partial p_z)_{\epsilon=\zeta} \neq 0$, the parameter β_0 vanishes and the only condition for observing quantum cyclotron resonance is $\zeta/\hbar\omega^2\tau \lesssim 1$; the possibility of achieving this condition was discussed above.

APPENDIX 1

We consider a series of the form

$$\sum_{n=N_1}^{N_2} f(n)\,\Delta p(n),$$

where

$$\Delta p(n) = p(n+1) - p(n), \quad f(n) = (\omega - \Omega(n) - i/\tau)^{-1};$$

assuming that

$$\Omega\tau \gg 1, \qquad \Delta\Omega = \Omega(n+1) - \Omega(n) \ll \Omega.$$

If ω does not coincide with any of the frequencies corresponding to the interval $[N_1, N_2]$, the series can be replaced approximately by the integral

$$\int_{p(N_1)}^{p(N_2)} f(p)\,dp.$$

If there is a number n_0 in the interval $[N_1, N_2]$, such that $\Omega(n_0) = \omega$, we must distinguish two cases: $\partial\Omega/\partial n \ll 1/\tau$ and $\partial\Omega/\partial n \gtrsim 1/\tau$.

In the first case the series is again approximately equal to $\int f\,dp$. When $\partial\Omega/\partial n \gtrsim 1/\tau$, the replacement of the sum by an integral is invalid for $n \approx n_0$. In this latter case, using the function $\cot \pi n$, we replace the series by an integral over a closed contour C in the complex plane, intersecting the real axis at the points $N_1 - 0$ and $N_2 + 0$; we then have

$$\Sigma \equiv \sum_{n=N_1}^{N_2} f(n)\,\Delta p(n) = \frac{1}{2i}\int_C \operatorname{ctg}\pi n\, f(n)\,\Delta p(n)\,dn$$

$$- \pi \sum \operatorname{Res}\,(f\operatorname{ctg}\pi n\,\Delta p).$$

After transformations which are accurate to ex-

ponential order, we get the following expressions for Σ:

$$\Sigma = \frac{1}{2i} \sum_{v=1}^{2} \int_{L_v} (\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n)\, p\,(n)\, \Delta f dn$$

$$+ f\,(N_2)\, p\,(N_2 + 1) - f\,(N_1 - 1)\, p\,(N_1) + \int_{N_1}^{N_2} p \Delta f dn$$

$$- \pi \sum \operatorname{Res} [f \Delta p\,(\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n)]. \qquad (A_1)$$

The contours L_1 and L_2 are straight lines parallel to the imaginary axis, passing respectively through the points $N_1 - 0$ and $N_2 + 1 + 0$. If the resonance frequency $\Omega(n_0) = \omega$ lies far from the edge of the spectrum, formula (A_1) reduces, to an accuracy of order $\Delta\Omega \Delta p / \Omega p$ to the form

$$\Sigma = \int_{p(N_1)}^{p(N_2)} f dp - \pi \sum \operatorname{Res} [f \Delta p\,(\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n)]. \quad (B_1)$$

The expression for Σ becomes complicated when n_0 is close to the edge of the spectrum, for example close to N_1. In this case

$$\Sigma = \frac{1}{2i} \int_{L_1} (\operatorname{ctg} \pi n + \operatorname{sign} \operatorname{Im} n)\,(p - p\,(N_1))\, \Delta f dn$$

$$+ \int_{N_1}^{\infty} \left(\Delta f + \frac{df}{dn} \right)(p - p\,(N_1))\, dn + \int_{p(N_1)}^{p(N_2)} f dp$$

$$- \pi \sum \operatorname{Res} [f \Delta p\,(\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n)]. \qquad (C_1)$$

In deriving formula (C_1) we used the invariance of the original expression with respect to the substitution $p(n) \to p(n) + p_0$.

In the preceding sections, in the course of intermediate calculations we repeatedly met with expressions of the type

$$\int_0^1 du \sum_n f(p, u)\, \Delta p(n).$$

In this connection we consider the integral

$$\int_0^1 du \operatorname{Res} [f \Delta p\,(\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n)],$$

where the residue is taken at the point $n(u)$, given by the equation

$$\omega - \Omega\,(\sigma n\,(u),\, p\,(n) - u\Delta p\,(n)) - i/\tau = 0.$$

Assuming that there is a simple pole of f at this point, we get

$$\int_0^1 du \operatorname{Res} [f \Delta p\,(\operatorname{ctg} \pi n + i \operatorname{sign} |\operatorname{Im} n)]$$

$$= \int_0^1 du \frac{dn}{du} \left(\frac{\operatorname{ctg} \pi n + i \operatorname{sign} \operatorname{Im} n}{(\partial\Omega/\partial p_z)_{s=\sigma n}} \right). \qquad (D_1)$$

In deriving this equation we used the fact that

$$\frac{\Delta p\,(n)}{d\Omega/du} = - \frac{dn/du}{(\partial\Omega/\partial p)_{S=\sigma n(u)}}.$$

APPENDIX 2

As was shown above, the current density in the bulk of the metal is determined by the equation $\mathbf{j}(y) = \mathbf{j}^{cl}(y) + \mathbf{j}^{qu}(y)$. The dependence of $\mathbf{j}^{cl}(y)$ on $\mathbf{E}(y)$ neglecting "spikes" was found earlier.[5] The dependence of $\mathbf{j}^{qu}(y)$ on electric field is given by (2.5). The equations (2.5) and (2.1) can be continued symmetrically into the region $y < 0$ if we set $\mathbf{E}(y) = \mathbf{E}(-y)$. This is a consequence of the central symmetry of the Fermi surface. The further course of solving the problem looks simplest if we go over to Fourier components of the current and electric field: $\mathbf{j}(k) = \mathbf{j}(-k)$ and $\mathscr{E}(k) = \mathscr{E}(-k)$. Carrying out the rather lengthy transformations to an accuracy of $(\delta/d)^{3/2}$, we get an integral equation for $\mathscr{E}(k)$:

$$- k^2 \mathscr{E}_m(k) + 2E'_m(0) = - \frac{4\pi i\omega}{c^2} \sum_{n=1}^{2} \Big\{ [A_{mn}^{(1)} - B_{mn} \sin kd_0$$

$$+ \lambda\,(C_{mn} - D_{mn} \sin kd_0)] \frac{\mathscr{E}_n(k)}{k}$$

$$- (B_{mn} + \lambda D_{mn}) \cos kd_0 \int_0^\infty \frac{\mathscr{E}_n(k')\,dk'}{(k - k')\sqrt{kk'}}$$

$$+ [A_{mn}^{(3)} - B_{mn} \sin kd_0$$

$$+ \lambda\,(C_{mn} - D_{mn} \sin kd_0)] \int_0^\infty \frac{\mathscr{E}_n(k')\,dk'}{(k + k')\sqrt{kk'}}$$

$$+ A_{mn}^{(4)} \int_0^\infty \frac{\ln k/k'}{(k^2 - k'^2)} \mathscr{E}_n(k')\,dk' \Big\}. \qquad (A_2)$$

The subscripts m and n take on the values 1 and 2, corresponding to the x and z components of the electric field;

$$\lambda\,(\omega,\, H_0) = \ln \frac{\sin \pi n_2}{\sin \pi n_1} - \pi i \varkappa \operatorname{sign} \left(\frac{\partial S}{\partial \Omega} \right)_{\varepsilon=\zeta},$$

the tensors $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ do not depend on k and are smooth functions of ω and H_0. The matrices $\hat{A}$, corresponding to classical cyclotron resonance neglecting "spikes," were found in [5]. The matrices $\hat{B}$, $\hat{C}$, and $\hat{D}$ have the following form:

$$\hat{B} = \pi i \varkappa \operatorname{sign} \left(\frac{\partial S}{\partial \Omega} \right)_{\varepsilon=\zeta} \hat{D},$$

$$C_{mn} = \frac{e^3 H_0 \omega}{i\hbar^3 c\,(\partial\Omega/\partial p_z)_S} \sum_v \frac{v_m(\eta_v)\, v_n(\eta_v)}{|v'_y(\eta_v)|},$$

$$D_{mn} = \frac{e^3 H_0 \omega}{i\hbar^3 c\,(\partial\Omega/\partial p_z)_S} \frac{v_m(\eta_1)\, v_n(\eta_2)\, e^{-i\beta} + v_m(\eta_2)\, v_n(\eta_1)\, e^{i\beta}}{\sqrt{|v'_y(\eta_1)\, v'_y(\eta_2)|}}$$

$(\beta = \omega \,|\, \eta_2 - \eta_1 |)$.

The quantities appearing in the tensors $\hat{B}$, $\hat{C}$, and $\hat{D}$ are taken on the "resonance" section of the Fermi surface $[\Omega(\zeta, p_z) = \omega]$. The terms containing $\sin kd_0$ and $\cos kd_0$ appear when we take "spikes" into account.

Equation (A$_2$) can be written symbolically:

$$\sum_m a_m \hat{L}_m \vec{\mathscr{E}}(k) = \mathbf{E}'(0). \qquad (B_2)$$

Here the a_m are constant coefficients, independent of k; the $\hat{L}_m$ are linear integral operators. Comparison of (A$_2$) and (B$_2$) with the corresponding "classical" expression shows that taking account of the quantization in the magnetic field does not change the form of the integral operators $\hat{L}_m$ in the equation for $\mathscr{E}(k)$. It then follows that in both the classical and quantum cases the electric field in the metal is determined by the same function $\mathscr{E}(k, a_m)$. It is understood that the dependence of a_m on ω and H_0 is completely different in these two cases: if $\gamma \lesssim 1$, the coefficients a_m are rapidly oscillating functions which are quasiperiodic in $1/H_0$ with the period $\Delta(1/H_0)$ found above.

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