RADIATIVE CORRECTIONS TO ENERGY LOSSES OF PARTICLES IN A MEDIUM

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Energy losses of particles in a medium are investigated by taking into account radiative corrections. It is shown that the macroscopic part of the radiative losses due to the presence of the medium may be very significant and comprise about 7% of the total losses. A density effect applies to the radiative corrections and occurs at energies larger than those at which the density effect in the main part of the losses arises. It is predicted that a decrease of ionization energy losses of about 7–10% should occur on the Fermi plateau. The theoretical and experimental results are compared. The problem of radiative corrections to Cerenkov radiation is discussed.

1. INTRODUCTION

1. In the analysis of ionization and Cerenkov[1)] energy losses of a particle in a medium [1] it is customary to take into consideration the first perturbation-theory term in e^2. It might appear that the reason for it is that for ionization losses, and also for general losses, which are determined essentially by ionization losses, we are dealing with energy transfers that are in considerable excess of the electron binding energy, since the relative contribution of the Cerenkov losses is small, i.e., in final analysis, we are dealing merely with radiative corrections to the scattering [2]. Inasmuch as the c.m.s. scattering angle is small (the particle passes through), the radiative corrections to the scattering are small. For example, in measurements made with photographic plates, when the only events recorded are those in which the energy transfer does not exceed ω_{max}[2)], the maximum c.m.s. scattering angle is of the order $\theta_{max} \sim \sqrt{\omega_{max}/\epsilon_p}$, where ϵ_p is the particle energy in the laboratory system. The radiative corrections to the scattering of an electron by an electron can be estimated from the known formulas (see the papers by Akhiezer and Polovin [2] or Redhead [3]). For the maximum value of the relative magnitude of the radiative corrections we obtain

$$\delta_R^{max} \sim e^2 \frac{\omega_{max}}{m} f\left(\frac{\varepsilon}{\Delta\varepsilon}\right).$$

The quantity f, which depends on $\Delta\epsilon$—the maximum energy of a soft quantum [more accurately, on $\ln(\epsilon/\Delta\epsilon)$]—is a number of the order of unity, for usually the experimental inaccuracy in the determination of the value of the energy is not small. The radiative corrections to scattering are consequently of the order of $(1/137)\,\omega_{max}/m_e$ (m_e is the electron mass); since $\omega_{max}/m_e \sim 10^{-1} - 10^{-2}$, the relative contribution of the radiative corrections to the polarization losses would amount to one-hundredth of a percent.

In the estimates presented above, however, we used for the radiative corrections a cross section in which integration was carried out over all values of the energies of the virtual quanta, including the long-wave macroscopic part; no account was taken here of the deviation of the dielectric constant of the medium from unity. This is incorrect, since, as is shown below, the macroscopic part of the radiative corrections greatly exceeds[3)] the "microscopic" part estimated above and determines the magnitude of the correction to the ionization and Cerenkov losses as being of the order of e^4.

The cause of this situation can be qualitatively understood from the fact that for the "microscopic" part of the radiative corrections the recoil upon scattering is appreciable and the greater this recoil the larger the corrections, which vanish when there is no scattering, whereas the macroscopic part of the radiative corrections does not vanish if recoil is neglected (just as Cerenkov radiation exists for a uniformly moving particle).

[1)]We take the term "Cerenkov losses" to mean energy lost to radiation of longitudinal or transverse Cerenkov quanta, i.e., both the Cerenkov radiation proper and the Bohr losses.

[2)]The order of ω_{max} is usually 1–5 keV and at the utmost 50 keV ($\hbar = c = 1$ everywhere).

[3)]The last statement pertains to sufficiently dense media (for example, photographic plates, etc.).

2. The macroscopic part of the radiative corrections to the energy loss is due, roughly speaking, to the fact that the mass of the particle in the medium differs from its mass in vacuum by an amount Δm. This question was considered in detail earlier [4]. Let us call attention to the behavior of Δm at large energies, an important factor in what follows. In particular, the transverse and longitudinal parts of the change in mass (Δm^t and Δm^l) increase linearly with increasing particle energy in the ultrarelativistic limit. With increasing energy, however, their sum tends to a constant limit $\Delta m = \Delta m^t + \Delta m^l = -e^2\omega_0$,[4] a limit independent of the natural frequencies of the medium (ω_0 is the plasma frequency). The energy range for which this saturation effect takes place, for the model with oscillators having a single natural frequency ω_s, is $\epsilon_p/m \gg \omega_s/\omega_0$. This criterion coincides with the criterion for the onset of the Fermi-density effect in losses for the oscillator model. Because the mass of the particle in the medium is different from the mass of the free particle and depends on its velocity, the square of the four-momentum of the particle is not equal to m^2.

3. As usual, in calculations in the higher perturbation-theory approximations expressions arise which diverge logarithmically on the lower limit (i.e., the infrared divergence). If the process takes place without a medium, then as is well known it is sufficient to consider a process with emission of soft quanta down to a certain frequency $\Delta\epsilon$, the width of which is specified by the experimental accuracy with which the energy of the particles participating in the process is determined [5]. The infrared divergence vanishes then, but the result contains $\Delta\epsilon$. In the presence of the medium, no such compensation of the infrared divergence by soft quanta can occur. This can be understood by recognizing that the emission of a "soft" quantum occurs only in the presence of recoil, albeit small, whereas the main process can be regarded at zero recoil.

A good illustrative example is that of radiative corrections to Cerenkov radiation [7]. Assume that the Cerenkov radiation condition $n(\omega)v > 1$ is satisfied in the entire low-frequency region of interest to us, down to $\omega = 0$, so that any soft quantum satisfies the Cerenkov-radiation condition. The emission of a soft quantum accompanying the main Cerenkov quantum is a process in which two Cerenkov quanta are emitted but there is no infrared divergence, since expansion in the number of quanta is perfectly permissible for the Cerenkov radiation (unlike bremsstrahlung). This is seen from the classical formula for Cerenkov radiation, which contains the factor ω ($\hbar = c = 1$):

$$I_\omega = e^2 v\omega (1 - 1/n^2v^2). \tag{1}$$

The emission of two Cerenkov quanta is therefore a quantum effect proportional to $e^4\omega^2/m^2$, which is neglibibly small compared with the radiative corrections to the Cerenkov radiation.

The infrared divergence in the radiative corrections, however, can be readily eliminated if it is recognized that for a particle moving in a medium we have $p^2 \neq m^2$ or that $E \neq \epsilon_p = \sqrt{p^2 + m^2}$ (compare with Sec. 2). Therefore elimination of the infrared divergence in radiative corrections to the Cerenkov radiation has much more in common with the corresponding problem for the Lamb shift ($p^2 \neq m^2$ as a result of interaction), than with the problem concerning radiative corrections to scattering.

2. MACROSCOPIC PART OF RADIATIVE CORRECTIONS TO ENERGY LOST BY PARTICLES IN A MEDIUM

1. For the sake of simplicity we disregard the exchange effect and assume that the particle is not identical with the particles of the medium. Let the field of the particles (electrons) of the medium be described by an operator $\hat{\varphi}$, the field of the investigated particle by an operator $\hat{\Psi}$, and the electromagnetic field by $\hat{A}$. Introducing the external sources of these fields, in analogy with the conventional technique (see, for example [8]), we obtain a system of equations in functional derivatives for the Green's functions $R(x_1, x_2)$, $G(x_1, x_2)$ and $D(x_1, x_2)$ of the fields φ, Ψ, and A. Here R is defined as the average over the ground state of the system of particles of the medium, and G is defined as the average over the vacuum; the D-function takes into account the polarization of the medium. Renormalization reduces to a choice of the constants in the multiplicative transformation

$$\widetilde{G} = Z_1^{-1}G, \quad \widetilde{R} = R/Z_1'^{-1}, \quad \widetilde{D} = Z_3^{-1}D,$$

$$\widetilde{\Gamma} = Z_1\Gamma, \quad \widetilde{\Gamma}' = Z_1'\Gamma \text{ [6]},$$

[4]An essential dependence of Δm on the energy of the particle as the latter moves along the axis of a hollow channel of radius a can appear if $a\omega_0 \gg 1$.

[5]We do not concern ourselves here with cases of very small $\Delta\epsilon$ or large particle energies, when it is necessary to sum the perturbation-theory series (see Abrikosov's paper[6]).

[6]We take account here of the fact that $Z_1 = Z_2$ and $Z_1' = Z_2'$.

where Γ and Γ' are the vertex parts of the Ψ and φ fields. Inasmuch as the change in the electromagnetic mass of the particle in the medium and its energy losses are observable effects, the choice of the constants Z_1, Z'_1, and Z_3, for example, should be carried out in such a way as to eliminate only the unobservable vacuum quantities. For example, Z_1 must be chosen from the condition that $\widetilde{G}$ have a pole when $i\hat{p} \to i\hat{p}_0 = m_e$ (m_e is the experimental mass also in the absence of the medium. From the equation for G [7] (in analogy with what was done in [8] in the absence of the medium)

$$\{Z_1 (i\hat{p} + m) + \widetilde{M}(p)\} \widetilde{G}(p) = 1, \tag{2}$$

it follows that when $i\hat{p} \to m_e$ and the density of the medium tends to zero we have

$$m = m_e - \widetilde{M}_v(p_0)/Z_1, \qquad Z_1 = 1 - \partial \widetilde{M}_v(p_0)/\partial i\hat{p}_0, \tag{3}$$

where the subscript "v" indicates that the corresponding quantity pertains to the vacuum (no medium). From (3) and (2) follows an equation for the renormalized $\widetilde{G}$:

$$(i\hat{p} + m_e + M^R(p))\widetilde{G}(p) = 1, \tag{4}$$

$$M^R(p) = \widetilde{M}(p) - \widetilde{M}_v(p_0) - (i\hat{p} + m_e)\,\partial\widetilde{M}_v(p_0)/\partial i\hat{p}_0, \tag{5}$$

$$\widetilde{M}(p) = -\frac{ie_e^2}{(2\pi)^4}\int Z_1\gamma_\mu \widetilde{G}(p+k)\,\widetilde{\Gamma}_v(p+k, k)\,\widetilde{D}_{\mu\nu}(k)\,d^4k; \tag{5a}$$

Z_3 is obtained from the requirement that the Green's function of the photon have a pole as $k^2 \to 0$ and the chemical potential $\to 0$. We do not write out the remaining equations.

2. Let us consider the low-frequency corrections to the low-frequency part of the losses. In first order in e^2 it is known that an account of the recoil [9] makes a negligibly small contribution $\sim \omega/\epsilon_p$. Let us also neglect terms with ω/ϵ_p in the next order of perturbation theory in e^2. We can obtain simple rules for the construction of the Green's functions of the particle in any order of perturbation theory, if we neglect in each perturbation-theory order the terms of order ω/ϵ_p compared with unity, i.e., if we neglect, roughly speaking, the recoil in all orders of perturbation theory.

Let us consider the equation for the non-renormalized $G(E, \mathbf{p})$, $p_\mu = \{\mathbf{p}, iE\}$:

$$(i\hat{p} + m + M)\,G(E, \mathbf{p}) = 1. \tag{6}$$

In the construction by iteration of the mass operator M zero-order Green's functions

$$G_0\Big(E - \sum_j \omega_j,\ p - \sum_j k_j\Big)$$

will appear. In the limit $\omega/\epsilon_p \ll 1$ we can neglect in G_0 the positron [8] part:

$$G_0(E, \mathbf{p}) = -\Lambda_p^+ (E - \varepsilon_p + i\delta)^{-1}, \quad \Lambda_p^\pm = (m \mp i\hat{p}_0)/2\varepsilon_p,$$

$$p_{\mu 0} = \{\mathbf{p}, i\varepsilon_p\}, \tag{7}$$

and consequently also the vacuum polarization in the D-function. Therefore the D-function is determined only by the dielectric constants ϵ^t and ϵ^l of the medium (see [4]). In addition, $\Lambda^+_{p-\Sigma k_j}$ can be replaced by Λ_p^+. As a result, in each term of the perturbation series for M there appear factors of the type $\gamma_i\Lambda_p^+\gamma_k\Lambda_p^+\gamma_s\Lambda_p^+\gamma_j\ldots$ They can be rewritten in a form that contains Λ_p^- to the left, and all the $\gamma_i\gamma_k\ldots$ to the right. This expression for M can be substituted into the expansion for G:

$$G = G_0 - G_0MG_0 + G_0MG_0MG_0 - \ldots, \tag{8}$$

recognizing that G_0 is specified by (7). Then the terms containing Λ_p^- will vanish by virtue of $\Lambda_p^+\Lambda_p^- = 0$, and the terms containing γ_i will enter in the form $\Lambda_p^+\gamma_i\Lambda_p^+ = -iv_i\Lambda_p^+$, where $\mathbf{v} = \mathbf{p}/\epsilon_p$ is the velocity of the particle. Thus, the entire dependence on the matrices γ_μ of the function G is determined by the factor Λ_p^+:

$$G(E, \mathbf{p}) = \Lambda_p^+ g(E, \mathbf{p}). \tag{9}$$

Substituting (9) in (6) and taking the trace of the γ-matrices, we obtain

$$[E - \varepsilon_p - \Sigma(E, \mathbf{p})]\,g(E, \mathbf{p}) = 1, \tag{10}$$

where

$$\Sigma = \tfrac{1}{2}\,\mathrm{Sp}\,M\Lambda_p^+. \tag{11}$$

Calculating the trace in accordance with the well known rules [5] and using the dispersion relations for the D-functions, which we express in terms of the imaginary parts of the retarded D, directly connected with the dielectric constants ϵ^l and ϵ^t, we obtain

$$\Sigma(p) = e^2\int_0^\infty d\omega \int d\mathbf{k}\, g(p+k)\,\gamma(p+k, k)\,d_0(k), \tag{12}$$

$$\gamma(p, k) = 1 + e^2\int_0^\infty d\omega' \int d\mathbf{k}' g(p+k')$$
$$\times\, \gamma(p+k', k)\, g(p+k'-k)$$
$$\times\, \gamma(p+k'-k, k')\, d_0(k') + \delta\gamma, \tag{13}$$

[7]The medium is assumed homogeneous and isotropic (translational symmetry exists).

[8]For the sake of simplicity we assume that we are considering a Ψ-field particle with $E > 0$.

$$d_0(\omega, \mathbf{k}) = \frac{2}{(2\pi)^4}\left\{\frac{[\mathbf{vk}]^2}{\mathbf{k}^2}\operatorname{Im} D_r^t(\omega, \mathbf{k}) - \operatorname{Im} D_r^l(\omega, \mathbf{k})\right\}, \quad (14)$$

$$D_r^t(\omega, \mathbf{k}) = \frac{4\pi}{\mathbf{k}^2 - \omega^2\varepsilon^t(\omega, \mathbf{k})}, \quad D_r^l(\omega, \mathbf{k}) = \frac{4\pi}{\mathbf{k}^2\varepsilon^l(\omega, \mathbf{k})},$$

$$k_\mu = \{\mathbf{k}, i\omega\}. \quad (15)$$

We use here the gauge obtained in our previous work [4].

Expansion of the mass operator $\epsilon(E, \mathbf{p})$ in powers of e^2 can be determined by the usual graphic method using the rules previously formulated [7]. Thus, in first order in e^2

$$\Sigma_1(E, \mathbf{p}) = \frac{2e^2}{(2\pi)^4}\int_0^\infty d\omega \int d\mathbf{k}\, \frac{1}{E - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega}$$

$$\times \left\{\frac{[\mathbf{vk}]^2}{\mathbf{k}^2}\operatorname{Im} D_r^t(\omega, \mathbf{k}) - \operatorname{Im} D_r^l(\omega, \mathbf{k})\right\}, \quad (16)^*$$

we obtain ($E = \epsilon_p + i\delta$, $\epsilon_p - \epsilon_{p-k} \approx \mathbf{k}\cdot\mathbf{v}$) for the real part the expression of [4] for Δm and the classical expression for the imaginary part (see [10]).

3. Let us consider the question of renormalization for $g(E, \mathbf{p})$. The equation for the renormalized $\tilde{g}(E, \mathbf{p})$ can be obtained in analogous fashion by using Eq. (14) for the renormalized $\tilde{G}(p)$. It is simpler, however, (with the same result) to carry out the renormalization directly in (10) [9]:

$$(E - \varepsilon_{\mathbf{p}} - \Sigma^R(E, \mathbf{p}))\,\tilde{g}(E, \mathbf{p}) = 1, \quad (17)$$

$$\Sigma^R(E, \mathbf{p}) = \tilde{\Sigma}(E, \mathbf{p}) - \tilde{\Sigma}_{\text{B}}(E, \mathbf{p})|_{E=\varepsilon_{\mathbf{p}}}$$

$$- (E - \varepsilon_{\mathbf{p}})\left.\frac{\partial \tilde{\Sigma}_{\text{B}}(E, \mathbf{p})}{\partial E}\right|_{E=\varepsilon_{\mathbf{p}}}, \quad (18)$$

$$\tilde{\Sigma}(E, \mathbf{p}) = e_e^2 \int_0^\infty d\omega \int d\mathbf{k}\, \tilde{g}(p+k)\,\tilde{\gamma}(p+k, k)\, d_0(k)\, Z_1, \quad (19)$$

$$\tilde{\gamma}(p, k) = Z_1 + e_e^2\int_0^\infty d\omega' \int d\mathbf{k}' Z_1 \tilde{g}(p+k')\,\tilde{\gamma}(p+k', k)$$

$$\times \tilde{g}(p+k'-k)\,\tilde{\gamma}(p+k'-k, k')\, d_0(k') + \delta\gamma. \quad (20)$$

The factor

$$Z_1 = 1 + \left.\frac{\partial \tilde{\Sigma}_{\text{B}}(E, \mathbf{p})}{\partial E}\right|_{E=\varepsilon_{\mathbf{p}}} \quad (21)$$

serves for the elimination of the overlapping divergences (as was shown by Fradkin [8]). As was noted in the introduction, to eliminate the infrared divergence it is essential to have $E \neq \epsilon_p$ in the medium. However, in the renormalization the vacuum values of Σ and $\partial\Sigma/\partial E$ for $E = \epsilon_p$ are subtracted from $\tilde{\Sigma}$. Let us see how to eliminate this difficulty.

We introduce for this purpose the photon mass λ and expand successively all the quantities in (17)—(21) in series in terms of e^2. All the terms of the series are finite, in view of the presence of λ. We can then verify that all the terms of the series contain only $d_0 - d_{0V}$. (It is simplest to verify this directly, for example for the terms e^4 and e^6.) The same series can be obtained from the non-renormalized equation (10), if we use $d_0 - d_{0V}$ in Σ in place of d_0. Actually this reduces to replacing D_r^t by $D_r^t - D_{r,V}^t$, where $D_{r,V}^t = D_r^t|_{\epsilon^t=1}$, since $\operatorname{Im} D_{r,V}^l = 0$. [10] Thus, all the corrections vanish when $\epsilon^t = \epsilon^l = 1$. The result is easy to understand. If we neglect recoil in all the orders, there is no effect in the vacuum. Our problem is to solve Eq. (10) with $d_0 = d_{0V}$ (henceforth, unless specially stipulated, we shall take d_0 to mean the indicated difference) by a method different from the series expansion employed here.

4. If we do not assume $E = \epsilon_p$, then the term of the mass operator Σ_2, proportional to e^4, does not contain an infrared divergence:

$$\Sigma_2(E, \mathbf{p}) = e^4\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d\mathbf{k}_1\, d\mathbf{k}_2\, d_0(\omega_1, \mathbf{k}_1)\, d_0(\omega_2, \mathbf{k}_2)$$

$$\times (E - \varepsilon_{\mathbf{p}-\mathbf{k}_1} - \omega_1)^{-1}(E - \varepsilon_{\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2} - \omega_1 - \omega_2)^{-1}$$

$$\times [(E - \varepsilon_{\mathbf{p}-\mathbf{k}_2} - \omega_2)^{-1} + (E - \varepsilon_{\mathbf{p}-\mathbf{k}_1} - \omega_1)^{-1}]. \quad (22)$$

Therefore the correct solution of Eq. (10) with account, say, of only Σ_1 and Σ_2 does not contain any difficulties of the infrared divergence type. For a practical solution of the dispersion equation it is convenient to use the iteration method. For the zeroth approximation we have $E = \epsilon_p$. We substitute this value into the mass operator which contains only the first term of the expansion in e^2, namely

$$\Sigma_1(E, \mathbf{p}) \approx \Sigma_1(\varepsilon_{\mathbf{p}}, \mathbf{p}),$$

as a result of which we obtain the first approximation

$$E_1 = \varepsilon_{\mathbf{p}} + \Sigma_1(\varepsilon_{\mathbf{p}}, \mathbf{p}).$$

To obtain the next iteration approximation we substitute E_1 in

$$\Sigma_1(E, \mathbf{p}) - \Sigma_1(\varepsilon_{\mathbf{p}}, \mathbf{p}) + \Sigma_2(E, \mathbf{p})$$

[9] Here ϵ_p contains m_e.

*$[\mathbf{vk}] = \mathbf{v} \times \mathbf{k}$.

[10] It must be noted that the dispersion relations are generally speaking true for $D_r^l - D_{r,V}^l$ (convergence on the large semi-circle is essential). However, by virtue of renormalization only this difference enters into the result.

and carry out the calculations in the first nonvanishing approximation. We obtain terms of the type e^4 and $e^4 \ln e^2$. The obtained value of E_2 is substituted into the difference of the mass operators of the preceding orders and the mass operator of the next order is evaluated, etc. The use of such a procedure gives an expansion of the effective energy spectrum of the particles in the medium in terms of e^2, with the expansion coefficients containing $\ln e^2$. The terms of the series decrease successively if the density of the medium is not very small (see below), i.e., if the polarization effect is sufficiently large.

We introduce

$$\frac{\delta\Sigma_1(E, \mathbf{p})}{\delta E} = \frac{\Sigma_1(E, \mathbf{p}) - \Sigma_1(\varepsilon_p, \mathbf{p})}{E - \varepsilon_p}. \quad (23)$$

We then obtain from (16), (22), and (23) for $\gamma_p = -2\,\mathrm{Im}E(\mathbf{p})$

$$\gamma_p = \gamma_p^{(1)}\left(1 - \mathrm{Re}\,\frac{\delta\Sigma_1}{\delta E} - \mathrm{Re}\frac{\partial\Sigma_1}{\partial E}\right) - \left(\mathrm{Im}\frac{\delta\Sigma_1}{\delta E} - \mathrm{Im}\frac{\partial\Sigma_1}{\partial E}\right)\mathrm{Re}(E_1 - \varepsilon_p), \quad (24)$$

where γ_p is the probability of energy loss with account of the e^4 terms, while $\gamma_p^{(1)}$ is the value in the e^2 approximation only. It can be shown that the imaginary parts of $\delta\Sigma_1/\delta E$ and $\partial\Sigma_1/\partial E$ coincide within the limits of the required accuracy, i.e., their difference is proportional to e^4:

$$\gamma_p = \gamma_p^{(1)}\left(1 - \frac{e^2}{\pi}\Delta\right), \quad (25)$$

where Δ describes the radiative correction.

5. The same factor describes the macroscopic part of the radiative corrections to short-range collisions. To prove this we write out the exact expressions for the mass operators of first and second order, M_1 and M_2, and solve the exact dispersion equation by perturbation theory, assuming a low frequency for one of the quanta and a high one for the other ($\omega \sim \epsilon_p$). Then, neglecting the second approximation of perturbation theory for M_1, which contains the matrix element of the transition to states with negative energy and is therefore small, we obtain

$$E_2 = E_1 + \frac{4e^4}{(2\pi)^4}\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d\mathbf{k}_1 \int d\mathbf{k}_2 D''_{\mu\nu,r}(\omega_1, \mathbf{k}_1) D''_{\lambda\sigma,r}(\omega_2, \mathbf{k}_2)(E - \omega_1 - \varepsilon_{p-k_1})^{-2}\{(E - \omega_2 - \varepsilon_{p-k_2})^{-1}$$
$$\times \langle\gamma_\mu\Lambda_p^+\gamma_\sigma\Lambda_{p-k_2}^+\gamma_\lambda\Lambda_p^+\gamma_\nu\rangle + (E + \omega_2 - \varepsilon_{p-k_2})^{-1} \times \langle\gamma_\mu\Lambda_p^+\gamma_\sigma\Lambda_{p-k_2}^-\gamma_\lambda\Lambda_p^+\gamma_\nu\rangle\} + (E_1 - \varepsilon_p)\langle\partial M_1/\partial E\rangle;$$
$$\Lambda_p^\pm = (m - i\hat{p}^\pm)/2\varepsilon_p, \qquad p_\mu^\pm = \{\mathbf{p}, \pm i\varepsilon_p\}. \quad (26)$$

Here D''_r is the imaginary part of the retarded Green's function and $\langle\ldots\rangle$ denotes averaging over the solutions of the free equation. Using the relations

$$\langle\gamma_i\Lambda_p^+\gamma_\sigma\Lambda_{p-k_2}^\pm\gamma_\lambda\Lambda_p^+\gamma_j\rangle = -v_i v_j\langle\gamma_\sigma\Lambda_{p-k_2}^\pm\gamma_\lambda\rangle, \quad (27)$$

$$\langle\gamma_4\Lambda_p^+\gamma_\sigma\Lambda_{p-k_2}^\pm\gamma_\lambda\Lambda_p^+\gamma_4\rangle = \langle\gamma_\sigma\Lambda_{p-k_2}^\pm\gamma_\lambda\rangle, \quad i, j = 1, 2, 3, \quad (28)$$

we obtain for the imaginary part of E_2 relation (25), in which $E_1 = \epsilon_p$. Expression (25) is obtained by perturbation theory using unperturbed functions that take into account the mass operator in the e^2 approximation (they were obtained in [4]).

6. We assume that the medium is transparent and neglect spatial dispersion in the corrections. [11] The quantity Δ can be resolved into transverse and longitudinal parts Δ^t and Δ^l (corresponding to the D^t and D^l terms in Σ_1):

$$\Delta = \Delta^t + \Delta^l. \quad (29)$$

Then $\Delta^l = 0$ under the assumptions made. Indeed, from

$$\mathrm{Im}\,D_r^l = -\frac{4\pi^2}{\mathbf{k}^2}\sum_s\left|\frac{\partial n^2(\omega_s)}{\partial\omega_s}\right|^{-1}\delta(\omega - \omega_s), \quad (30)$$

where ω_s are the zeroes of the refractive index $n(\omega)$, it follows that

$$\frac{\delta\Sigma_1^l}{\delta E} = \frac{\partial\Sigma_1^l}{\partial E} = \frac{ie^2}{v}\sum_s\omega_s^{-1}\left|\frac{\partial n^2(\omega_s)}{\partial\omega_s}\right|^{-1}. \quad (31)$$

The fact that (31) is imaginary proves our statement. Further, from the equation

$$\mathrm{Im}\,D_r^t = 4\pi^2\delta(k^2 - \omega^2 n^2(\omega)) \quad (32)$$

it follows that

$$\Delta = \Delta^t = \mathrm{Re}\int_{n^2>0}^{\infty}\Delta_\omega^t\,d\omega - \mathrm{Re}\int_0^\infty\Delta_{\omega,\mathrm{B}}^t\,d\omega, \quad \Delta_{\omega,\mathrm{B}}^t = \Delta_\mathrm{B}^t|_{n=1}, \quad (33)$$

$$\Delta_\omega^t = \frac{v}{2(E_1 - \varepsilon_p)}\left(1 - \frac{1}{n^2v^2}\right)\ln\frac{\omega - (E_1 - \varepsilon_p)(1 - nv)^{-1}}{\omega - (E_1 - \varepsilon_p)(1 + nv)^{-1}} + \frac{3}{\omega n} + \frac{4\omega - 3(E_1 - \varepsilon_p)}{2\omega^2n^2v^2}\ln\frac{E_1 - \varepsilon_p - \omega(1 - nv)}{E_1 - \varepsilon_p - \omega(1 + nv)}. \quad (34)$$

3. EFFECT OF DENSITY IN THE RADIATIVE CORRECTIONS

1. It is well known that a density effect is expected for the energy losses calculated in the e^2

[11] For ultrarelativistic particles, for example, this is precisely the region of frequencies and wave vectors that makes the greatest contribution to the corrections, as will be shown later. We emphasize that the approximation employed in the corrections can be useful for a strongly absorbing medium, where the absorption must be taken into account in $\gamma_p^{(1)}$ but not in Δ. This is connected with the fact that the regions of frequencies that are essential for $\gamma_p^{(1)}$ and Δ can be different.

approximation, namely that the losses should exhibit a plateau. For the oscillator model

$$n^2(\omega) = 1 - \omega_0^2 / (\omega^2 - \omega_s^2), \qquad (35)$$

which was calculated by Fermi [1], the criterion for the onset of the density effect is

$$\varepsilon_p/m \gg \xi \sqrt{\omega_0^2 + \omega_s^2}/\omega_0, \qquad (36)$$

where $\xi \sim 1$. With account of the radiative corrections, the energy losses have the form

$$w = w^{(1)}(1 - e^2\Delta/\pi), \qquad (37)$$

where $w^{(1)}$ are the losses in the e^2 approximation.

At large energies saturation, the value of which depends on the density of the medium, also sets in for Δ (the density effect). The criterion for the appearance of the density effect in radiative corrections is more stringent, namely that ξ in (36) must greatly exceed unity in this case (ξ is proportional to e^{-2} = .137). Actually it turns out that $\xi \sim 10 - 40$. This means that after "reaching" the Fermi plateau, a decrease in the losses should be observed. Inasmuch as Δ can reach relatively large numbers—up to 50 [12)], the radiative correction can reach 5—10%.

2. For the simplest oscillator model, described by formula (35), the integral expressing the quantity Δ cannot be calculated in terms of known tabulated functions. Only the limiting cases $\omega_s \ll \omega_0$ and $\omega_s \gg \omega_0$ can be considered. In the former case one can put in the limit $\omega_s = 0$. We then obtain the result of [8]. The value of Δ as a function of ϵ_p/m is plotted in the figure for two values of $|\lambda| = \omega_0^{-1}|E_1 - \epsilon_p|$, namely $|\lambda| = 1/10$ and $|\lambda| = 1/50$. In the limiting case of nonrelativistic energies Δ^t is small, $\Delta^t \sim v^2$ (the square of the ratio of the particle through the velocity of light):

$$\Delta^t = \frac{2}{3} v^2 \ln \frac{4}{|\lambda|^2}. \qquad (38)$$

In the untrarelativistic limit when $\epsilon_p/m \ll 1/|\lambda|$ we have

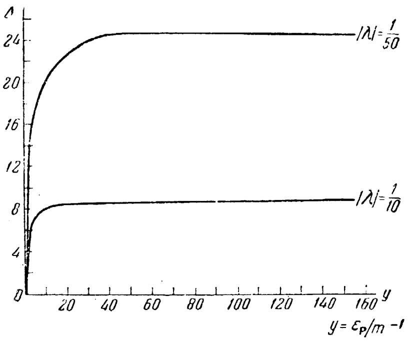
$$\Delta^t(\varepsilon_p) = 2\ln^2\frac{\varepsilon_p}{m} + 2\left(\ln\frac{2\varepsilon_p}{m} - 1\right)\left(\ln\frac{m^2}{\varepsilon_p^2|\lambda|^2} - 1\right) + 2\ln 2(2 - \ln 2) - \frac{\pi^2}{6}, \qquad (39)$$

and when

$$\varepsilon_p/m \gg 1/|\lambda|, \qquad (40)$$

$$\Delta^t(\varepsilon_p) = \frac{7}{2} - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{1}{2{,}72|\lambda|^2}. \qquad (41)$$

[12)]This expression is suitable only if the radiative correction $e^2\Delta/\pi$ is itself small, for in the opposite case it is necessary to sum perturbation-theory series.



In this case the criterion for the onset of the density effect in the radiative corrections is Eq. (40). If $\omega_s > \omega_0$, Δ^t can be expressed in the nonrelativistic limit in terms of elliptic integrals. In the limit $\omega_s \gg \omega_0$ and $v \ll 1$ we have

$$\Delta^t = \frac{4}{3} v^2 \left\{2\ln\frac{\omega_s^2}{\omega_0^2|\lambda|} - 1\right\}. \qquad (42)$$

In the ultrarelativistic limit when $\omega_s \gg \omega_0$ the greatest interest is attached to the dependence of the radiative corrections on the energy in the region of the "Fermi plateau," $\epsilon_p/m \gg \omega_s/\omega_0$.[13)] When the last inequality is satisfied, it is possible to carry out calculation with logarithmic accuracy, discarding terms of order of unity, compared with the large logarithm. The characteristic logarithm is in this case $\ln(1/|\lambda|)$. It reaches values on the order of 5—6 (see below). This means that for the radiative corrections, which reach an order of 8—10% in the vicinity of the "plateau," the result is accurate to 2—3%. This, for example, is sufficient for a comparison of the theory with the experiment. It is important that in calculating with logarithmic accuracy it is possible to obtain a result for any $n(\omega)$, since the main contribution to the integral is made by frequencies for which n is close to unity, and consequently $n \sim 1 - \omega_0^2/2\omega^2$, and the region of angles of greatest importance in the integration is $x = \cos\theta \sim 1$. Using this fact, we can replace the exact expression for Δ by the approximate expression

$$\Delta = v^2 \operatorname{Re}\int_0^\infty dy \int^1 dx\,(1 - x)\,\{4y^3\,[2y^2(1 - v + 1 - x) + 1]^{-1}\,[2\lambda y - 2y^2\,(1 - v + 1 - x) - 1]^{-1} - 4y^2\,[2\lambda y - (1 - v + 1 - x)\,2y^2 - 1]^{-2} - (1 - v + 1 - x)^{-1}\,[\lambda - (1 - v + 1 - x)\,y]^{-1} + y\,[\lambda - (1 - v + 1 - x)\,y]^{-2}\}, \qquad (43)$$

[13)]The last inequality can also be obtained by a simple estimate. The characteristic frequencies correspond to the fact that $1 - nv$ still differs from $1 - v$. For $n \sim 1 - \omega_0^2/2\omega^2$ we obtain $\omega \sim \omega_0\epsilon_p/m$. In conjunction with $\omega \gg \omega_s$ this yields $\epsilon_p \gg m\omega_s/\omega_0$.

where $y = \omega/\omega_0$ and the values of the lower limits are inessential, inasmuch as the main contribution is made by integration near the upper limits. In the limit $\epsilon_p/m \ll 1/|\lambda|$ we obtain

$$\Delta(\varepsilon_p) = 2\ln\frac{\varepsilon_p}{m}\ln\frac{m}{\varepsilon_p|\lambda|^2}, \tag{44}$$

which coincides with (39) with logarithmic accuracy. In the limit $\epsilon_p/m \gg 1/|\lambda|$ we have

$$\Delta(\varepsilon_p) = 2\ln^2\frac{1}{|\lambda|}, \tag{45}$$

which coincides with (41) with logarithmic accuracy. The dependence on the specific characteristics of the medium is only via $|\lambda|$.

3. The value of $|\lambda|$ is determined by the effective energy spectrum of the particle in the medium in the first order in e^2. The real part of λ (see the introduction) is small in the region of the Fermi plateau $\epsilon_p/m \gg \omega_s/\omega_0$. Thus (see [11]),

$$|\lambda| = \frac{e^2}{\omega_0(2\pi)^3}\int_0^\infty d\omega\int d\mathbf{k}\,\delta(\omega - \varepsilon_p + \varepsilon_{p-k})\Big[2\,\mathrm{Im}\,D_r^t(\omega,\mathbf{k}) \times\left(1 - \frac{m^2 - (\mathbf{pk}) + (\mathbf{pk})^2/k^2}{\varepsilon_p\varepsilon_{p-k}}\right) - \mathrm{Im}\,D_r^l(\omega,\mathbf{k})\left(1 + \frac{\varepsilon_p^2 - (\mathbf{pk})}{\varepsilon_p\varepsilon_{p-k}}\right)\Big]. \tag{46}$$

For a transparent medium, neglecting the spatial dispersion, we obtain in the case of the oscillator model (35)

$$|\lambda| = |\lambda^t| + |\lambda^l|; \quad |\lambda^l| = \frac{e^2\omega_0}{v\sqrt{\omega_0^2+\omega_s^2}}\ln\frac{k_{max}v}{\sqrt{\omega_0^2+\omega_s^2}}, \tag{47}$$

$$|\lambda^t| = e^2 v\left\{\left(1 - \frac{1}{v^2}\right)\frac{\omega_s}{\omega_0} + \frac{\omega_0}{v^2\sqrt{\omega_0^2+\omega_s^2}}\ln\frac{\sqrt{\omega_s^2+\omega_0^2}+\omega_s}{\omega_0}\right\}, \tag{48}$$

and when $\omega_s \gg \omega_0$ and $v \to 1$ we get

$$|\lambda| = e^2\frac{\omega_0}{\omega_s}\ln\frac{2k_{max}}{\omega_0}. \tag{49}$$

In the case of an arbitrary number of oscillators we have

$$\varepsilon(\omega) = 1 + \sum_i \frac{\omega_0^2 f_i}{\omega_{si}^2 - \omega^2},$$

and if the different natural frequencies are sufficiently well separated, then

$$|\lambda| = e^2\frac{\omega_0}{\overline{\omega}_s}\ln\frac{2k_{max}}{\omega_0}, \quad \frac{1}{\overline{\omega}_s} = \sum_i\frac{f_i}{\omega_{si}}. \tag{50}$$

Thus, $|\lambda|$ decreases with increasing ω_s/ω_0, while $\ln(1/|\lambda|)$ increases.

The parameter k_{max} in (47)—(50) is chosen to permit the neglect of spatial dispersion. However, an analysis of (46) shows that the main contribution to (46) is made by relatively small k, unlike the analogous calculation for energy losses given in [11].[14)] To find k_{max} it is essential to take into account spatial dispersion, a factor which is important in the classical region. By way of an example we can cite the calculated $|\lambda|$ for a degenerate electron gas. An account of spatial dispersion for the classical ϵ^l and ϵ^t (see [10]) yields for ultrarelativistic velocities

$$|\lambda| = e^2\left(\ln\frac{v}{v_0} + \frac{\pi\sqrt{3}}{8} + \sqrt{6\pi}\right) = e^2\ln\frac{0.93v}{d\omega_0}. \tag{51}$$

Comparing with (47), we obtain $k_{max} = 0.93/d$ for $\omega_s = 0$, where $d = v_0/\sqrt{3}\omega_0$ is the Debye radius and v_0 is the particle velocity on the Fermi surface. Inasmuch as k_{max} enters under the logarithm sign, we can approximately assume that in the general case $k_{max} = \langle\omega\rangle/v_{av}$, where v_{av} is some mean velocity of the electrons of the medium.

4. Let us estimate the influence of bremsstrahlung on the radiative corrections. Bremsstrahlung, as is well known, is influenced by multiple scattering (see the paper of Landau and Pomeranchuk [12]) and by the polarization of the medium (see Ter-Mikaelyan [13]). The most detailed investigation of bremsstrahlung with account of both effects was made by Migdal [14]. From [14] we find that it is necessary to add to the calculated value $|\lambda|$ the term

$$|\lambda^{\mathrm{T}}| = \frac{1}{\omega_0}\int_0^{\varepsilon_p}\frac{d\omega}{\omega L(s)}\frac{\Phi(s)}{(1+\varepsilon_p^2\omega_0^2/m^2\omega^2)}\left[\frac{1}{3}\frac{\omega^2}{\varepsilon_p^2}G(s) + \frac{2}{3}\left(1 + \left(1 - \frac{\omega}{\varepsilon_p}\right)^2\right)\right], \tag{52}$$

where

$$\frac{1}{\omega_0 L(s)} = \frac{\omega_0 Z e^4}{\pi m}\ln\frac{190}{Z^{1/3}s^{1/3}}, \quad s^2 = \frac{m^4 L(s)\,\omega}{8E_s^2\varepsilon_p(\varepsilon_p - \omega)}\left(1 + \frac{\varepsilon_p^2}{m^2}\frac{\omega_0^2}{\omega^2}\right)^2$$

and $E_s = m\sqrt{4\pi/e^2}$; $G(s)$ and $\Phi(s)$ are Migdal functions [14]. As estimate of $|\lambda^b|$ can be obtained by approximating (52) in various regions of ω: from zero to ω^*, from ω^* to ω^{**}, and from ω^{**} to ϵ_p, where

$$\omega^* = {}^1\!/_2\,\omega_0^{4/3}\varepsilon_p^{2/3}E_s^{-2/3}L^{1/3}; \quad \omega^{**} = 8E_s^2\varepsilon_p^2 m^{-4}L^{-1}.$$

The integrals $|\lambda^b|_1$, $|\lambda^b|_2$, and $|\lambda^b|_3$ over the first, second, and third regions respectively have the following estimates:

[14)] Qualitatively this follows from the fact that m is close to unity in the quantum region, so that the result should be proportional to n − 1 (or Imϵ), i.e., to ω_0^2; from dimensionality considerations $|\lambda|$ is proportional to $\omega_0 e^2/m$, which is small compared with (50).

$$|\lambda^b|_1 \approx \frac{Z\omega_0 e^4}{m}\left(\frac{\omega_0 L m^3}{\varepsilon_p E_s^2}\right)^{2/3} \ln\frac{190}{Z^{1/3}}, \quad |\lambda^b|_2 \approx \frac{16}{\omega_0 L},$$
$$|\lambda^b|_3 \approx \frac{4}{3\omega_0 L}\left(\ln\frac{m^4 L}{8E_s^2 \varepsilon_p} - \frac{5}{8}\right); \quad L = L(1). \tag{53}$$

The integral over the second region vanishes when $\epsilon_p/m < \omega_0 Lm^2/8E_s^2$. A comparison of $|\lambda^b|$ with the values of $|\lambda|$ calculated above shows that even for elements corresponding to the end of the periodic table $|\lambda^b|$ is approximately two orders of magnitude smaller than $|\lambda|$.

5. Let us estimate the influence of multiple scattering on the radiative corrections to the particle energy losses. For the main part of the losses, as shown by Ter-Mikaelyan [15], multiple scattering does not come into play, owing to the presence of the Fermi density effect. The density effect cuts off precisely those impact parameters for which multiple scattering can play any role. Let us show that multiple scattering does likewise not influence the radiative corrections to the losses.

The reason for it is the density effect considered here for the radiative corrections. The large values of the corrections come from the region where the following quantity is small [see (43)]:

$$\omega(1 - nvx) + \omega_0|\lambda| \approx \left(1 - v + \frac{\theta^2}{2} + \frac{\omega_0^2}{2\omega^2}\right)\omega + \omega_0|\lambda|,$$
$$x = \cos\theta. \tag{54}$$

Using multiple-scattering theory [16] we get

$$\langle\theta^2\rangle = E_s^2\, l/\varepsilon_p^2 L.$$

Let us assume that l is equal to the path on which the particle's own field is formed, $l \sim \epsilon_p^2/\omega m^2$. Taking into account the fact that $\omega \sim \omega_0\epsilon_p/m$, we find that $\langle\theta^2\rangle$ for multiple scattering becomes comparable with the angles θ^2 which are significant in (54) when

$$\frac{\varepsilon_p}{m} > \frac{\varepsilon_p^{sc}}{m} = \frac{1}{4Ze^2}\frac{m}{\omega_0}\Big/\ln\frac{190}{Z^{1/3}}. \tag{55}$$

This quantity exceeds appreciably the value at which the density effect sets in for the radiative losses. For example, for electrons the criterion (59) has the form $\epsilon_p \gtrsim 10$ Bev, whereas the density effect for the radiative corrections occurs when $\epsilon_p^{d.e} \sim 200$ Mev. Therefore multiple scattering does not influence the radiative corrections [15)].

[15)]The estimate can also be obtained with logarithmic accuracy by calculation, if, following [12], we replace $\langle f(\theta^2)\rangle$ by $f(\langle\theta^2\rangle)$. The corrections for multiple scattering reach a maximum when $\epsilon_p = (\epsilon_p^{sc}\epsilon_p^{d.e})^{1/2}$. Then their contribution to Δ reaches a value on the order of unity, which should be discarded in calculations with logarithmic accuracy. Here $\epsilon_p^{d.e}$ is the value of the energies at which the density effect sets in in radiative corrections.

4. DISCUSSION OF RESULTS

1. Ionization losses are frequently investigated experimentally in photographic plates, where the main substance is AgBr. For AgBr it is possible to estimate the value of $1/\overline{\omega}_s$ from the known ionization potentials. We get $\omega_0/\overline{\omega}_s \sim 0.2$—$0.3$. Therefore the drop in the ionization-loss curve should occur for electrons at $\epsilon_p/m \sim 200$. The radiative corrections after saturation is reached should be ~ 5—10%.

It turns out that the predicted drop in the ionization losses for ultrarelativistic electrons has already been observed experimentally (see [17]). Comparison [18] of the experimental data with the theoretical results developed above shows that agreement exists between theory and experiment within the limits of experimental error and the accuracy of the theoretical calculations. We note here that agreement exists in the values of the corrections and in the values of the energies for which saturation in the corrections sets in. Both quantities are determined by the single parameter $|\lambda|$. The theoretical value of $|\lambda|$ can be obtained from a model, the choice of which is always connected with a certain approximation. The model of several oscillators or of a group of non-overlapping absorption bands (due to the transition of the electron from the bound state to the continuum [19]) gives a value of λ which is in good agreement with the experimental results. If the absolute value of the radiative correction increases, the characteristic saturation energy of the radiative corrections should also increase. The experimental material is so far insufficient for a check on this statement.

2. The macroscopic part of the radiative corrections depends only on ϵ_p/m, and for heavy particles, say mesons, we can predict theoretically an ionization vs. momentum curve analogous to that observed for electrons. An estimate of the "micropart" of the radiative corrections from known formulas [5] shows them to be of the order of $e^2\omega_{max}/2\pi m_e$ and they are consequently always small if the experimental apparatus registers only events with energy transfers ω_{max} small compared with the electron rest energy m_e. Corrections of the type considered here should be taken into account in interpretation of data obtained with bubble and cloud chambers.

3. To verify the calculated radiative corrections to the intensity of the Cerenkov radiation it is necessary to measure rather accurately the spectral intensity of the emitted Cerenkov radiation.

The literature data on the observed deviations of the intensity of the Cerenkov radiation from the classical Tamm-Frank formula [20] were apparently in error [21].

4. We summarize: 1) the radiative corrections to the energy losses are far from small ($\sim 5-10\%$); 2) a density effect exists for the corrections and sets in at energies larger than those for which the density effect in the main part of the losses is reached; 3) the theory of radiative corrections to the ionization losses is confirmed by experiment, and 4) the radiative corrections to the Cerenkov radiation can be experimentally observed.

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