ON ASYMPTOTIC REPRESENTATIONS OF THE GREEN'S FUNCTIONS AND VERTEX FUNCTION IN QUANTUM ELECTRODYNAMICS

P. I. FOMIN

Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R.

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Asymptotic representations of the type of those of Gell-Mann and Low[2] are obtained for the electron Green's function and the vertex function as regularized according to Dyson, with the dependence on the fictitious photon mass λ_0 taken into account. The derivation is based on the renormalizability of quantum electrodynamics and the existence of limits of the unrenormalized functions for zero values of the electron mass and the fictitious photon mass λ_0. Consequences of the asymptotic representations which follow from the assumption that the functions can be expanded in power series in e^2 are considered.

1. The asymptotic properties of the Green's functions and the vertex function in the high-momentum region have been studied in a number of papers.[1-7]

Gell-Mann and Low have obtained the following asymptotic representations for the photon and electron Green's functions as regularized according to Dyson:

$$d(k^2/m^2, e^2) = e^{-2}F(\psi(e^2)k^2/m^2), \qquad |k^2| \gg m^2, \quad (1)$$

$$s(p^2/m^2, e^2) = A(e^2)H(\psi(e^2)p^2/m^2), \qquad |p^2| \gg m^2, \quad (2)$$

where F, ψ, A, H are unknown functions. In this work, however, no account was taken of the fact that both the electron Green's function as regularized according to Dyson and the vertex function have infrared divergences. If we proceed in the usual way to remove the divergence by ascribing a "mass" λ_0 to the photon, then s will depend on λ_0, and therefore the problem arises of finding the asymptotic representation of s with the dependence on λ_0 taken into account.

It is also an interesting problem to find the analogous representation for the vertex function.

Both of these problems are solved in the present paper. We shall show that the following asymptotic representations hold for the regularized electron Green's function and vertex function[1]:

$$s(p, m, \lambda_0, e^2) = r\left(\frac{\lambda_0^2}{m^2}, e^2\right)H\left(\psi(e^2)\frac{p^2}{m^2}\right), \quad (3)$$

$$\Gamma(p, q, m, \lambda_0, e^2) = r^{-1}\left(\frac{\lambda_0^2}{m^2}, e^2\right)B\left(\psi(e^2)\frac{pq}{m^2}, \psi(e^2)\frac{p^2}{m^2}, \psi(e^2)\frac{q^2}{m^2}\right), \quad (4)$$

where r, H, B are unknown functions and ψ is connected with d by the relation (1).

We note that the results (3) and (4) cannot be obtained by direct use of the method of Gell-Mann and Low. In the present paper we develop a somewhat different method, which is of greater generality. We start from Dyson's relations (5)—(8), which express the renormalizability of quantum electrodynamics, and the fact that there exist finite limits of the unrenormalized functions for $m \to 0$ and $\lambda_0 \to 0$.

These assumptions are sufficient for the derivation of both the results (3) and (4) and the formula (1) for the transverse function d.

The great generality of these assumptions allows us to suppose that the asymptotic representations (1), (3), and (4) are valid both within and outside of the framework of perturbation theory. If we further assume that d, s, and Γ can be expanded in power series in e^2, then on the basis of Eqs. (1), (3), and (4) we can get more detailed information about these functions. For example, this postulate alone is enough to determine the asymptotic form of the function d up to a constant. Furthermore the actual expansion parameter is not e^2, but $e^2 \ln(k^2/m^2)$. This conclusion, which is usually obtained on the basis of concrete perturbation-theory calculations, is thus a consequence of

[1]Here and in what follows we mean by s and Γ respectively any of the scalar dimensionless functions in terms of which the complete electron Green's function and the complete vertex function can be expressed. The photon Green's function is chosen in purely transverse form, and the corresponding scalar function is denoted by d.

the extremely general assumptions indicated above.

What has been said also applies in equal degree to the function $H(\psi(e^2)p^2/m^2)$, which expresses the dependence of the function s on p^2. Much less detailed results can be obtained regarding the vertex function, because of the greater arbitrariness in the representation (4) as compared with Eqs. (1) and (3), which is due to the larger number of arguments in Γ.

2. Dyson's relations [8] connecting the regularized functions d, s, and Γ and the renormalized charge e with the corresponding unregularized functions d_0, s_0, and Γ_0 and the "bare" charge e_0 can be written in the following form

$$d_0(k, m, \Lambda, e_0^2) = Z_3(m, \Lambda, e^2)\, d(k^2/m^2, e^2), \quad (5)$$

$$s_0(p, m, \lambda_0, \Lambda, e_0^2) = Z(m, \lambda_0, \Lambda, e^2)\, s(p, m, \lambda_0, e^2) \quad (6)$$

$$\Gamma_0(p, q, m, \lambda_0, \Lambda, e_0^2) = Z^{-1}(m, \lambda_0, \Lambda, e^2)\, \Gamma(p, q, m, \lambda_0, e^2), \quad (7)$$

$$e_0^2 = e^2 Z_3^{-1}(m, \Lambda, e^2). \quad (8)$$

Here Λ is an invariant cutoff momentum; the absence of the argument Λ in the regularized functions expresses the fact that they remain finite for $\Lambda \to \infty$. The mass renormalization has been carried out in the functions d_0, s_0, and Γ_0, but they are not completely regularized and diverge for $\Lambda \to \infty$.

From Eqs. (5)—(8) there follow the relations:

$$d(k^2/m^2, e^2)/d(q^2/m^2, e^2) = d_0(k, m, \Lambda, e_0^2)/d_0(q, m, \Lambda, e_0^2), \quad (9)$$

$$\frac{s(p, m, \lambda_0, e^2)}{s_0(q, m, \lambda_0, e^2)} = \frac{s_0(p, m, \lambda_0, \Lambda, e_0^2)}{s_0(q, m, \lambda_0, \Lambda, e_0^2)}, \quad (10)$$

$$\frac{\Gamma(p, q, m, \lambda_0, e^2)}{\Gamma(p', q', m, \lambda_0, e^2)} = \frac{\Gamma_0(p, q, m, \lambda_0, \Lambda, e_0^2)}{\Gamma_0(p', q', m, \lambda_0, \Lambda, e_0^2)}, \quad (11)$$

$$s(k, m, \lambda_0, e^2)\, \Gamma(p, q, m, \lambda_0, e^2) = s_0(k, m, \lambda_0, \Lambda, e_0^2)\, \Gamma_0(p, q, m, \lambda_0, \Lambda, e_0^2), \quad (12)$$

$$e^2 d(k^2/m^2, e^2) = e_0^2 d_0(k, m, \Lambda, e_0^2). \quad (13)$$

As is easily verified, the integrals corresponding to the Feynman diagrams for d_0, s_0, and Γ_0 in any order in e_0^2 have limits both for $m \to 0$ and for $\lambda_0 \to 0$ (limits of s_0 and Γ_0 for $\lambda_0 \to 0$ exist only if $p^2 \neq m^2$, $q^2 \neq m^2$, and we shall assume these inequalities).

The quantities s_0 and Γ_0 will have finite limits for $\lambda_0 \to 0$ also if by the use of the relations (8) they are represented as functions of e^2 (instead of e_0^2), since Eq. (8) does not contain λ_0. From this one easily concludes that the left members of Eqs. (10), (11), and (12) also have finite limits for $\lambda_0 \to 0$, whereas it is known that the terms of the perturbation-theory series for s and Γ themselves do not have such limits. This means that the dependence of s and Γ on λ_0 must be of the form

$$s(p, m, \lambda_0, e^2) = r(\lambda_0^2/m^2, e^2)\, t(p^2/m^2, e^2), \quad (14)$$

$$\Gamma(p, q, m, \lambda_0, e^2) = \alpha\left(\frac{\lambda_0^2}{m^2}, e^2\right) \beta\left(\frac{pq}{m^2}, \frac{p^2}{m^2}, \frac{q^2}{m^2}, e^2\right) \quad (|p^2 - m^2|, \ |q^2 - m^2| \gg \lambda_0^2), \quad (15)$$

and also we can suppose that

$$r(\lambda_0^2/m^2, e^2)\, \alpha(\lambda_0^2/m^2, e^2) = 1. \quad (16)$$

Furthermore, the limits of the quantities d_0, s_0, and Γ_0 for $m \to 0$ will also exist if they are represented as functions of $e_\lambda^2 = e_0 d_0(\lambda, m, \Lambda, e_0^2)$ (instead of e_0^2), since in d_0 we can set $m = 0$.

On the other hand, by Eq. (13),

$$e_\lambda^2 = e^2 d(\lambda^2/m^2, e^2). \quad (17)$$

Therefore we can assert that also the left members of Eq. (9)—(12), when by means of Eq. (17) they are represented as functions of e_λ^2 instead of e^2, have limits for $m \to 0$.

We thus arrive at the following asymptotic functional equations in the high-momentum region (all scalars, k^2, p^2, pq, etc., large in comparison with m^2):

$$d\left(\frac{k^2}{m^2}, e^2\right) \Big/ d\left(\frac{q^2}{m^2}, e^2\right) = f\left(\frac{k^2}{\lambda^2}, \frac{q^2}{\lambda^2}, e_\lambda^2\right), \quad (18)$$

$$t\left(\frac{p^2}{m^2}, e^2\right) \Big/ t\left(\frac{q^2}{m^2}, e^2\right) = g\left(\frac{p^2}{\lambda^2}, \frac{q^2}{\lambda^2}, e_\lambda^2\right), \quad (19)$$

$$\frac{\beta(pq/m^2, p^2/m^2, q^2/m^2, e^2)}{\beta(p'q'/m^2, p'^2/m^2, q'^2/m^2, e^2)} = \gamma\left(\frac{pq}{\lambda^2}, \frac{p^2}{\lambda^2}, \frac{q^2}{\lambda^2}, \frac{p'q'}{\lambda^2}, \frac{p'^2}{\lambda^2}, \frac{q'^2}{\lambda^2}, e_\lambda^2\right), \quad (20)$$

$$t\left(\frac{k^2}{m^2}, e^2\right) \beta\left(\frac{pq}{m^2}, \frac{p^2}{m^2}, \frac{q^2}{m^2}, e^2\right) = h\left(\frac{pq}{\lambda^2}, \frac{p^2}{\lambda^2}, \frac{q^2}{\lambda^2}, \frac{k^2}{\lambda^2}, e_\lambda^2\right), \quad (21)$$

where certain unknown functions appear in the right members and the connection between e_λ^2 and e^2 is given by the relation (17).

The solutions of the equations (18) and (19) are of the forms [2]:

$$e^2 d(k^2/m^2, e^2) = F(\psi(e^2)\, k^2/m^2), \quad (22)$$

$$t(p^2/m^2, e^2) = A(e^2)\, H(\psi(e^2)\, p^2/m^2), \quad (23)$$

where F, ψ, A, H are arbitrary functions of one variable.

When Eqs. (22) and (23) are used it follows from Eqs. (20) and (21) that

$$\beta\left(\frac{pq}{m^2},\frac{p^2}{m^2},\frac{q^2}{m^2},e^2\right)=A^{-1}(e^2)\,B\left(\psi(e^2)\frac{pq}{m^2},\psi(e^2)\frac{p^2}{m^2},\psi(e^2)\frac{q^2}{m^2}\right),\quad(24)$$

where B is some function of three variables.

Using Eqs. (14)—(16), (23), and (24) and including $A(e^2)$ in the function r, we find that in the high-momentum region s and Γ have the structures (3) and (4).

3. In the derivation of the asymptotic representations (22), (3), and (4) perturbation theory is essentially used only in the proof of the initial relations (5)—(8), which express the renormalizability of quantum electrodynamics. If it turns out that renormalizability is not connected with the use of perturbation theory, then this will also be true of the results (22), (3), and (4).

If, on the other hand, we remain within the framework of perturbation theory and assume that the functions d, s, and Γ can be expanded in power series in e^2, then we can get additional information about them from Eqs. (22), (3), and (4).

Let us first consider the function d. We write Eq. (22) in the form

$$e^2d=\Phi(\varphi(e^2)+x),\qquad x=\ln(k^2/m^2).\quad(25)$$

It follows from Eq. (25) that

$$\frac{\partial}{\partial x}(e^2d)\Big/\frac{\partial}{\partial e^2}(e^2d)=1/\varphi'(e^2).\quad(26)$$

Substituting in Eq. (26) the expansions

$$d=\sum_{n=0}^{\infty}e^{2n}d_n(x),\quad(27)$$

$$[\varphi'(e^2)]^{-1}=\sum_{m=0}^{\infty}e^{2m}a_m,\quad(28)$$

we find that $a_0 = 0$ and

$$d'_n(x)=\sum_{m=0}^{n}(m+1)\,a_{n-m+1}d_m(x).\quad(29)$$

Using the fact that $d_0(x) = 1$ and taking $x \gg 1$, we find that $a_1 = 0$ and

$$d_n(x)=(a_2x)^n+O(x^{n-1}).\quad(30)$$

Thus for $\ln(k^2/m^2) \gg 1$ the actual expansion parameter in Eq. (27) is $e^2\ln(k^2/m^2)$.

This fact is usually derived from the results of concrete calculations made with perturbation theory. Here we have shown that it follows automatically from Eq. (22) if we make the additional assumption that the function d can be expanded in power series in e^2, and thus that it is connected with the renormalizability of quantum electrodynamics.[2)]

Substituting Eq. (30) in Eq. (27) and summing, we get

$$d(k^2/m^2,e^2)=[1-a_2e^2\ln(k^2/m^2)]^{-1}.\quad(31)$$

Comparison with the result of calculations in first approximation in e^2 gives $a_2 = (3\pi)^{-1}$, and we arrive at the well known formula first obtained by Landau, Abrikosov, and Khalatnikov.[1]

When we confine ourselves to the main term in Eq. (28) we easily find that

$$\ln\psi(e^2)\equiv\varphi(e^2)=-1/a_2e^2=-3\pi/e^2.\quad(32)$$

Let us treat the function s in an analogous way. For this purpose it is more convenient to use the following representation, which is equivalent to Eq. (3)

$$s(p,m,\lambda_0,e^2)=r'(\lambda_0^2/m^2,e^2)\,H(\psi(e^2)\,p^2/m^2)\,H^{-1}(\psi(e^2)).\quad(33)$$

It is easy to show, in analogy with Eq. (31) that with the assumption of expansibility in series and keeping only the leading powers of $\ln(p^2/m^2)$ in each order in e^2 we get the following result:

$$H\left(\psi(e^2)\frac{p^2}{m^2}\right)=c_0\left[1+ce^2\left(1-\frac{e^2}{3\pi}\ln\frac{p^2}{m^2}\right)^{-1}\right],\quad(34)$$

$$s(p,m,\lambda_0,e^2)=r'\left(\frac{\lambda_0^2}{m^2},e^2\right)\left[1+\frac{ce^4}{3\pi}\ln\frac{p^2}{m^2}\left(1-\frac{e^2}{3\pi}\ln\frac{p^2}{m^2}\right)^{-1}\right].\quad(35)$$

From a comparison of Eq. (35) with the results of calculations in the first orders in e^2 (cf. [4,6])

$$s=1+\frac{3e^2}{4\pi}\ln\frac{\lambda_0^2}{m^2}-\frac{e^4}{32\pi^2}\ln\frac{p^2}{m^2}+\ldots$$

we find

$$c=-3/32\pi,\qquad r'\left(\frac{\lambda_0^2}{m^2},e^2\right)=1+\frac{3e^2}{4\pi}\ln\frac{\lambda_0^2}{m^2}+\ldots$$

Let us now turn to the vertex function. We write Eq. (4) in the form

$$\Gamma=r^{-1}(\lambda_0^2/m^2,e^2)\,b(\varphi+x,\varphi+y,\varphi+z),$$
$$\varphi=-3\pi/e^2,\quad x=\ln(pq/m^2),$$
$$y=\ln(p^2/m^2),\ z=\ln(q^2/m^2).\quad(36)$$

The fact that b is an unknown function of three variables keeps us from reaching such definite conclusions about Γ as we reached about d and s on the basis of Eqs. (22) and (33). For definiteness let us consider the case in which $|pq| \gg |p^2|$, $|q^2| \gg m^2$. In an approximation using only terms

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2)This matter has also been treated by Eriksson[9] by means of the method of the renormalization group.

of the type $(e^2L^2)^n$ [L is a general designation for large logarithms of the type of $\ln(pq/p^2)$, $\ln(p^2/m^2)$, and so on], Sudakov [5] obtained for this case the following result:

$$\Gamma_\sigma = \gamma_\sigma \exp\left(-\frac{e^2}{2\pi}\ln\left|\frac{pq}{p^2}\right|\ln\left|\frac{pq}{q^2}\right|\right). \qquad (37)$$

One cannot obtain this result on the basis of Eq. (36), knowing only the first terms of the expansion in e^2, since each term of the expansion (37) has the structure (36) to the accuracy considered. The representation (36) does, however, allow us to improve the result (37) somewhat. In fact, an obvious extension of Eq. (37) which corresponds to Eq. (36) and is symmetrical in y and z is

$$\Gamma_\sigma = \gamma_\sigma \exp\left\{\frac{3}{2}\frac{(x-y)(x-z)}{\alpha(\varphi+x)+\beta(2\varphi+y+z)}\right\}, \qquad (38)$$

where $\alpha + 2\beta = 1$. Keeping only terms of the types $(e^2L^2)^n$ and $e^2L(e^2L^2)^n$, we get

$$\Gamma_\sigma = \gamma_\sigma\left\{1 - \frac{e^4}{6\pi^2}\ln\left|\frac{pq}{p^2}\right|\ln\left|\frac{pq}{q^2}\right|\left(\alpha\ln\left|\frac{pq}{m^2}\right| + \beta\ln\left|\frac{p^2\cdot q^2}{m^4}\right|\right)\right\} \times \exp\left(-\frac{e^2}{2\pi}\ln\left|\frac{pq}{p^2}\right|\ln\left|\frac{pq}{q^2}\right|\right). \qquad (39)$$

For $\alpha = 0$, $\beta = 1/2$ this result is in agreement with the result obtained by Vaks [7] by direct calculations.

It must be emphasized that in Eq. (39) we have taken into account not all possible terms of the type $e^2L(e^2L^2)^n$, but only those associated with the contribution from vacuum polarization.

4. In conclusion we point out that the use of the Dyson relations (5)—(8) and of the fact that the unrenormalized functions have finite limits for $m \to 0$ also allows one to prove (cf. [10]) the existence of finite limits for $m \to 0$ of the generalized Green's functions and vertex function introduced by Bogolyubov and Shirkov, [3,4] which are used in the formulation of the renormalization group, which has for its purpose the improvement of the perturbation-theory formulas. It is precisely the possibility of going to the limit $m \to 0$ in the functional equations of the renormalization group that is the decisive fact that allows one to determine by means of these equations the asymptotic forms of the Green's functions [3)] in the high-momentum region.

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3)In contrast with this fact, and contrary to the conclusion of Blank and Shirkov, [11] the asymptotic form of the vertex function cannot be determined in this way because of the comparatively large number of arguments contained in this function.

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