QUANTUM VORTICES IN A FERMION SYSTEM

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Superconductivity differential equations for the inhomogeneous case and an arbitrary temperature are derived. It is shown that in a system of fermions in the superfluid state quantum vortices arise which possess properties similar to those of vortices in a boson system.

At the present time it has been established in many experiments [1] that in rotating superfluid He II there are produced vortex filaments, that is, certain singular lines, around which the superfluid part of the liquid, according to Onsager and Feynman, rotates with velocity $v = n\hbar/mr$, $n = 1, 2, 3, \ldots$ [2]. Recently Pitaevskiĭ presented a quantum-mechanical derivation of the main properties of such vortex filaments for a Bose gas with weak repulsion interaction between the atoms [3]. On the other hand, it is known from modern superconductivity theory that a bound Cooper pair of fermions behaves in many respects like a boson. It is therefore natural to expect that under certain conditions there will be formed in a superfluid Fermi gas quantum vortices with properties similar to vortices in the Bose gas.

In the present work we derive, by a method previously proposed [4], the basic properties of quantum vortices in a Fermi gas for an unbounded system of fermions (for example, for nuclear matter) at any temperature $T < T_c$, including $T = 0$, where T_c is the critical temperature.

1. FUNDAMENTAL EQUATIONS

Inasmuch as in the presence of a vortex the fermion system is not homogeneous in space, we start from the following system of equations for the Fourier components of the temperature Green's functions $G_\omega(\mathbf{r}, \mathbf{r}')$ and $F_\omega(\mathbf{r}, \mathbf{r}')$ [5]:

$$\left\{i\omega + \frac{1}{2m^*}\frac{\partial^2}{\partial \mathbf{r}^2} + \mu\right\} G_\omega(\mathbf{r}, \mathbf{r}') + \Delta_T(\mathbf{r}) F^+_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$
$$\left\{-i\omega + \frac{1}{2m^*}\frac{\partial^2}{\partial \mathbf{r}^2} + \mu\right\} F^+_\omega(\mathbf{r}, \mathbf{r}') - \Delta^*_T(\mathbf{r}) G_\omega(\mathbf{r}, \mathbf{r}') = 0; \quad (1)$$

$$\Delta^*_T(\mathbf{r}) = gT\sum_{\omega'} F^+_\omega(\mathbf{r}, \mathbf{r}), \quad \omega = \pi(2n+1)T, \quad n = \ldots -1, 0, 1, \ldots, \quad (2)$$

where m^* is the reduced mass, which in our approximation takes into account the influence of the self-consistent field, μ is the chemical potential, and g is the energy of the pairing interaction.

For what follows it is convenient to change over to a system of integral equations. Introducing for this purpose the Fourier component of the fermion Green's function for the normal state, $\widetilde{G}_\omega(\mathbf{r} - \mathbf{r}')$, using the formula

$$\left\{i\omega + \frac{1}{2m^*}\frac{\partial^2}{\partial \mathbf{r}^2} + \mu\right\} \widetilde{G}_\omega(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (3)$$

we obtain in lieu of (1) the following system of integral equations

$$G_\omega(\mathbf{r}, \mathbf{r}') = \widetilde{G}_\omega(\mathbf{r} - \mathbf{r}') - \int \widetilde{G}_\omega(\mathbf{r} - \mathbf{s})\Delta_T(\mathbf{s}) \times \widetilde{G}_{-\omega}(\mathbf{s} - \mathbf{l})\Delta^*_T(\mathbf{l}) G_\omega(\mathbf{l}, \mathbf{r}')\, d^3s\, d^3l, \quad (4)$$

$$F^+_\omega(\mathbf{r}, \mathbf{r}') = \int \widetilde{G}_{-\omega}(\mathbf{r} - \mathbf{s})\Delta^+_T(\mathbf{s})\widetilde{G}_\omega(\mathbf{s} - \mathbf{r}')\, d^3s - \int \widetilde{G}_{-\omega}(\mathbf{r} - \mathbf{s})\Delta^*_T(\mathbf{s})\widetilde{G}_\omega(\mathbf{s} - \mathbf{l})\Delta_T(\mathbf{l}) F^+_\omega(\mathbf{l}, \mathbf{r}')\, d^3l\, d^3s. \quad (5)$$

In the region $|\omega| \ll \mu$ we have

$$\widetilde{G}_\omega(\mathbf{r} - \mathbf{r}') = -\frac{m^*}{2\pi R}\exp\left\{ip_0\frac{\omega}{|\omega|} - \frac{|\omega|}{v_0}\right\} R, \quad R = |\mathbf{r} - \mathbf{r}'|, \quad v_0 = (2\mu/m^*)^{1/2}. \quad (3')$$

From (5) with allowance for (2) we can obtain the equation for the gap $\Delta^*_T(\mathbf{r})$:

$$g^{-1}\Delta^*_T(\mathbf{r}) = T\sum_\omega \int \widetilde{G}_{-\omega}(\mathbf{r} - \mathbf{s})\Delta^*_T(\mathbf{s})\widetilde{G}_\omega(\mathbf{s} - \mathbf{r})\, d^3s - T\sum_\omega \int \widetilde{G}_{-\omega}(\mathbf{r} - \mathbf{s})\Delta^*_T(\mathbf{s}) \times \widetilde{G}_\omega(\mathbf{s} - \mathbf{l})\Delta_T(\mathbf{l}) F^+_\omega(\mathbf{l}, \mathbf{r})\, d^3l\, d^3s. \quad (6)$$

Equation (4) for the homogeneous case can be written in symbolic form as

$$\widetilde{G}_\omega = G'_\omega/(1 - \Delta^0_T F^{+\prime}_\omega), \quad (7)$$

where the homogeneous Green's functions G'_ω and

F'_ω of the superfluid system have the form

$$G'_\omega(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \frac{-i\omega-\xi}{\omega^2+\xi^2+\Delta_T^{02}} e^{i\mathbf{pr}}\, d^3\mathbf{p}, \qquad (8)$$

$$F'_\omega(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \frac{\Delta_T^0 e^{i\mathbf{pr}}}{\omega^2+\xi^2+\Delta_T^{02}}\, d^3\mathbf{p}, \qquad (9)$$

where $\xi = p^2/2m^* - \mu$; Δ_T^0 is the homogeneous energy gap at temperature T. Let us rewrite equation (6) with allowance for (7) in symbolic form

$$g^{-1}\Delta_T^* = T\sum_\omega \Big\{G_{-\omega}\Delta_T^* G'_\omega + \widetilde{G}_{-\omega}\Delta_T^* G'_\omega \Delta_T^0 F_\omega^{+'} - \widetilde{G}_{-\omega}\Delta_T^* G'_\omega \Delta_T F_\omega^+ + \widetilde{G}_{-\omega}\Delta_T^* G'_\omega \frac{1}{1-\Delta_T^0 F_\omega^{+'}}\Delta_T^0 F_\omega^{+'}\Delta_T^0 F_\omega^{+'} - \widetilde{G}_{-\omega}\Delta_T^* G'_\omega \frac{1}{1-\Delta_T^0 F_\omega^{+'}}\Delta_T^0 F_\omega^{+'}\Delta_T F_\omega^{+}\Big\}. \qquad (10)$$

Equation (10) is exact. Bearing in mind that a solution of (10) for an inhomogeneous energy gap must contain also the homogeneous solution corresponding to the ordinary temperature dependence of the gap, we construct the first iteration in the following manner. We put

$$F_\omega^+(\mathbf{l},\mathbf{r}) = F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,\Delta_T^*(\mathbf{l})/\Delta_T^0 \qquad (11)$$

and replace $\Delta_T(\mathbf{r})$ by Δ_T^0 in the fourth and fifth members of the right half of (10). In place of (10) we obtain here an approximate equation which takes into account the inhomogeneity only up to third order in $\Delta_T(\mathbf{r})$:

$$g^{-1}\Delta_T^*(\mathbf{r}) = T\sum_\omega \Big\{\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})\,\Delta_T^*(\mathbf{s})\,G'_\omega(\mathbf{s}-\mathbf{r})\,d^3\mathbf{s} + \int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})\,\Delta_T^*(\mathbf{s})\,G'_\omega(\mathbf{s}-\mathbf{l})\,\Delta_T^0 F_T^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l} - \frac{1}{\Delta_T^0}\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})\,\Delta_T^*(\mathbf{s}) \times G'_\omega(\mathbf{s}-\mathbf{l})\,\Delta_T(\mathbf{l})\,\Delta_T^*(\mathbf{l})\,F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l}\Big\}. \qquad (12)$$

If we now assume that $\Delta_T^*(\mathbf{r})$ is a slowly varying function of the coordinates, then by expanding $\Delta_T^*(\mathbf{r})$ in a series up to second order in the derivatives, we reduce (12) to the form

$$\Big\{\frac{1}{2m^*}\Big[1+\frac{A}{C}|\Delta_T(\mathbf{r})|^2\Big]\frac{\partial^2}{\partial\mathbf{r}^2} + \frac{E}{C}\frac{\partial|\Delta(\mathbf{r})|^2}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{r}} + \frac{B}{C}\Big[1-\frac{|\Delta_T(\mathbf{r})|^2}{\Delta_T^{02}}\Big] + \frac{D}{2C}\frac{\partial^2|\Delta_T(\mathbf{r})|^2}{\partial\mathbf{r}^2}\Big\}\Delta_T^*(\mathbf{r}) = 0; \qquad (13)$$

$$A = \frac{m^*}{3\Delta_T^0}T\sum_\omega\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{r}-\mathbf{s})^2 \times G'_\omega(\mathbf{s}-\mathbf{l})\,F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l},$$

$$B = \Delta_T^0 T\sum_\omega\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})\,G'_\omega(\mathbf{s}-\mathbf{l})\,F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l},$$

$$C = \frac{m^*}{3}T\sum_\omega\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{r}-\mathbf{s})^2 G'_\omega(\mathbf{s}-\mathbf{r})\,d^3\mathbf{s} + \frac{m^*}{3}T\sum_\omega\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{r}-\mathbf{s})^2 \times G'_\omega(\mathbf{s}-\mathbf{l})\,\Delta_T^0 F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l},$$

$$D = \frac{T}{3\Delta_T^0}\sum_\omega\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s}) \times G'_\omega(\mathbf{s}-\mathbf{l})(\mathbf{l}-\mathbf{r})^2 F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l},$$

$$E = \frac{T}{\Delta_T^0}\sum_\omega\int \widetilde{C}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{s}-\mathbf{r}) \times G'_\omega(\mathbf{s}-\mathbf{l})(\mathbf{l}-\mathbf{r})\,F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l}. \qquad (14)$$

We can confine ourselves to terms up to the second derivative if the series obtained by integrating the Taylor series of $\Delta_T^*(\mathbf{r})$ converges rapidly. The condition for the convergence is

$$\frac{1}{(2n)!}\left|\frac{\partial^{2n}\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^{2n}}H_{2n}\right| \gg \frac{1}{(2n+2)!!}\left|\frac{\partial^{2(n+1)}\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^{2(n+1)}}H_{2(n+1)}\right|, \qquad (15)$$

where

$$H_{2n} = \frac{1}{3^n}T\sum_\omega\Big\{\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{s}-\mathbf{r})^{2n}G'_\omega(\mathbf{s}-\mathbf{r})\,d^3\mathbf{s} + \Delta_T^0\int \widetilde{G}_{-\omega}(\mathbf{r}-\mathbf{s})(\mathbf{s}-\mathbf{r})^{2n} \times G'_\omega(\mathbf{s}-\mathbf{l})\,F_\omega^{+'}(\mathbf{l}-\mathbf{r})\,d^3\mathbf{s}\,d^3\mathbf{l}\Big\}. \qquad (16)$$

The integral H_{2n} has the following order of magnitude

$$H_{2n} \sim (v_0/\pi T)^{2n} \quad \text{for } T\to T_c,$$
$$H_{2n} \sim (v_0/\Delta_T^0)^{2n} \quad \text{for } T\to 0.$$

Therefore, in particular, for the fourth derivative to be small, we have

$$\frac{\partial^2\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^2} \gg 0.08\left(\frac{v_0}{\pi T}\right)^2\frac{\partial^4\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^4}, \qquad T\to T_c, \qquad (17)$$

$$\frac{\partial^2\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^2} \gg 0.2\left(\frac{v_0}{\Delta_T^0}\right)^2\frac{\partial^4\Delta_T^*(\mathbf{r})}{\partial\mathbf{r}^4}, \qquad T\to 0. \qquad (18)$$

For further simplification of (13), we neglect the terms containing the products of the derivatives of $\Delta_T(\mathbf{r})$ by the function itself. The conditions under which this neglect is possible actually coincide with condition (17) and (18). Then (13) simplifies considerably:

$$\left\{\frac{1}{2m^*}\frac{\partial^2}{\partial\mathbf{r}^2} + \frac{B}{C}\left[1-\frac{|\Delta_T^*(\mathbf{r})|^2}{\Delta_T^{02}}\right]\right\}\Delta_T^*(\mathbf{r}) = 0. \qquad (19)$$

Let us find the explicit form of the coefficients B and C contained in (19). In calculating the corresponding integrals by means of formulas (14) it is necessary to take into account the momentum cutoff contained in the definition of the Hamiltonian in the model of Bardeen, Cooper, and Schrieffer, since it is necessary to take into account only the interaction of fermions which are contained in a layer of thickness $\tilde{\omega}$ at the Fermi surface, that is,

$$(E_F - \tilde{\omega}) \leqslant \bar{\xi} \leqslant (E_F + \tilde{\omega}).$$

The calculation of the integrals B and C in the approximation $\Delta_T \ll \tilde{\omega} \ll \mu$ entails no difficulty. We obtain

$$C = \frac{2\pi}{3} m^* v_0^2 N_T(0)\, T \sum_\omega \Big\{ \frac{1}{(\omega + \sqrt{\omega^2 + \Delta_T^{02}})\sqrt{\omega^2 + \Delta_T^{02}}} + \Delta_T^{02} \frac{3\sqrt{\omega^2 + \Delta_T^{02}} + \omega}{2(\omega + \sqrt{\omega^2 + \Delta_T^{02}})^3 (\omega^2 + \Delta_T^{02})^{3/2}} \Big\},$$

$$B = \frac{\pi}{2} N_T(0)\, T \sum_\omega \frac{\Delta_T^{02}}{(\omega^2 + \Delta_T^{02})^{3/2}}, \qquad (20)$$

where $N_T(0)$ is the level distribution density at the Fermi surface.

In the limiting cases $T \rightarrow T_c$ and $T = 0$ we obtain for the ratio of the coefficients B/C

$$\frac{B}{C} = \begin{cases} {}^3/_2\, \Delta_T^{02}/\mu, & T \rightarrow T_c \\ {}^3/_8\, \Delta_0^{02}/\mu, & T = 0 \end{cases}. \qquad (21)$$

Introducing $\psi(\mathbf{r}) = \Delta_T^*(\mathbf{r})/\Delta_T^0$ and $l^2 = c/2m^*B$, we rewrite (19) in the form

$$\Big\{ l^2 \frac{\partial^2}{\partial \mathbf{r}^2} + 1 - |\psi(\mathbf{r})|^2 \Big\} \psi(\mathbf{r}) = 0. \qquad (22)$$

2. QUANTUM VORTEX

Equation (22) has the same form as the Ginzburg-Landau equation. In cylindrical coordinates r, φ, and z it admits of a solution in the form

$$\psi(\mathbf{r}) = e^{i\varphi} R(r), \qquad (23)$$

which represents one vortex filament at the center of a cylinder of radius R_0. In this case $R(r)$ satisfies the equation

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} - \frac{1}{\rho^2} R + R - R^3 = 0, \qquad (24)$$

and $\rho = r/l$.

Equation (24) was solved numerically by Ginzburg and Pitaevskiĭ [6] in connection with the phenomenological theory of superfluidity near the critical point. Asymptotically $R(\rho) \rightarrow \rho$ as $\rho \rightarrow 0$, and for $\rho \gg 1$ we have

$$R(\rho) = 1 - \tfrac{1}{2}\rho^{-2}. \qquad (25)$$

In all the intermediate points we approximate the solution in the form

$$R(\rho) = 1 - e^{-0.7\rho}. \qquad (25')$$

The quantity l has the meaning of the internal radius of the vortex and its dimension is on the order of the Cooper pair ($\sim 10^{-12}$ cm for nuclear matter and $\sim 10^{-4}$ cm for metal at $T = 0$). The temperature dependence of l is determined essentially by the temperature dependence of the homogeneous energy gap:

$$\frac{1}{l} = \sqrt{\frac{3m^*}{\mu}} \begin{cases} \frac{8\pi^2}{7\zeta(3)} \left(T^2 \ln \frac{T_c}{T}\right)^{1/2}, & T \rightarrow T_c \\ \frac{1}{2}\left(\Delta_0^0 - \sqrt{2\pi T \Delta_0^0}\, e^{-\Delta_0^0/T}\right), & T \rightarrow 0. \end{cases} \qquad (26)$$

We see from (26) that as $T \rightarrow T_c$ the inside radius of the vortex tends to infinity and the vortical state becomes destroyed, as is the state of superfluidity itself.

The most essential region is the region near $T = 0$, which could not be considered in the macroscopic theories, owing to the unknown dependence of the parameters of (22) on the energy gap and on the temperature. We note, however, that the radial dependence of the solution in the form of (25) and (25′) is compatible with the condition (17)—(18) only in the temperature range where $\ln(T_c/T) \ll 4$.

In the limit as $T \rightarrow 0$ the solution is valid only in the form (25) for $\rho \gg 1$, that is, away from the core of the vortex. In order to determine the correct radial dependence of the vortex near the core, it is necessary to consider the equation with high derivatives with respect to r.

Let us find now the velocity component v_φ and the energy per unit length of the vortex filament. As is well known, these quantities can be determined with the aid of the density matrix. The temperature Green's function is connected with the density matrix $\rho(\mathbf{r}, \mathbf{r}')$ in the following fashion:

$$\rho(\mathbf{r}, \mathbf{r}') = T \sum_\omega G_\omega(\mathbf{r}, \mathbf{r}') + \tfrac{1}{2}\delta(\mathbf{r} - \mathbf{r}'). \qquad (27)$$

The part of $\rho(\mathbf{r}, \mathbf{r}')$ due to the superfluid addition has in first approximation the form

$$\delta\rho(\mathbf{r}, \mathbf{r}') = -\Delta(\mathbf{r})\, \Delta^*(\mathbf{r}')\, T \times \sum_\omega \int \tilde{G}_\omega(\mathbf{r} - \mathbf{s})\, \tilde{G}_{-\omega}(\mathbf{s} - \mathbf{l})\, G'_\omega(\mathbf{l} - \mathbf{r})\, d^3s\, d^3l. \qquad (28)$$

The average component of the velocity v_φ of particles that participate in the vortex, and the additional energy on top of the homogeneous superfluid state, will be determined from the formulas

$$v_\varphi = \mathrm{Sp}_\varphi \Big\{ -\frac{i}{m^*} \nabla_\varphi \delta\rho(\mathbf{r}, \mathbf{r}') \Big\} \Big/ \mathrm{Sp}_\varphi \{\delta\rho(\mathbf{r}, \mathbf{r}')\}, \qquad (29)$$

$$\Delta E = \mathrm{Sp}\left\{-\frac{1}{2m^*}\nabla_r^2\left[\delta\rho(\mathbf{r},\mathbf{r}') - \delta\rho(\mathbf{r},\mathbf{r}')_{\Delta_T^0}\right]\right\} + \mathrm{Sp}\left\{\frac{\Delta_T^2(\mathbf{r}) - \Delta_T^{02}}{2g}\right\}. \quad (30)$$

Using formula (23) in the calculation of v_φ, we obtain

$$v_\varphi = 1/m^* r,$$

that is, the same variation for the velocity component v_φ as for a Bose system.

Calculating ΔE, for example, with radial dependence (25′), we obtain

$$\Delta E = \frac{7\pi N_T(0)\mu}{12m^*} + \frac{\pi}{m^*}\left\{\ln\frac{0.7R_0}{l} + 0.22\right\}\frac{\Delta_T^{02}}{4\mu g}.$$

From the last expression for ΔE we see that in general outline at $T = 0$ it is analogous to the formula for ΔE, obtained by Pitaevskiĭ for a Bose system [3].

In conclusion we emphasize that Eq. (19) is interpolative for the entire region of the existence of the vortex, and describes the vortex only qualitatively.

[1] Andronikashvili, Mamaladze, Matinyan, and Tsakadze, UFN **73**, 3 (1961), Soviet Phys. Uspekhi **4**, 1 (1961).

[2] L. Onsager, Nuovo cimento **6**, Suppl. 2, 249 (1947). R. P. Feynman, Progress in Low Temperature Physics, 1, Amsterdam (1957); Revs. Modern Phys. **29**, 205 (1957).

[3] L. P. Pitaevskiĭ, JETP **40**, 646 (1961), Soviet Phys. JETP **13**, 451 (1961).

[4] L. P. Rapoport and A. G. Krilovetskiĭ, DAN SSSR **145**, 771 (1962), Soviet Phys. Doklady **7**, 703 (1963).

[5] L. P. Gor'kov, JETP **36**, 1918 (1959), Soviet Phys. JETP **9**, 1364 (1959).

[6] V. L. Ginzburg and L. P. Pitaevskiĭ, JETP **34**, 1240 (1958), Soviet Phys. JETP **7**, 858 (1958).

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