ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE ON THE FIRST "UNPHYSICAL" SHEET

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The analytic properties of the partial waves and of the scattering amplitude on the first "unphysical" sheet are investigated. A proof of the Mandelstam representation is given for each term of the expansion of the scattering amplitude in terms of a parameter.

RECENTLY, many authors [1-5] have directed their attention to the study of the analytic properties of the scattering amplitude on the second sheet of its Riemann surface. Using the Mandelstam representation and the unitarity condition, one can continue the scattering amplitude as a function of the energy on to the second sheet of its Riemann surface (the first "unphysical" sheet) and study its analytic properties there.

The study of the analytic properties of the scattering amplitude on the second Riemann sheet is very important for the following reasons. It follows from the Mandelstam representation that the scattering amplitude has only real singularities (poles and cuts) on the first Riemann sheet. On the other hand, the scattering amplitude must have complex poles corresponding to resonance states. It was conjectured by Peierls [1] that the complex poles of the scattering amplitude lie on the second Riemann sheet. This idea of Peierls has been confirmed in many investigations.

Peierls himself [1] discussed the scattering amplitude for a finite potential and showed how the poles corresponding to resonance states enter in the scattering amplitude. Gunson and Taylor, [2] Zimmermann, [3] Oehme, [4] and Goldberger and coworkers [5] investigated the partial waves and the complete scattering amplitude in quantum field theory on the second sheet and proved the possibility of the existence of poles on this sheet. Besides poles, there are also additional cuts on the second sheet, and the dispersion relations on the second sheet have a form which differs from that of the dispersion relations on the first sheet.

In this note, following the work of Zimmermann, [3] we investigate the scattering amplitude for a Yukawa-type potential as a function of energy on the second Riemann sheet. In this case the Mandelstam representation exists and it is not necessary to postulate its validity as in the relativistic case. Moreover, a more complete study can be made than in the relativistic case, since the unitarity condition has a particularly simple form.

In the first section we investigate the analytic properties of the partial waves on the first as well as on the second sheet and show that complex poles appear on the second sheet. In the second section the complete scattering amplitude on the second sheet is discussed. It is shown that, besides the complex poles, there exists an additional cut. This cut arises in the following way. On the first sheet, the scattering amplitude is given as a sum of two analytic functions which coincide with its imaginary and real parts for real positive values of the energy. The discontinuities of the imaginary and real parts along the additional cut cancel each other on the first sheet, but add on the second sheet. In the third section it is shown that each term of the expansion of the scattering amplitude in terms of a parameter satisfies the Mandelstam representation.

1. Let us consider a potential of the type

$$V(r)=\int_m^\infty \sigma(\mu)\frac{e^{-\mu r}}{r}d\mu. \tag{1}$$

The scattering amplitude for potentials of this type has the spectral representation of Mandelstam [6-8]

$$f(s, t)=\int_{m^2}^\infty \frac{\rho(t')}{t'+t}dt'+\int_0^\infty ds'\int_{4m^2+m^4/s'}^\infty dt'\frac{\rho(s', t')}{(s'-s)(t'+t)}, \tag{2}$$

where $s=k_f^2=k_i^2$, $t=(k_f-k_i)^2$, k_f is the momentum after the scattering, and k_i the momentum before the scattering.

Using the unitarity condition

$$\operatorname{Im} f(s, t) = \frac{\sqrt{s}}{4\pi} \int d\Omega' f^*(s, (\mathbf{k}_f - \mathbf{k}')^2) f(s, (\mathbf{k}' - \mathbf{k}_i)^2),$$

$$s > 0, \quad \mathbf{k}'^2 = s \tag{3}$$

and the expansion of the scattering amplitude $f(s,t) = f(s, \cos\theta)$ in Legendre polynomials

$$f(s, \cos\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(\cos\theta),$$

$$\cos\theta = 1 - t/2s \tag{4}$$

(P_l is a Legendre polynomial), we obtain the following relation, which is basic for the following discussion:

$$\operatorname{Im} f_l(s) = \sqrt{s}\, |f_l(s)|^2, \quad s > 0. \tag{5}$$

We note that the representation (1) implies that the functions $f_l(s)$ are analytic in s in the complex plane with the cuts

$$\operatorname{Im} s = 0, \quad \operatorname{Re} s \geqslant 0, \quad \operatorname{Re} s \leqslant -m^2/4. \tag{6}$$

Let us focus our attention on the function $f_l(s)$ and investigate its analytic properties on the first "unphysical" sheet.

Using (5), it is easily shown, following Zimmermann,[3] that the function $f_l(s)$ can be written in the form

$$f_l(s) = F_l(s) + i\sqrt{s}\, G_l(s). \tag{7}$$

The functions $F_l(s)$ and $G_l(s)$ are analytic in the complex plane with the cut

$$\operatorname{Im} s = 0, \quad \operatorname{Re} s \leqslant -m^2/4, \tag{8}$$

except for poles. On the second Riemann sheet the function $f_l(s)$ is given by the formula $f_l^{(2)}(s) = F_l(s) - i\sqrt{s}\, G_l(s)$, and is therefore analytic in the complex plane with the cuts (6), except for poles. It should be noted that in formula (7) the poles of $F_l(s)$ and $G_l(s)$ cancel mutually.

2. Let us now study the scattering amplitude $f(s,t)$ on the second Riemann sheet. The function $f(s,t)$ can be written in the form

$$f(s, t) = F(s, t) + i\sqrt{s}\, G(s, t),$$

where

$$F(s, t) = \sum_{l=0}^{\infty} (2l+1) F_l(s) P_l\left(1 - \frac{t}{2s}\right),$$

$$G(s, t) = \sum_{l=0}^{\infty} (2l+1) G_l(s) P_l\left(1 - \frac{t}{2s}\right).$$

For real t in the interval $-m^2 < t \leq 0$, the series defining $F(s,t)$ and $G(s,t)$ converge and the functions $F(s,t)$ and $G(s,t)$ are analytic in the complex plane with the cut (8), except for poles.

On the second Riemann sheet the function $f(s,t)$ is given by

$$f^{(2)}(s, t) = F(s, t) - i\sqrt{s}\, G(s, t) \tag{9}$$

and is analytic in the complex plane with the cuts (6) and the poles of the function $G(s,t)$.

The function $f(s,t)$, with $-m^2 < t$, is analytic in s in the complex plane with the cut

$$\operatorname{Im} s = 0, \quad \operatorname{Re} s \geqslant 0, \tag{10}$$

and for $\operatorname{Im} s = 0$, $\operatorname{Re} s < 0$, it is analytic and real.

Since $f(s,t) = F(s,t) + i\sqrt{s}\, G(s,t)$ and the functions $F(s,t)$ and $G(s,t)$ have on the cut (8) a discontinuity equal to twice their imaginary parts, $\operatorname{Im} f(s,t)$ may vanish on this cut if the conditions

$$\operatorname{Im} F(s, t) = -\sqrt{-s}\, \operatorname{Im} G(s, t) \text{ or}$$
$$\operatorname{Im} F(s, t) = \operatorname{Im} G(s, t) = 0$$

are fulfilled.

Let us show now that it is the first possibility which corresponds to our case, i.e., that $\operatorname{Im} f(s,t)$ vanishes for $\operatorname{Im} s = 0$, $\operatorname{Re} s \leq -m^2/4$, because $\operatorname{Im} F(s,t) = -\sqrt{-s}\, \operatorname{Im} G(s,t)$.

We obtain from (2)

$$F(s, t) = \int_{m^2}^{\infty} \frac{\rho(t')}{t'+t} dt' + \mathrm{P}\int_0^{\infty} \frac{ds'}{s'-s} \int_{4m^2+m^4/s'}^{\infty} dt' \frac{\rho(s', t')}{t'+t},$$

$$\sqrt{s}\, G(s, t) = \pi \int_{4m^2+m^4/s}^{\infty} dt' \frac{\rho(s, t')}{t'+t}. \tag{11}$$

Since $\sqrt{s}\, G(s,t)$, for $-m^2 < t \leq 0$, is analytic in the complex plane with the cuts (6) except for poles, we can show that $F(s,t)$ is analytic in s in the complex plane with the cut (8) except for poles. The proof follows directly from (11), and we show that the imaginary part of $F(s,t)$ is equal to the imaginary part of $-\sqrt{-s}\, G(s,t)$ on the cut (8).

Indeed, let us consider the function

$$F_1(s, t) = \int_{m^2}^{\infty} \frac{\rho(t')}{t'+t} dt' + \frac{1}{\pi}\int_0^{\infty} \frac{ds'}{s'-s} \sqrt{s'}\, G(s', t) - i\sqrt{s}\, G(s, t) \tag{12}$$

and show that 1) it coincides with $F(s,t)$ for $s > 0$, and 2) it is analytic in the complex plane with the cut (8). The first assertion is proved immediately by going to the limit of real s. Possible singularities of the function $F_1(s,t)$ are the cuts (6) and poles. We show now that the function $F_1(s,t)$ only has the cut (8) and poles.

Indeed, the discontinuity of $F_1(s,t)$ for $s > 0$ is equal to

$$\lim_{\varepsilon\to 0}(F_1(s+i\varepsilon,t)-F_1(s-i\varepsilon,t))$$
$$=\frac{1}{\pi}\mathrm{P}\int_0^\infty\frac{ds'}{s'-s}\sqrt{s'}\,G(s',t)+i\sqrt{s}G(s,t)-i\sqrt{s}G(s,t)$$
$$+\int_{m^2}^\infty\frac{\rho(t')}{t'+t}dt'-\frac{1}{\pi}\mathrm{P}\int_0^\infty\frac{ds'}{s'-s}\sqrt{s'}G(s',t)+i\sqrt{s}G(s,t)$$
$$-i\sqrt{s}G(s,t)-\int_{m^2}^\infty\frac{\rho(t')}{t'+t}dt'=0. \qquad (13)$$

The discontinuity of $F_1(s,t)$ for $\mathrm{Im}\ s=0$, $\mathrm{Re}\ s \le -m^2/4$ is equal to $-2\sqrt{-s}\ \mathrm{Im}\ G(s,t)$, as follows from formula (12).

Thus the discontinuities of the functions $-i\sqrt{s}\times G(s,t)$ and $F(s,t)$ are equal to one another for $\mathrm{Im}\ s=0$, $\mathrm{Re}\ s\le -m^2/4$. Using (9), we find that the discontinuity of $f^{(2)}(s,t)$ on the cut (8) is nonvanishing and equal to

$$4\,\mathrm{Im}\,F(s,t)=-4\sqrt{-s}\,\mathrm{Im}\,G(s,t).$$

We see that the function $f^{(2)}(s,t)$ has a number of properties which are different from the properties of the function $f(s,t)$. First, $f^{(2)}(s,t)$ can have poles in the complex plane, and second, $f^{(2)}(s,t)$ has an additional cut (8) and hence an additional term in the dispersion relations on the "unphysical" sheet.[9] We note that this additional cut does not occur in the case of a finite potential,[10] as is easily seen by letting m go to infinity.

The region of analyticity of the function $f^{(2)}(s,\cos\theta)$ in s and $\cos\theta$ can be determined by the method of Zimmermann.[3]

3. In this section we show that all terms $f_n(s,t)$ of the expansion of the scattering amplitude in terms of a parameter λ,

$$f(s,t)=\sum_{n=1}^\infty\lambda^n f_n(s,t)$$

have a double spectral representation, provided that they satisfy ordinary dispersion relations and

$$f_1(s,t)=f_1(t)=\int_{m^2}^\infty\frac{\rho(t')}{t'+t}dt'.$$

The proof is by induction.

The unitarity condition (3) leads to

$$\mathrm{Im}\,f_i(s,t)=\frac{\sqrt{s}}{4\pi}\sum_{n=1}^{i-1}\int d\Omega' f^*_{i-n}(s,(\mathbf{k}_f-\mathbf{k}')^2)\,f_n(s,(\mathbf{k}'-\mathbf{k}_i)^2),$$
$$s>0. \qquad (14)$$

If $f_n(s,t)$ has a double spectral representation (2) for $n=1,\ldots,i-1$, then it follows from (14) that $\mathrm{Im}\ f_i(s,t)$ is analytic in t in the complex plane with the cut $\mathrm{Im}\ t=0$, $\mathrm{Re}\ t\le -4m^2-m^4/s$. Indeed, let us substitute in (14) the expression

$$f_n(s,t)=\int_{m^2}^\infty\frac{\varphi_n(t',s)}{t'+t}dt', \qquad (15)$$

where $\varphi_n(t,s)$ is the discontinuity of $f_n(s,t)$ (with $s>0$) in t on the cut $\mathrm{Im}\ t=0$, $\mathrm{Re}\ t\le -m^2$. We obtain

$$\mathrm{Im}\,f_i(s,t)=\frac{\sqrt{s}}{4\pi}\sum_{n=1}^{i-1}\int d\Omega'\int_{m^2}^\infty\frac{\varphi^*_{i-n}(s,t_1')}{t_1'+t_1}dt_1'\int_{m^2}^\infty\frac{\varphi_n(s,t_2')}{t_2'+t}dt', \qquad (16)$$

where

$$t_1=(\mathbf{k}_f-\mathbf{k}')^2,\quad t_2=(\mathbf{k}'-\mathbf{k}_i)^2.$$

Formula (16) can also be written in the form[11]

$$\mathrm{Im}\,f_i(s,t)$$
$$=\frac{\sqrt{s}}{16\pi s^2}\sum_{n=1}^{i-1}\int_{m^2}^\infty\int_{m^2}^\infty\Big\{\varphi^*_{i-n}(s,t_1')\,\varphi_n(s,t_2')\int_{-1}^1 d\cos\varphi\int_{-1}^1 d\cos\psi$$
$$\times\frac{\Theta(-K(\cos\varphi,\cos\psi,\cos\theta))}{\sqrt{-K(\cos\varphi,\cos\psi,\cos\theta)}}$$
$$\times\frac{1}{t_1'/2s+1-\cos\varphi}\,\frac{1}{t_2'/2s+1-\cos\psi}\Big\}dt_1'\,dt_2'$$
$$=\frac{\sqrt{s}}{16\pi s^2}\sum_{n=1}^{i-1}\int_{m^2}^\infty\int_{m^2}^\infty\varphi^*_{i-n}(s,t_1')\,\varphi_n(s,t_2')$$
$$\times H\Big(1+\frac{t_1'}{2s},1+\frac{t_2}{2s},\cos\theta\Big)dt_1'dt_2', \qquad (17)$$

where $\Theta(x)$ is the unit step function and

$$H(\xi,\eta,z)=-\frac{\pi}{\sqrt{K(\xi,\eta,z)}}\ln\frac{z-\xi\eta+\sqrt{K(\xi,\eta,z)}}{z-\xi\eta-\sqrt{K(\xi,\eta,z)}},$$
$$K(\xi,\eta,z)=\xi^2+\eta^2+z^2-2\xi\eta z-1.$$

The function $H(\xi,\eta,z)$ is analytic in z in the complex plane with the cut

$$\mathrm{Im}\,z=0,\qquad \mathrm{Re}\,z\geqslant\xi\eta+\sqrt{(\xi^2-1)(\eta^2-1)}. \qquad (18)$$

It is easily seen that the cut (18) is in our case determined by

$$\mathrm{Im}\cos\theta=0,\qquad \mathrm{Re}\cos\theta\geqslant 1+2m^2/s+m^4/2s^2.$$

Recalling that $\cos\theta=1-t/2s$, we find that $\mathrm{Im}\ f_i(s,t)$ is analytic in t in the complex plane with the cut

$$\mathrm{Im}\,t=0,\qquad \mathrm{Re}\,t\leqslant -4m^2-m^4/s.$$

If $\mathrm{Im}\ f_i(s,t)$ (with $s>0$) is analytic in t in the complex plane with the cut $\mathrm{Im}\ t=0$, $\mathrm{Re}\ t\le -4m^2-m^4/s$, it follows from the ordinary dispersion relation in s that $f_i(s,t)$ is analytic in s and t in the two complex planes with the cuts

$$\operatorname{Im} s = 0, \quad \operatorname{Re} s \geqslant 0, \quad \operatorname{Im} t = 0, \quad \operatorname{Re} t \leqslant -4m^2.$$

Therefore, if $f_1(s,t), \ldots, f_{i-1}(s,t)$ have a Mandelstam representation, and $f_i(s,t)$ obeys an ordinary dispersion relation, we can conclude on the basis of the unitarity relation that $f_i(s,t)$ also has a Mandelstam representation. Since $f_1(t)$ has a Mandelstam representation by assumption, we find by this procedure that all $f_i(s,t)$ do likewise. (For simplicity we disregard the fact that the $f_i(s,t)$ have different cuts and choose the maximal cut.)

We have thus obtained the result of Bowcock and Martin[12] from the unitarity condition.

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