ANGULAR DISTRIBUTION OF THREE PARTICLES PRODUCED NEAR THE THRESHOLD

V. V. ANISOVICH and L. G. DAKHNO

A. F. Ioffe Physico-technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor July 7, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 198-202 (January, 1963)

The amplitude for production of three particles near threshold is considered. The angular distribution of the produced particles is given to an accuracy of terms quadratic in the particle momenta in the final state. The angular distributions in the $\pi + N \to N + \pi + \pi$ and $K + N \to N + K + \pi$ reactions are discussed.

SEVERAL authors [1-6] have considered the amplitude for the production of three particles near threshold with total orbital angular momentum $L = 0$. If the reaction is considered sufficiently close to the threshold, the amplitude can be expanded in powers of the momenta relative to the motion of the produced particles. For the $\pi + N \to N + \pi + \pi$ and $\gamma + N \to N + \pi + \pi$ reactions, the amplitude with $L = 0$ was calculated to an accuracy up to terms quadratic in the momenta [2,3,5], and for τ decay the amplitude was calculated to an accuracy up to the third power. [1,4,6] In these papers it was shown that the experimental study of the foregoing reactions makes it possible to find the amplitude for pion scattering at zero energy. The first experimental data have recently been obtained. [7]

If we are interested in the dependence of the cross section on the angles between the directions of the outgoing particles and particle momenta in the initial state, it is necessary to consider the production of particles in states with $L > 0$. The present work is devoted to this question.

The amplitude for the production of three particles near threshold with total angular momentum L decreases with increasing L—the main term in the amplitude is of the order K^L (K is a quantity equal in order of magnitude to the c.m.s. momenta of the produced particles). Similarly to the case with $L = 0$, the amplitude for $L > 0$ can be expanded close to threshold in powers of the momenta of the produced particles. It then turns out that the main terms of the amplitude with angular momentum L (which are of the order K^L) are expressed through $L+1$ undetermined constants. But the subsequent corrections to this amplitude (of the order K^{L+1}) are expressed through the same undetermined constants and through the scattering amplitudes for different pairs of produced particles at zero energy. Hence the structure of the amplitude with $L > 0$ is similar to the amplitude with $L = 0$ (see [1-6]). By studying the experimental dependence of the cross section on the angles between the directions of the momenta in the initial and final states, we can also obtain information about the size of the scattering amplitude for the produced particles at zero energy.

In this article we consider the amplitude for the production of three particles with an accuracy to terms quadratic in the momenta. The amplitude with $L = 0$ has been considered earlier to such an accuracy. [2,3,4] Therefore we should consider amplitude terms up to the second power with $L = 1$ and $L = 2$ (amplitudes with larger L are of higher order).

The amplitude for the production of three particles (five-point function) depend on five independent invariants. As such invariants we can choose, for example, the squares of the energy associated with the relative motion of pairs of particles in the final state s_{12}, s_{13}, s_{23} [$s_{il} = (\sqrt{m_i^2 + k_i^2} + \sqrt{m_l^2 + k_l^2})^2 - (\mathbf{k}_i + \mathbf{k}_l)^2$, where m_i and $\mathbf{k}_i$ are the mass and momentum of the i-th particle in the c.m.s.] and two momentum transfers t_1 and t_2 [$t_i = (\omega - \sqrt{m_i^2 + k_i^2})^2 - (\mathbf{P} - \mathbf{k}_i)^2$, where ω and $\mathbf{P}$ are the total energy and momentum of one of the particles in the initial state]. The notation is explained in Fig. 1.

Near the threshold for the production of three particles, the invariants s_{il} and t_i can be expanded in powers of the final-state particle mo-

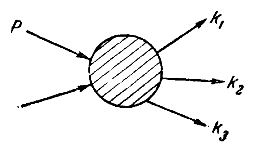

FIG. 1

menta. Then $s_{il} = (m_i + m_l)^2 + (m_i + m_l) \times k_{il}^2/u_{il}$ (k_{il} is the momentum of the relative motion of the i-th and l-th particle, while u_{il} is their reduced mass), $t_i = \omega^2 - P^3 + m_i^2 - 2m_i\omega + 2Pk_iz_i \equiv t_{i0} + 2Pk_iz_i$ (z_i is the cosine of the angle between the directions of the momenta **P** and $\mathbf{k}_i$).

Since the amplitude for the production of three particles near threshold is an analytic function of t_1 and t_2, it can be expanded in a series in $t_1 - t_{10}$ and $t_2 - t_{20}$:

$$A\,(k_{12}k_{13}k_{23}z_1z_2) = A_0\,(k_{12}k_{13}k_{23}) + A_{10}\,(k_{12}k_{13}k_{23})\,k_1z_1 + A_{01}(k_{12}k_{13}k_{23})\,k_2z_2 + A_{20}\,(k_{12}k_{13}k_{23})\,k_1^2\,z_1^2 + A_{11}\,(k_{13}k_{13}k_{23})\,k_1k_2z_1z_2 + A_{02}\,(k_{12}k_{13}k_{23})\,k_2^2z_2^2 + \dots \tag{1}$$

The functions $A_0, A_{10}, A_{01}, \dots,$ do not, in general, vanish when the total energy of the three particles in the final state is equal to zero. In the c.m.s., the momenta of the produced particles lie in one plane, since $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. The position of this plane relative to the direction of the momentum of the particles in the initial state **P** is characterized by two angles ϑ and φ which can, for example, be connected with z_1 and z_2 as follows:

$$z_1 = \sin\vartheta\cos\varphi,\quad z_2 = \sin\vartheta\cos(\varphi - \gamma); \tag{2}$$

γ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. Expanding $A(k_{12}k_{12}k_{23}z_1z_2)$ in terms of $Y_{LM}(\vartheta, \varphi)$, we obtain the amplitudes for the production of three particles with different total angular momenta L.

We have to consider amplitudes with L = 1 and L = 2. The amplitude with L = 1 is determined to an accuracy up to quadratic terms by the terms $A_{10}(k_{12}k_{13}k_{23})k_1z$; and $A_{01}(k_{12}k_{23}k_{13})k_2z_2$. The terms $A_{20}k_1^2z_1^2$, $A_{11}k_1k_2z_1z_2$ and $A_{02}k_2^2z_2^2$ contribute to the amplitudes with L = 0 and L = 2. The terms of the expansion not written in (1) are of order greater than the second power of the momenta of the produced particles.

We calculate the amplitude for production with L = 1 to an accuracy of terms quadratic in the momenta. We first rewrite the terms in (1) of interest to us in more symmetric form, where we make use of the fact that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$:

$$A_{10}(k_{12}k_{13}k_{23})\,k_1z_1 + A_{10}\,(k_{12}k_{13}k_{23})\,k_2z_2 = A_1\,(k_{12}k_{13}k_{23})\,k_1z_1 + A_2\,(k_{12}k_{23}k_{13})k_2z_2 + A_3\,(k_{12}k_{13}k_{23})\,k_3z_3. \tag{3}$$

The values of A_1, A_2, and A_3 in the zero-order approximation relative to the momenta of the produced particles will be denoted by α_1, α_2, and α_3, respectively. The imaginary and real parts of the complex constants α_l are related by the unitarity condition[8]

$$\alpha_l = \rho_l e^{i\delta}, \tag{4}$$

where δ is the scattering phase shift in the initial state at the threshold energy and ρ_l is an undetermined real constant.

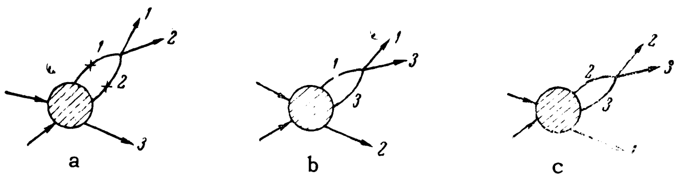

FIG. 2

The corrections to A_1, A_2, and A_3 proportional to k_{il} are obtained from consideration of the diagrams shown in Fig. 2.[4] These corrections are calculated in the standard way from the dispersion relations for the relative momenta of pairs of particles.[5] For example, the correction from diagram 2a to $A_1k_1z_1 + A_2k_2z_2 + A_3k_3z_3$ is

$$\frac{k_{12}^2}{\pi}\int_0^\infty dk_{12}'^2\,\frac{B_{12}(k_{12}')}{k_{12}'^2\,(k_{12}'^2 - k_{12}^2 - i\varepsilon)}; \tag{5}$$

$B_{12}(k_{12})$ is the absorption part of diagram 2a corresponding to the real lines marked by crosses in the diagram. It is given by

$$B_{12}\,(k_{12}) = \int_{-1}^{1}\frac{dz}{2}(\alpha_1k_1'z_1' + \alpha_2k_2'z_2' + \alpha_3k_3z_3)\,k_{12}a_{12}. \tag{6}$$

In place of the shaded region we have inserted expression (4); $\mathbf{k}_1'$ and $\mathbf{k}_2'$ are the momenta of particles 1 and 2 in the intermediate state, z is the cosine of the angle between the relative momentum of particles 1 and 2 in the intermediate state $\mathbf{k}_{12}'$ and the momentum **P**; z_1' and z_2' are the cosines of the angles between momenta $\mathbf{k}_1'$, $\mathbf{k}_2'$, and **P**; a_{12} is the scattering length for particles 1 and 2.

Since in the c.m.s.

$$\mathbf{k}_2' = -[m_2/(m_1 + m_2)]\mathbf{k}_3 - \mathbf{k}_{12}',\quad \mathbf{k}_1' = -[m_1/(m_1 + m_2)]\,\mathbf{k}_3 + \mathbf{k}_{12}', \tag{7}$$

the integration in the expression for the absorption part can be performed without difficulty:

$$B_{12}\,(k_{12}) = k_3z_3a_{12}k_{12}\,(\alpha_3 - [m_1/(m_1 + m_2)]\alpha_1 - [m_2/(m_1 + m_2)]\,\alpha_2). \tag{8}$$

Inserting (8) into the dispersion integral (5), we find that it is equal to $iB_{12}(k_{12})$. The remaining corrections from diagrams 2b and 2c are calculated in a similar way. Hence the expression for $A_1k_1z_1 + A_2k_2z_2 + A_3k_3z_3$ to an accuracy of terms quadratic in the momenta of the particles in the final state has the form

$$A_1k_1z_1 + A_2k_2z_2 + A_3k_3z_3 = k_3z_3\left[\alpha_3 + ik_{12}a_{12}\left(\alpha_3 - \frac{m_1}{m_1+m_2}\alpha_1 - \frac{m_2}{m_1+m_2}\alpha_2\right)\right] + k_2z_2\left[\alpha_2 + ik_{12}a_{13}\left(\alpha_2 - \frac{m_1}{m_1+m_3}\alpha_1 - \frac{m_3}{m_1+m_3}\alpha_3\right)\right] + k_1z_1\left[\alpha_1 + ik_{23}a_{23}\left(\alpha_1 - \frac{m_2}{m_2+m_3}\alpha_2 - \frac{m_3}{m_2+m_3}\alpha_3\right)\right]. \quad (9)$$

We note that it follows from (4) and from the equality $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ that the part of the amplitude under consideration depends essentially only on two independent combinations of α_1, α_2, and α_3.

In the zero-order approximation, A_1, A_2, and A_3 are equal to α_1, α_2, and α_3, which are constants about which nothing had been known earlier (except that their imaginary and real parts are related to each other through the unitarity condition). These constants should be determined from experiment. But the first momentum corrections for A_1, A_2, and A_3 are already expressed through the same constants α_1, α_2, and α_3 and through the paired scattering amplitudes at zero energy a_{12}, a_{13}, and a_{23}. In this sense, the situation here is quite similar to the case of the amplitude with total orbital angular momentum $L = 0$.

As we already mentioned, the amplitude still has terms that are quadratic in the momenta with $L = 2$. In the approximation considered, the amplitude with $L = 2$ is determined by three undetermined constants. However, since the dependence on z_1 and z_2 in this amplitude is different than in the amplitude with $L = 1$, they can be separated experimentally.

We consider the angular distribution in the $\pi + N \rightarrow N + \pi + \pi$ reactions. Here we will be interested in terms proportional to z_1, z_2, and z_3. In the amplitude these terms can arise only from the initial states $S_{1/2}$ and $D_{3/2}$. In the production of a meson on unpolarized nucleons the initial $S_{1/2}$ state does not make any contribution to the cross section in terms linear in z_1, z_2, and z_3. In this case we can take $\alpha_l = \Sigma_T \beta_{lT} e^{i\delta_{T3}}$, where δ_{T3} is the phase shift for meson-nucleon scattering in the $D_{3/2}$ state with isospin T at threshold energies and β_{lT} are real numbers.

As can be proved with the aid of formula (9), the coefficients of k_1z_1, k_2z_2, and k_3z_3 in the zero-order approximation for the cross section are proportional to $\mathrm{Re}(\lambda^*\alpha_l)$ (λ is the amplitude for the production of three particles with $L = 0$ for $k_{il} = 0$). The amplitude λ is equal to $\Sigma_T \rho_T e^{i\alpha_{T1}}$, where α_{T1} are the phase shifts for meson-nucleon scattering in the $P_{1/2}$ state at threshold energy and ρ_T are real numbers.[2] As in the case of the phase shifts δ_{T3}, the phase shifts α_{T1} are small. Therefore the coefficients of k_1z_1, k_2z_2, and k_3z_3 in the zero-order approximation are proportional to $\cos(\alpha_{T1} - \delta_{T3}) \sim 1$. It is seen from (9) that in the cross section the first corrections to the coefficients of k_1z_1, k_2z_2, and k_3z_3 (proportional to the first power of k_{il}) contain the quantity $\mathrm{Im}(\lambda^*\alpha_l) \sim \sin(\alpha_{T1} - \delta_{T3})$, which is small. Hence, in these corrections an additional small order of magnitude occurs and the corrections can be neglected in the approximation. This additional small order of magnitude occurs, in particular, as a result of the smallness of the phase shifts α_{T1}, which, generally speaking, is a chance effect. In other reactions, for example, $K + N \rightarrow N + K + \pi$, the N+K phase shift in the $P_{1/2}$ state at threshold energy can turn out to be a large quantity. In this case the coefficients of k_1z_1, k_2z_2, and k_3z_3 in the cross section will contain terms linear in k_{12}, k_{23}, and k_{13}; after these coefficients are found experimentally, the scattering amplitudes for pairs of particles at zero energy can be determined. The formulas prove to be quite convenient, owing to the fact that in this reaction, as in the reaction $\pi + N \rightarrow N + \pi + \pi$, the coefficients α_l can be considered to be real numbers.

The authors thank A. A. Ansel'm, V. N. Gribov, V. M. Shekhter, and I. M. Shmushkevich for helpful discussions and comments.

[1] V. N. Gribov, Nuclear Phys. **5**, 653 (1958).

[2] A. A. Ansel'm and V. N. Gribov, JETP **36**, 1890 (1959), Soviet Phys. JETP **9**, 1345 (1959).

[3] A. A. Ansel'm and V. N. Gribov, JETP **37**, 501 (1959), Soviet Phys. JETP **10**, 354 (1960).

[4] V. N. Gribov, JETP **41**, 1221 (1961), Soviet Phys. JETP **14**, 871 (1962).

[5] Anisovich, Ansel'm, and Gribov, JETP **42**, 224 (1962), Soviet Phys. JETP **15**, 159 (1962).

[6] Anisovich, Anselm, and Gribov, Nuclear Phys. **38**, 132 (1962).

[7] Batusov, Bunyatov, Sidorov, and Yarba, JETP **40**, 460 (1961), Soviet Phys. JETP **13**, 320 (1961).

[8] E. Fermi, Nuovo cimento Suppl. **2**, 1 (1955); V. N. Gribov, JETP **33**, 1431 (1957), Soviet Phys. JETP **6**, 1102 (1958).

Translated by E. Marquit
36