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MICROCOVARIANCE AND MICROCAUSALITY IN QUANTUM THEORY

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The conditions of microcovariance in quantum theory, which reflect the fact that space-time is pseudoeuclidean at all points, are derived. The conditions of microcausality in relativistic quantum theory are also formulated. The derivations of all of these conditions are based only on the fundamental principles of quantum theory and the geometrical properties of space-time. In particular, there are no assumptions about the point character of particles or the validity of any postulates of quantum field theory. In relativistic quantum field theory a new class of physical quantities, called dynamic moments, is introduced. By means of the dynamic moments the matrix elements of currents can be connected with the scattering matrix in an exact way, without bringing in postulates of quantum field theory. As applied to the general conditions of microcovariance and microcausality obtained in this paper, the use of dynamic moments makes it feasible to develop an experimental program for studying the limits of validity of existing ideas about the structure of space-time. It is shown that the existence of a Hamiltonian description follows from the existence of the scattering matrix and the pseudoeuclidean character of space-time.

1. INTRODUCTION

THE purpose of this paper is the general mathematical formulation of the conditions of microcovariance and microcausality in relativistic quantum theory. Here we mean by microcovariance the set of conditions that reflect the fact that space time is pseudoeuclidean at all points. The physical meaning of the requirement of microcovariance is discussed in Sec. 2. A mathematical formulation of this requirement is given in Sec. 3. The result of Sec. 4 is a derivation of the condition of microcausality which does not depend on any postulates at all from quantum field theory.

The conditions of microcovariance and microcausality are conditions imposed on the operator of the energy-momentum tensor of the physical system (one of the conditions is the existence of this operator). These conditions cannot be used directly, since the components of the energy-momentum tensor, like any other local quantities, are not accessible to experimental observation in microscopic regions of space. To overcome this difficulty, which is discussed in Sec. 5, methods are developed in the later sections of the paper for obtaining physical quantities accessible to experimental observation from the energy-momentum tensor and from other local quantities. This set of methods also does not depend on the postulates of quantum field theory.

In Sec. 6 the one-particle and two-particle matrix elements of the operators for the energy-momentum tensor and the electromagnetic current are parametrized—i.e., expressed in terms of invariant form-factors. In Sec. 7 these matrix elements are connected with the scattering matrix by means of a new class of physical quantities, called dynamic moments. In Sec. 8 some analytic properties of the matrix elements of the energy-momentum tensor are discussed. In Sec. 9 the connection between the energy-momentum tensor and the electromagnetic characteristics of a particle is established. In Sec. 10 it is shown that the existence of a Hamiltonian description follows from the existence of the scattering matrix and the pseudoeuclidean character of space-time.

A reader interested only in the use (and not in the derivation) of the conditions of microcovariance and microcausality can omit all of the developments connected with the use of the generally covariant formalism, and by taking on faith the formulas listed at the end of Sec. 4 can begin reading with Sec. 5.

2. SPACE-TIME DESCRIPTION IN RELATIVISTIC QUANTUM THEORY

One of the most important problems of modern high-energy physics is to settle the question of the limits of applicability of present ideas about the structure of space-time. There is a widespread opinion that there exists an "elementary length" within which there is a decided change in the form of the space-time structure. In order, however, to make possible the experimental investigation of this question it is necessary to formulate the conditions imposed on the quantum theory of space-time structure.

Let us see to what extent these conditions have been formulated in the existing theory. In principle it would seem that one could hope to get information about changes in space-time structure at small distances from studies

a) of quantum electrodynamics at small distances;

b) of dispersion relations;

c) of the conditions of relativistic invariance.

A breakdown of quantum electrodynamics at small distances would, however, indicate not that ideas about space and time had ceased to hold, but that the scheme of calculations in quantum electrodynamics had failed, or else that some additional interaction had become important.

In just the same way, violations of unproved dispersion relations (for example, those of Mandelstam) would indicate only that there are limits to the applicability of these relations. Even the experimental detection of a violation of proved dispersion relations would, strictly speaking, not be associated with the structure of space, but would only indicate limitations on the usefulness of the assumption of the existence of asymptotic fields possessing local properties and at the same time capable of being expressed in a definite way in terms of operators for the creation and absorption of particles (cf. e.g., [1-3]). This assumption is equivalent to the assertion that particles have (in a certain sense) a point nature. Therefore it can be expected that it can be applied unconditionally only in the study of problems in which effects associated with the structure of the particles are small. Thus the violation of proved dispersion relations would show the necessity of including structure effects, rather than a change of the properties of space-time.

Finally, the conditions of relativistic invariance, which are of the form (cf. e.g., [4])

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i(\delta_{\mu\sigma}M_{\lambda\nu} + \delta_{\mu\lambda}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\lambda} + \delta_{\lambda\nu}M_{\sigma\mu}),$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\nu\delta_{\mu\lambda} - P_\mu\delta_{\nu\lambda}), \qquad [P_\mu, P_\nu] = 0, \tag{1}$$

are already directly connected with the structure of space-time. Violations of these conditions would indicate without doubt changes in the structure of space-time.

Let us, however, examine more closely in what kind of physical processes a violation of the relations (1) would manifest itself. A violation of these relations would indicate lack of equivalence of different inertial coordinate systems, i.e., violation of the symmetry of the laws of nature with respect to the inhomogeneous Lorentz group. Obviously the first manifestation of such a change of the geometry would be that mass and spin would cease to be exact quantum numbers even for a single free particle, and would change during free motion. To detect this kind of change of the geometry one naturally needs experiments not at extremely small distances, but on the contrary at extremely large distances. Thus we conclude that changes of the geometry associated with violation of the conditions (1) of relativistic invariance must be sought not in experiments with high-energy accelerators, but in astronomical observations. An example is the red shift. The assertion just made can also be explained by means of the uncertainty relation: to detect small deviations from the relations (1) it is necessary to measure extremely small changes of momentum, which of course can be done best at extremely large distances, and not at large energies.

If, on the other hand, the space-time structure is decidedly altered at small distances (inside particles), and remains pseudoeuclidean far from particles, this does not change the relations (1), which can therefore be called conditions for the <u>macrocovariance</u> of quantum theory. We shall give the name of <u>microcovariance</u> conditions to the conditions for the pseudoeuclidean nature of space at all points, which we shall derive in what follows. It follows from what has been said that the relations (1) provide no possibilities for determining a change of the geometry at small distances.[1)]

Summarizing, we can say that at present an experimental test of the limits of applicability of existing ideas about space-time cannot be made, since it is not known what the test should comprise. It is even not excluded that some unrecognized signs of a breakdown of geometry at small distances are already contained in existing experimental data. The question arises as to how to formulate the

[1)]If it is found that in high-energy reactions there are violations of the conservation laws while the masses and spins of free particles remain integrals of the motion, this will be evidence not of a change of the geometry, but of the breakdown of one or more principles of quantum theory.

conditions of microcovariance of quantum theory, which reflect the pseudoeuclidean nature of space-time in all arbitrarily small regions.

It turns out that the key to the solution of the problem is the examination of the group of transformations to various curvilinear coordinate systems:

$$x^{\mu} = f^{\mu}(x'). \tag{2}$$

The role of this group in relativistic quantum theory has already been discussed in [5], where in particular we obtained the commutation relations for the operators $R_{\mu}(x)$ which produce the transformation of the state vector Ψ

$$\Psi = \left(1 + i\int d^4x \xi^{\mu}(x) R_{\mu}(x)\right)\Psi' \tag{3}$$

in an infinitesimal coordinate transformation of the type (2):

$$x^{\mu} = x'^{\mu} + \xi^{\mu}(x'). \tag{4}$$

These commutation relations are of the form

$$[R_{\mu}(x),\ R_{\nu}(y)] = i\{R_{\nu}(x)\,\partial/\partial x^{\mu} - R_{\mu}(y)\,\partial/\partial y^{\nu}\}\,\delta^4(x-y) \tag{5}$$

and are the conditions for general covariance of the quantum theory. They reflect the symmetry group of the general theory of relativity.

It is on the basis of the conditions (5) (which are also discussed from a somewhat different point of view in [6]) that a quantum theory of gravitation must be constructed. In the present paper, however, the conditions (5) cannot be used directly, since we shall not be concerned with gravitational effects, which evidently play a negligibly small part in the structure of elementary particles, and we shall regard the metric tensor simply as a prescribed external field. Then the various curvilinear coordinate systems are already not physically equivalent, although of course the corresponding transformations for the state vectors must exist as before.

The necessity of considering transitions to different curvilinear coordinate systems even in a theory with a fixed (for example, pseudoeuclidean) metric can be explained in various ways. Suppose, for example, we are considering the scattering of two particles. At minus infinity in time these particles are spatially separated, so that each may be regarded as free, and the concept of momentum has a meaning for each particle. It is known that the momentum is defined as the operator for an infinitesimal displacement. In the case considered, in order to define the momentum of one of the particles one must make the displacement not in the whole space, but only in a half-space which contains the first particle and not the second. The result is necessarily to make the coordinate system a curvilinear one. It can be seen from this example that only the use of transitions to various curvilinear coordinate systems makes possible a space-time description of events, which is lacking in the pure S-matrix theory.

There is a widespread opinion that a space-time description is not necessary at all. We cannot agree with this approach for two reasons. First, the restriction to the pure S-matrix aspect decidedly impoverishes the theory, since it discards all properties associated with the microstructure of space. One could with equal success, for example, declare that the conditions (1) for macrocovariance are unnecesary, since even without them one can get a complete description of the experimental facts, dealing with each inertial system by itself. The only difference is that the conditions (1) are well known, and the conditions of microcovariance are not. Second, there exists a broad class of experiments (essentially connected with space-time relations) for the description of which the S-matrix formalism is inadequate.

As examples, we may mention collisions in the presence of an external field, and the experiment of Wu, Lee, et al.,[7] who studied the development of a resonance line in time (the shape of the line radiated turned out to depend on the distance between the source and the detector, if the time of propagation of the radiation from source to detector is comparable with the lifetime of the level). Besides transformations to curvilinear coordinates, it is useful to bring into the treatment small changes of the metric. For example, if we make the metric slightly different from a pseudoeuclidean one and different at different points, then in this way we mark, as it were, all of the points of space, without much disturbance of the processes being studied (here we make use of the smallness of gravitational effects).

Thus for the derivation of conditions of microcovariance which take into account the pseudoeuclidean nature of space at all points it is necessary to consider transformations to various curvilinear coordinate systems, and it is useful to consider small variations of the metric.

3. CONDITIONS OF MICROCOVARIANCE OF QUANTUM THEORY

We shall give a mathematical formulation of the arguments developed in the preceding section.

The following fundamental propositions are taken as a starting point:

1. The theory is a quantum theory. This means that the states of physical systems are described by state vectors Ψ, which depend on certain variables, and that the equations of motion are linear in Ψ and the physical quantities are bilinear in Ψ^* and Ψ, i.e., are averages of operators.

2. Space-time is pseudoeuclidean at all points. Besides flat space-time, we shall also consider for auxiliary purposes the more general case of a four-dimensional space which is pseudoeuclidean at infinity (in space and in both directions of time) and is a Riemannian space with a fixed metric at finite distances.

3. There exists a scattering matrix S which takes the initial state Ψ_i into the final state Ψ_f:

$$\Psi_f = S\Psi_i. \tag{6}$$

Here it is understood that at infinity in time there remain only free stable particles which are infinitely far separated in space. No distinction between elementary and compound particles is made here. Our treatment, unlike the usual one, does not associate asymptotic <u>in</u> and <u>out</u> fields with these stable particles.

To these propositions we may subsequently add the conditions of microcausality and of positiveness of the energy eigenvalues (cf. Sec. 4). It is very probable, however, that these conditions are closely connected with each other and that they are both consequences of the three fundamental propositions stated above.

Proposition (3) includes within itself the conditions (1) of macroscopic invariance (but, of course, does not reduce to them). Therefore it has the consequence that the state of a system of n free particles with the masses $\kappa_1, \ldots, \kappa_n$ and the spins $s_1, \ldots, s_n$ is described by a state vector

$$\Psi^{\varkappa_1, \ldots, \varkappa_n;\, s_1, \ldots, s_n,\, \alpha}_{m_1, \ldots, m_n}(\mathbf{p}_1, \ldots, \mathbf{p}_n), \tag{7}$$

in which the variables are the three-dimensional momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$, the spin projections $m_1, \ldots, m_n$, and possibly some other invariant variables (for example, charges), which are denoted by the index α (cf. e.g., [8]). This is, in particular, the structure of the state vectors Ψ_i and Ψ_f of Eq. (6).

For the general case of a fixed metric which goes to the pseudoeuclidean metric at infinity the S matrix depends on the variables of the initial state i and those of the final state f, and is a functional of the metric tensor $g^{\mu\nu}(x)$ (here it is more convenient to consider the contravariant tensor); that is, it is of the form

$$\langle f \,|\, S\,[g^{\mu\nu}(x)] \,|\, i\rangle. \tag{8}$$

We emphasize that here, as before, the sets of variables i, f describe the free motion.

An infinitesimal transformation of the metric

$$g^{\mu\nu}(x) = g'^{\mu\nu}(x) + \delta g^{\mu\nu}(x), \tag{9}$$

where $\delta g^{\mu\nu}(x) \to 0$ at infinity, corresponds to a transformation of the S matrix with respect to the functional variable $g^{\mu\nu}(x)$ only:

$$S = \left(1 - \int d^4x\, \delta g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)}\right) S'. \tag{10}$$

Let us now introduce operators $T^i_{\mu\nu}(x)$, $T^f_{\mu\nu}(x)$, defining them by the relations

$$T^i_{\mu\nu}(x) = \frac{2}{i} S^+ \frac{\delta S}{\delta g^{\mu\nu}(x)} \frac{1}{\sqrt{-g}}, \quad T^f_{\mu\nu}(x) = \frac{2}{i} \frac{\delta S}{\delta g^{\mu\nu}(x)} S^+ \frac{1}{\sqrt{-g}}, \tag{11}$$

where g is the determinant of the components of the covariant metric tensor. In what follows we shall sometimes for simplicity denote the operator $T^i_{\mu\nu}(x)$ by simply $T_{\mu\nu}(x)$. It follows from Eq. (11) that

$$S T^i_{\mu\nu}(x) S^+ = T^f_{\mu\nu}(x), \tag{12}$$

so that the operators $T^i_{\mu\nu}$ and $T^f_{\mu\nu}$ describe the same physical quantity in the representations of the initial and final state vectors—that is, their matrix elements are of the forms

$$\langle i \,|\, T^i_{\mu\nu}(x) \,|\, i'\rangle, \quad \langle f \,|\, T^f_{\mu\nu}(x) \,|\, f'\rangle. \tag{13}$$

By means of Eq. (11) we can bring the transformation to a form in which the transformation operator acts not on the functional variable $g^{\mu\nu}(x)$, but on the variables of the initial or final state:

$$\begin{aligned} S &= S' \left\{1 - \frac{i}{2}\int d^4x \sqrt{-g}\, \delta g^{\mu\nu}(x)\, T^i_{\mu\nu}(x)\right\} \\ &= \left\{1 - \frac{i}{2}\int d^4x \sqrt{-g}\, \delta g^{\mu\nu}(x)\, T^f_{\mu\nu}(x)\right\} S'. \end{aligned} \tag{14}$$

Instead of the tensor $T_{\mu\nu}(x)$ itself it is frequently more convenient to consider the corresponding tensor density $\tau_{\mu\nu}(x)$:

$$\tau_{\mu\nu}(x) = \sqrt{-g}\, T_{\mu\nu}(x). \tag{15}$$

In pseudoeuclidean space we of course have $(-g)^{1/2} = 1$ and $\tau_{\mu\nu}(x) = T_{\mu\nu}(x)$.

We shall now show that $T_{\mu\nu}(x)$ is the operator for the energy-momentum tensor of the system, i.e., that it is conserved and is connected by the appropriate integral relations with the total four-momentum and four-dimensional angular momentum. To prove the conservation of $T_{\mu\nu}(x)$ we consider the transformation of the tensor $g^{\mu\nu}(x)$ under the infinitesimal coordinate transformation (4):

$$g^{\mu\nu}(x) = g'^{\mu\nu}(x) + g'^{\mu\lambda}(x)\frac{\partial\xi^{\nu}}{\partial x^{\lambda}} + g'^{\lambda\nu}(x)\frac{\partial\xi^{\mu}}{\partial x^{\lambda}} - \frac{\partial g'^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}. \quad (16)$$

If $\xi^{\mu}(x) \to 0$ at infinity, this coordinate transformation will change neither the state vectors Ψ_i, Ψ_f nor the matrix S which connects them (the scattering cross section does not change owing to our bending the coordinate system somewhere in between).

When we substitute Eq. (16) in Eq. (14) and use the fact that the S matrix must be unchanged, we get

$$S = S\left\{1 - i\int d^4x\sqrt{-g}\,\Big(g^{\nu\lambda}(x)\frac{\partial\xi^{\mu}(x)}{\partial x^{\lambda}}T^{i}_{\mu\nu}(x) - \frac{1}{2}\frac{\partial g^{\lambda\nu}(x)}{\partial x^{\mu}}\xi^{\mu}(x)\,T^{i}_{\nu\lambda}(x)\Big)\right\}. \quad (17)$$

When we now integrate by parts and use the arbitrariness of $\xi^{\mu}(x)$, we have

$$\frac{\partial}{\partial x^{\lambda}}\left(g^{\nu\lambda}(x)\,T_{\mu\nu}(x)\sqrt{-g}\right) + \frac{1}{2}\frac{\partial g^{\lambda\nu}(x)}{\partial x^{\mu}}T_{\nu\lambda}(x)\sqrt{-g} = 0. \quad (18)$$

The left member of Eq. (18) is the covariant derivative $T^{\nu}_{\mu;\nu}$ of the mixed tensor $T^{\nu}_{\mu} = g^{\nu\lambda}T_{\mu\lambda}$, so that we can write the equation in the form

$$T^{\nu}_{\mu;\,\nu} = 0. \quad (19)$$

In flat space covariant derivatives become ordinary derivatives and Eq. (18) reduces to

$$\partial T_{\mu\nu}(x)/\partial x^{\nu} = 0. \quad (20)$$

Thus the conservation of $T_{\mu\nu}(x)$ has been proved.

Let us now consider a coordinate transformation (4) such that

$$\xi^{\mu}(x) \to 0 \ \text{for}\ x^0 \to +\infty,$$
$$\xi^{\mu}(x) \to \xi^{\mu}_0 = \text{const}\ \text{for}\ x^0 \to -\infty. \quad (21)$$

In this case the relation (17) is altered in two respects. First, owing to the fact that the shift ξ^{μ}_0 has occurred at minus infinity the initial state vector Ψ_i is subjected to the transformation

$$\Psi_i = (1 + i\xi^{\mu}_0 P_{\mu})\,\Psi'_i, \quad (22)$$

which leads to a corresponding transformation of the S matrix with respect to the variables of the initial state i. Second, when we integrate by parts in the transformation for S there remains an integral over the hypersurface in the infinite past:

$$\int d^4x\sqrt{-g}\left(g^{\nu\lambda}\frac{\partial\xi^{\mu}}{\partial x^{\lambda}}T_{\mu\nu} - \frac{1}{2}\frac{\partial g^{\lambda\nu}}{\partial x^{\mu}}\xi^{\mu}T_{\nu\lambda}\right) = \int_{x^0\to-\infty} d\sigma_{\lambda}\,g^{\nu\lambda}\xi^{\mu}_0 T_{\mu\nu}\sqrt{-g}. \quad (23)$$

[The other terms after the integration by parts are zero because of Eq. (19).] The resulting change of the matrix on account of the combined action of the transformations (22) and (23) must be zero:

$$S\left(1 - i\xi^{\mu}_0 P_{\mu})(1 + i\xi^{\mu}_0\int d\sigma_{\lambda}T_{\mu\nu}g^{\nu\lambda}\sqrt{-g}\right) = S. \quad (24)$$

When we use the facts that ξ^{μ}_0 is arbitrary and that the space becomes flat for $x^0 \to -\infty$, we find from this that

$$\int T_{\mu\nu}(x)\,d\sigma_{\nu} = P_{\mu}. \quad (25)$$

In an analogous way, if we suppose that for $x^0 \to -\infty$ we take $\xi_{\mu} \to \epsilon_{\mu\nu}x^{\nu}$, where $\epsilon_{\mu\nu}$ is a small constant antisymmetric tensor, we can obtain the relation

$$\int (x_{\mu}T_{\nu\lambda} - x_{\nu}T_{\mu\lambda})\,d\sigma_{\lambda} = M_{\mu\nu}. \quad (26)$$

The relations (20), (25), and (26) prove the correctness of the definition (11) for the operator for the energy-momentum tensor.

We can now formulate the conditions of microcovariance of the S matrix: if space-time is everywhere pseudoeuclidean, there exists an operator $T_{\mu\nu}(x)$ which satisfies the conservation law (20) and determines the distribution of matter in space and time.

The conditions we have found are not exhaustive. New relations can be obtained, for example, by considering the covariant properties of the operator $T_{\mu\nu}(x)$. Under the coordinate transformation (4) the average $\overline{T}_{\mu\nu}(x)$ of the tensor operator $T_{\mu\nu}(x) \equiv T^{i}_{\mu\nu}(x)$,

$$\overline{T}_{\mu\nu}(x) = \langle\Psi^{+}_i T_{\mu\nu}(x)\,\Psi_i\rangle \quad (27)$$

transforms as a covariant tensor of the second rank:

$$\overline{T}_{\mu\nu}(x) = \frac{\partial x'^{\lambda}}{\partial x^{\mu}}\frac{\partial x'^{\varepsilon}}{\partial x^{\nu}}\overline{T}'_{\lambda\varepsilon}(x')\,| = \overline{T}'_{\mu\nu}(x) - \frac{\partial\xi^{\lambda}(x)}{\partial x^{\mu}}\overline{T}'_{\lambda\nu}(x) - \frac{\partial\xi^{\lambda}(x)}{\partial x^{\nu}}\overline{T}'_{\mu\lambda}(x) - \frac{\partial\overline{T}'_{\mu\nu}(x)}{\partial x^{\lambda}}\xi^{\lambda}(x). \quad (28)$$

The operator $T_{\mu\nu}(x)$ depends on the left and on the right on the variables of the initial state, and in addition is a function of the point x and a functional of the metric tensor $g^{\mu\nu}(x)$ —that is, it is of the form

$$\langle i \mid T_{\mu\nu} [x, g^{\mu\nu}(x)] \mid i' \rangle. \qquad (29)$$

On the other hand, if $\xi^\mu(x) \to 0$ for $x^0 \to -\infty$, the state vector Ψ_i is unchanged under the action of the coordinate transformation (4), so that the entire change of the operator $T_{\mu\nu}(x)$ under this transformation comes through the functional variable $g^{\alpha\beta}(x)$. For this same reason the equation (28) can also be regarded as an operator equation.

The change of $g^{\alpha\beta}(x)$ under the action of the transformation (4) is written out in Eq. (16), and the change of $T_{\mu\nu}(x)$ when there is a change of $g^{\alpha\beta}(x)$ is obviously given by the relation

$$T_{\mu\nu}(x) = \left(1 - \int d^4y\, \delta g^{\alpha\beta}(y) \frac{\delta}{\delta g^{\alpha\beta}(y)}\right) T''_{\mu\nu}(x). \qquad (30)$$

Substitution of Eq. (16) in Eq. (30) and integration by parts gives

$$T_{\mu\nu}(x) = \Big\{1 - 2\int d\sigma_\alpha \xi^\varepsilon(y) g^{\lambda\alpha}(y) \frac{\delta}{\delta g^{\lambda\varepsilon}(y)} + 2\int d^4y \xi^\varepsilon(y) \frac{\partial}{\partial y^\alpha}\left(g^{\lambda\alpha}(y)\frac{\delta}{\delta g^{\lambda\varepsilon}(y)}\right) + \int d^4y \xi^\varepsilon(y) \frac{\partial g^{\lambda\alpha}(y)}{\partial y^\varepsilon} \frac{\delta}{\delta g^{\lambda\alpha}(y)}\Big\} T''_{\mu\nu}(x). \qquad (31)$$

The condition for microcovariance of the tensor $T_{\mu\nu}(x)$ is now

$$T'_{\mu\nu}(x) = T''_{\mu\nu}(x), \qquad (32)$$

where $T'_{\mu\nu}(x)$ is given by the relation (28) in operator form. When we now use the arbitrariness of $\xi^\mu(x)$ and confine ourselves to the case in which $\xi^\mu(x) \to 0$ in all directions, we can put Eq. (32) in the form

$$2\frac{\partial}{\partial y^\alpha}\left(g^{\varepsilon\alpha}(y)\frac{\delta T_{\mu\nu}(x)}{\delta g^{\lambda\varepsilon}(y)}\right) + \frac{\partial g^{\varepsilon\alpha}(y)}{\partial y^\lambda}\frac{\delta T_{\mu\nu}(x)}{\delta g^{\varepsilon\alpha}(y)} = \left(T_{\lambda\nu}(x)\frac{\partial}{\partial x^\mu} + T_{\mu\lambda}(x)\frac{\partial}{\partial x^\nu} + \frac{\partial T_{\mu\nu}(x)}{\partial x^\lambda}\right)\delta^4(x-y). \qquad (33)$$

The functional derivative $\delta T_{\mu\nu}(x)/\delta g^{\lambda\epsilon}(y)$ is connected with the components $T_{\mu\nu}(x)$ themselves through the condition of unitarity of the S matrix. We can establish this connection by using the definition (11) and the identity

$$dg = g g^{\mu\nu} dg_{\mu\nu} = -g g_{\mu\nu} dg^{\mu\nu}. \qquad (34)$$

It follows from Eqs. (11) and (34) that

$$\frac{1}{\sqrt{-g(y)}}\frac{\delta T_{\mu\nu}(x)}{\delta g^{\lambda\sigma}(y)} = -\frac{i}{2} T_{\lambda\sigma}(y) T_{\mu\nu}(x) + \frac{2}{i}\frac{1}{\sqrt{-g(x)}\sqrt{-g(y)}} S^+ \frac{\delta^2 S}{\delta g^{\mu\nu}(x)\delta g^{\lambda\sigma}(y)} - \frac{1}{2\sqrt{-g}} T_{\mu\nu}(x) g_{\lambda\sigma}\delta(x-y). \qquad (35)$$

When we now write out the analogous relation for $(-g(x))^{-1/2}\delta T_{\lambda\sigma}(y)/\delta g^{\mu\nu}(x)$ and subtract one equation from the other we get

$$2\left(\frac{1}{\sqrt{-g(y)}}\frac{\delta T_{\mu\nu}(x)}{\delta g^{\lambda\sigma}(y)} - \frac{1}{\sqrt{-g(x)}}\frac{\delta T_{\lambda\sigma}(y)}{\delta g^{\mu\nu}(x)}\right) = i[T_{\mu\nu}(x), T_{\lambda\sigma}(y)] + \frac{1}{\sqrt{-g}}\delta(x-y)\left(T_{\lambda\sigma}(y) g_{\mu\nu} - T_{\mu\nu}(x) g_{\lambda\sigma}\right). \qquad (36)$$

The relation (36) takes a somewhat simpler form if we write it in terms of the tensor densities (15):

$$2\left(\frac{\delta\tau_{\mu\nu}(x)}{\delta g^{\lambda\sigma}(y)} - \frac{\delta\tau_{\lambda\sigma}(y)}{\delta g^{\mu\nu}(x)}\right) = i[\tau_{\mu\nu}(x), \tau_{\lambda\sigma}(y)]. \qquad (37)$$

The relations (33) determine the microcovariant properties of the operator $T_{\mu\nu}(x)$. For flat space these are of the form

$$\frac{\partial}{\partial y^\alpha}\frac{\delta\tau_{\mu\nu}(x)}{\delta g^{\lambda\alpha}(y)} = -\frac{1}{2}\left\{T_{\lambda\nu}(x)\frac{\partial}{\partial x^\mu} + T_{\lambda\mu}(x)\frac{\partial}{\partial x^\nu} + \frac{\partial T_{\mu\nu}(x)}{\partial x^\lambda}\right\}\delta(x-y). \qquad (38)$$

Further relations of microcovariance and unitarity can be obtained in analogous ways.

4. THE CONDITIONS OF MICROCAUSALITY

The general formulation of the condition of microcausality is that any perturbation applied at a four-dimensional point affects the distribution of matter only in the upper light cone of that point. We can take as the perturbation a change of the metric, since it affects the motion of all forms of matter (for example, a change of the electromagnetic field affects only the motion of charged particles). Since the distribution of matter is described by the energy-momentum tensor, it is obvious that the desired condition of microcausality is of the form

$$\delta\tau_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y) = \delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y) = 0 \qquad (39)$$

in the upper light cone of the point y, i.e., for

$$y^0 > x^0, \quad (y-x)^2 \equiv (\mathbf{x}-\mathbf{y})^2 - (x^0 - y^0)^2 > 0. \qquad (40)$$

If follows from Eqs. (39) and (37) that the causality condition can be written in the form

$$[T_{\mu\nu}(x), T_{\lambda\sigma}(y)] = 0 \text{ for } (x-y)^2 > 0. \qquad (41)$$

We note that in Eqs. (39) and (41) the commutator is not zero for x = y, i.e., the right members of these relations are not zero, but certain quasi-local (cf. [1]) operators.

The condition (39) is of the form of the Bogolyubov causality condition, [1] but is written for the "gravitational" current $T_{\mu\nu}(x)$, which gives it universal applicability, since all types of matter have inertial properties.

There is one further physical condition which it is reasonable to impose on the theory—the condition that the eigenvalues of the energy density are positive. It can be written in the form

$$\langle \Psi^* T_{00}(x) \Psi \rangle \geqslant 0 \qquad (42)$$

at all points and for all state vectors. Obviously the relation (42) is closely connected with causality. At present, however, it is not clear just what is the relation between the conditions (39) and (42). It is not excluded, for example, that they are not only equivalent, but even superfluous, in the sense that the conditions of microcovariance and of freedom from internal contradictions inevitably lead to the existence of causality.

Integrating Eq. (38) with respect to d^4y over a convex four-volume containing the point x, and using Eq. (39), we get

$$\frac{\partial T_{\mu\nu}(x)}{\partial x^{\lambda}} = i\,[T_{\mu\nu}(x), \int T_{\lambda\varepsilon}(y)\, d\sigma_{\varepsilon}], \qquad (43)$$

where the integral is taken over only the part of the hypersurface which is inside the lower light cone of the point x. This integral can of course also be extended to an infinite spacelike hypersurface, which leads to the relation

$$[T_{\mu\nu}(x),\ P_{\lambda}] = -i\partial T_{\mu\nu}(x)/\partial x^{\lambda}. \qquad (44)$$

In an analogous way one can obtain the commutation relations between $T_{\mu\nu}(x)$ and the four-dimensional angular-momentum operator:

$$[T_{\mu\nu}(x),\ M_{\lambda\sigma}] = x^{\sigma}\frac{1}{i}\frac{\partial T_{\mu\nu}(x)}{\partial x^{\lambda}} - x^{\lambda}\frac{1}{i}\frac{\partial T_{\mu\nu}(x)}{\partial x^{\sigma}}$$
$$- i\,(T_{\sigma\nu}\delta_{\mu\lambda} - T_{\lambda\nu}\delta_{\mu\sigma} + T_{\mu\sigma}\delta_{\lambda\nu} - T_{\nu\sigma}\delta_{\mu\lambda}). \qquad (45)$$

The commutators (44) and (45) define the usual covariance properties of $T_{\mu\nu}(x)$ in flat space.

For applications to flat space the main results of the arguments given so far are expressed by the relations (11), (20), (25), (26), (35), (37)—(39), (41)—(44). It must be pointed out that the relations of microcovariance and microcausality have been written out for the operator of the energy-momentum tensor $T_{\mu\nu}(x)$. Although this tensor indeed has a clear physical meaning, it is impossible in practice to measure it in microscopic regions of space. Therefore the problem arises of going over from the tensor $T_{\mu\nu}(x)$ to quantities accessible to direct measurements, i.e., to the S matrix. It is desirable that this transition be made exactly, without bringing in any propositions of quantum field theory.

5. THE PHYSICAL INTERPRETATION

Our purpose now is to solve the problem of the connection between the quantities described by space-time relations in relativistic quantum theory and quantities accessible to direct experimental measurement. Up to now this important problem has been solved only for individual special cases (cf. e.g., [1-3]), and it has been essential to bring in certain propositions of quantum field theory, for which the limits of applicability are not clear.

Three types of physical quantities are used at present in relativistic quantum theory:

a) momenta, spin projections, and variables of the type of charge, for individual particles;

b) local quantum fields;

c) currents.

A virtue of quantities of class a) is that asymptotically (i.e., for $x_0 \to \pm\infty$) they are integrals of the motion. Therefore they are accessible to direct experimental observation—that is, they can serve as the variables of the S matrix. These quantities, however, have an extremely important shortcoming: they (like the concept of the number of particles itself) have meaning only asymptotically, at infinity in the time. Therefore with momenta and spin projections of individual particles one can describe not the interaction process itself, but only its results.

Quantities of class b) (local quantized fields) of course exist at all times and thus describe the details of the interaction process. These fields, however, cannot be directly measured in experiments with elementary particles, since they are not asymptotic integrals of the motion (the values of the fields at various points cannot be variables of the S matrix). Therefore in working with quantized fields one must know how to go over from these fields to the momenta and spin projections of individual particles. The recipes available here are essentially based on assumptions of local field theory, for which the domain of applicability is evidently restricted to processes which do not involve the structural properties of the particles.

Like the fields, the current operators [quantities of class c)] describe details of the interaction process, but are not directly measurable quantities. From the currents, however, one can already obtain some experimentally measurable quantities, which we shall call static moments, without bringing in quantum field theory. Such quantities are the charge, the magnetic moment, the mean-square radius of the charge distribution, and other such single-particle characteristics. Then from a given matrix element $\langle p | j_\mu(x) | p' \rangle$ of the current operator between two single-particle states one can reconstruct the spatial distribution of the charges and currents inside the particle. As applied to the scattering of two particles, however, the study of the static moments already gives nothing, since the magnetic moment, the mean-square radius, and other such quantities are not asymptotic integrals of the motion for a system of two interacting particles.

Therefore recipes for obtaining scattering amplitudes from currents depend in an essential way on quantum field theory (cf. e.g., [1]). These recipes not only are of restricted applicability, but also are not complete, since in them one uses only the Fourier components of the currents on the mass shell, whereas from the correspondence principle it follows that the current operators also have meaning off the mass shell (this incompleteness corresponds in particular to the appearance of the "unphysical" region in dispersion relations). In fact, from the point of view of the correspondence principle a matrix element $\langle p_1, p_2 | j_\mu(x) | p_1', p_2' \rangle$, for example, specified between two-particle states (at minus infinity in the time), contains an immense amount of information, since it describes the distribution of charges and currents at all points of space-time for reactions at arbitrary energies which begin with the collision of the two particles. Thus the prescription of this operator must determine the angular distributions and the electromagnetic structure in all of the particles produced in the reactions and possessing any electromagnetic properties. Meanwhile, by means of existing methods one can obtain from this matrix element only the amplitude of the reaction of the production of one photon in the collision of the two particles, and this with an accuracy restricted by the limits on the applicability of field-theoretical methods.

As will be shown below, from a prescribed two-particle matrix element one can extract much broader physical information (elastic-scattering cross section, inelastic-scattering cross section, electromagnetic structures of particles produced in the collision of the particles), if one brings into the theory new physical quantities called dynamic moments. The dynamic moments are a natural generalization of the static moments (see above) to the case of a system of several colliding and emerging particles and, as will be shown in Sec. 7, are the most convenient physical quantities for the description of relativistic quantum processes.

The introduction of the dynamic moments is especially important for the operator $T_{\mu\nu}(x)$ of the energy-momentum tensor, since, as was shown in Secs. 3 and 4, it is in terms of this tensor that the conditions of microcovariance and microcausality of quantum theory are expressed.

Direct measurements of the components $\hat{T}_{\mu\nu}(x)$ at various points are in practice impossible. Therefore to obtain from the conditions of microcovariance and microcausality consequences accessible to experimental verification it is necessary to solve the nontrivial problem of obtaining from $T_{\mu\nu}(x)$ the largest possible number of directly measurable quantities. In other words, it is necessary to establish the connection between $T_{\mu\nu}(x)$ and the scattering matrix. Here we cannot use the connection between $T_{\mu\nu}(x)$ and S through the definition (11), since we are concerned with the operators S and $T_{\mu\nu}(x)$ at only one point of the functional space of the metric tensor—for the flat metric.

In the spirit of present quantum field theory one would have to call the operator $T_{\mu\nu}(x)$ the "gravitational current," and treat the relations of covariance and causality for $T_{\mu\nu}(x)$ as some sort of restrictions on processes of the type of the scattering of a graviton by a nucleon (cf. e.g., [1-3]). This sort of interpretation will not be used, however, for two reasons. First, the validity of the relations connecting the elements of the S matrix with the commutators of the currents is restricted by the limited applicability of certain postulates of quantum field theory. Second, processes involving gravitons are far beyond the scope of present experimental techniques.

Therefore we shall deal with the tensor $T_{\mu\nu}(x)$ only in its direct meaning, i.e., as the operator for the corresponding physical quantities, defined at a point, and shall use only its inertial (and not its gravitational!) properties. To extract physical information from the matrix elements of this tensor we shall introduce the corresponding static and dynamic moments.

6. PARAMETRIZATION OF THE ENERGY-MOMENTUM TENSOR AND OF OTHER CURRENTS

The matrix elements of the tensor $T_{\mu\nu}(x)$ [or of another current, for example $j_\mu(x)$] depend on a number of Lorentz-noninvariant kinematical variables: the momenta and spin projections of individual particles. Therefore the problem arises of parametrization of $T_{\mu\nu}(x)$, i.e., of its conversion into some set of invariant functions (usually called form-factors). The methods for parametrization of current operators do not differ in principle from the set of methods for parametrization of the S matrix, which have been worked out in detail (cf., e.g., [9]). This matter will be treated in detail in a separate paper. Here we shall confine ourselves to a few of the simplest cases.

First of all we note that if the four-momentum on the left is P_λ and that on the right is P'_λ, then according to Eq. (44),

$$\langle P \mid T_{\mu\nu}(x) \mid P'\rangle = e^{ix_\lambda (P'_\lambda - P_\lambda)} \langle P \mid T_{\mu\nu}(0) \mid P'\rangle. \quad (46)$$

Similarly

$$\langle P \mid j_\mu(x) \mid P'\rangle = e^{ix_\lambda (P'_\lambda - P_\lambda)} \langle P \mid j_\mu(0) \mid P'\rangle. \quad (47)$$

The conservation condition (20) and the analogous relation for $j_\mu(x)$ take the forms

$$(P'_\nu - P_\nu) \langle P \mid T_{\mu\nu}(0) \mid P'\rangle = 0, \quad (48)$$

$$(P'_\mu - P_\mu) \langle P \mid j_\mu(0) \mid P'\rangle = 0. \quad (49)$$

It is not hard to show that for a small number of spinless particles with masses $\kappa_1, \kappa_2, \ldots$ and momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$ the operators $T_{\mu\nu}(0)$ and $j_\mu(0)$ that satisfy Eqs. (48) and (49) can be parametrized in the following way:

a) no particles on the right nor on the left:

$$\langle 0 \mid T_{\mu\nu}(x) \mid 0\rangle = 0, \quad \langle 0 \mid j_\mu(x) \mid 0\rangle = 0; \quad (50)$$

b) one particle on the left, none on the right:

$$\langle \mathbf{p} \mid T_{\mu\nu}(0) \mid 0\rangle = (2\pi)^{-3} (2E)^{-1/2} (p_\mu p_\nu + \delta_{\mu\nu}\varkappa^2) f, \quad (51)$$

$$\langle \mathbf{p} \mid j_\mu(0) \mid 0\rangle = 0, \quad (52)$$

where $f = \text{const}$, $E = (\mathbf{p}^2 + \kappa^2)^{1/2}$, $p_4 = iE$;

c) one particle on the left and one on the right (i.e., the operator for one particle):

$$\langle \mathbf{p} \mid T_{\mu\nu}(0) \mid \mathbf{p}'\rangle = (2\pi)^{-3} (4EE')^{-1/2} \{\{p_\mu p'_\nu + p'_\mu p_\nu - \delta_{\mu\nu} (p_\lambda p'_\lambda + \varkappa^2)\} F_1(t) + \{(p_\mu - p'_\mu)(p_\nu - p'_\nu) + 2\delta_{\mu\nu} (p_\lambda p'_\lambda + \varkappa^2)\} F_2(t)\}, \quad (53)$$

$$\langle \mathbf{p} \mid j_\mu(0) \mid \mathbf{p}'\rangle = e (2\pi)^{-3} (4EE')^{-1/2} (p_\mu + p'_\mu) F(t), \quad (54)$$

where $t = -(p_\mu - p'_\mu)^2 = 2(\kappa^2 + p_\mu p'_\mu)$, and F_1, F_2, F are invariant form-factors. From the conditions that the integrals of $T_{\mu 0}(x)$ and $j_0(x)$ over three-dimensional space must give respectively the four-momentum p_μ and the charge e, we find that

$$F_1(0) = 1, \quad F(0) = 1; \quad (55)$$

d) two particles on the left, none on the right:

$$\langle \mathbf{p}\mathbf{p}' \mid T_{\mu\nu}(0) \mid 0\rangle = (2\pi)^{-3} (4EE')^{-1/2} \{-\{p_\mu p'_\nu + p'_\mu p_\nu + \delta_{\mu\nu} (\varkappa^2 - p_\lambda p'_\lambda)\} F'_1(t) + \{(p_\mu + p'_\mu)(p_\nu + p'_\nu) + 2\delta_{\mu\nu} (\varkappa^2 - p_\lambda p'_\lambda)\} F'_2(t)\}, \quad (56)$$

$$\langle \mathbf{p}\mathbf{p}' \mid j_\mu(0) \mid 0\rangle = e (2\pi)^{-3} (4EE')^{-1/2} (p_\mu - p'_\mu) F'(t), \quad (57)$$

where $t = -(p + p')^2$.

If crossing symmetry exists, then the form-factors F'_1, F'_2, F' are the respective analytic continuations of F_1, F_2, F. It is not obvious beforehand whether or not crossing symmetry follows from the conditions of microcovariance and microcausality in the general case of particles which have structure. The connection does exist for free particles described by local quantized fields. For example, for a scalar field $\varphi(x)$ the Lagrangian function L is

$$L = -\frac{1}{2}\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} - \frac{1}{2}\varkappa^2\varphi^2 + \alpha\frac{\partial}{\partial x_\mu}\left(\varphi\frac{\partial\varphi}{\partial x_\mu}\right), \quad (58)$$

and the corresponding energy-momentum tensor is of the form (53) and (56), where $F_1 = F'_1 = 1$ and $F_2 = F'_2 = \alpha$. Here crossing symmetry is necessary for the validity of the microcausality condition (41).

We can make one further remark concerning the Lagrangian (58). The last term is a complete divergence, and thus does not affect the equations of motion. It follows from Eq. (53), however, that this term is not physically unimportant. It has an effect, for example, in the scattering of a particle in an external gravitational field. The physical meaning of the form-factor $F_2(t)$ can be found from an examination of the various static moments (i.e., quantities of the type of moment of inertia) of the particle.

To conclude this section let us consider the more complicated case of the parametrization of $T_{\mu\nu}(x)$ and $j_\mu(x)$ between two two-particle states:

$$\langle \mathbf{p}_1\mathbf{p}_2 \mid T_{\mu\nu}(x) \mid \mathbf{p}'_1\mathbf{p}'_2\rangle = \exp\{ix_\lambda (p'_{1\lambda} + p'_{2\lambda} - p_{1\lambda} - p_{2\lambda})\} \times (2\pi)^{-3} (16E_1E_2E'_1E'_2)^{-1/2} \langle \mathbf{p}_1\mathbf{p}_2 \mid F_{\mu\nu} \mid \mathbf{p}'_1\mathbf{p}'_2\rangle, \quad (59)$$

$$\langle \mathbf{p}_1\mathbf{p}_2 \mid j_\mu(x) \mid \mathbf{p}'_1\mathbf{p}'_2\rangle = \exp\{ix_\lambda (p'_{1\lambda} + p'_{2\lambda} - p_{1\lambda} - p_{2\lambda})\} \times (2\pi)^{-3} (16E_1E_2E'_1E'_2)^{-1/2} \langle \mathbf{p}_1\mathbf{p}_2 \mid G_\mu \mid \mathbf{p}'_1\mathbf{p}'_2\rangle. \quad (60)$$

Generally speaking, the form-factors in this case will depend on six invariants s, s′, t, t′, u, u′, which can be chosen in the following way:

$$s = -(p_1 + p_2)^2 = \varkappa_1^2 + \varkappa_2^2 - 2p_1p_2,$$

$$s' = -(p'_1 + p'_2) = \varkappa_1^2 + \varkappa_2^2 - 2p'_1p'_2,$$

$$t = -(p_1 - p'_1)^2 = 2\varkappa_1^2 + 2p_1p'_1,$$

$$t' = -(p_2 - p_2')^2 = 2\varkappa_2^2 + 2p_2p_2',$$
$$u = -(p_1 - p_2')^2 = \varkappa_1^2 + \varkappa_2^2 + 2p_1p_2',$$
$$u' = -(p_2 - p_1')^2 = \varkappa_1^2 + \varkappa_2^2 + 2p_1'p_2. \quad (61)$$

Besides the invariants listed in Eq. (61), there are also two singular invariants:

$$2E_1\delta^3(\mathbf{p}_1 - \mathbf{p}_1') = \text{inv}, \qquad 2E_2\delta^3(\mathbf{p}_2 - \mathbf{p}_2') = \text{inv}. \quad (62)$$

The matrix elements $F_{\mu\nu}$ and G_μ in Eqs. (59) and (60) break up into sums of three terms, corresponding to the free first particle, the free second particle, and the interaction

$$F_{\mu\nu} = F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} + F_{\mu\nu}^{int}, \quad (63)$$
$$G_\mu = G_\mu^{(1)} + G_\mu^{(2)} + G_\mu^{int}. \quad (64)$$

This separation follows from the fact that the free currents and the interaction currents are singular in different ways. Namely, the free currents contain the three-dimensional δ functions (62), and, as will be shown below, the interaction currents contain only the one-dimensional δ function. The result is that the conservation laws (48), (49) are satisfied by each term separately, and not just by their sum.

The general form of the free currents is determined by the relations (53), (54). For example,

$$\langle \mathbf{p}_1\mathbf{p}_2 \mid G_\mu^{(1)} \mid \mathbf{p}_1'\mathbf{p}_2' \rangle = 2e_1E_2\delta^3(\mathbf{p}_2 - \mathbf{p}_2')(p_{\mu 1} + p_{\mu 1}')\, G^{(1)}(t). \quad (65)$$

It is not clear whether, for example, the invariants that depend on the variables of the second particle can (or perhaps must) be involved in $F^{(1)}$. There is the condition that the total $T_{00}(x)$ must be positive, which must be satisfied,[4] and a number of further conditions. For example, it is obvious that $T_{\mu\nu}^{int}(x)$ can have nonzero values only at points where $T_{\mu\nu}^{(1)}(x)$ and $T_{\mu\nu}^{(2)}(x)$ are not zero. Questions of this type need further investigation.

To obtain the general form of $F_{\mu\nu}^{int}$, G_μ^{int} it is convenient to go over from the four-momenta p_1, p_2, p_1', p_2' to four combinations of these quantities:

$$K = p_1 + p_2 - p_1' - p_2', \qquad K' = -(p_1 + p_2 + p_1' + p_2'),$$
$$K'' = -p_1 + p_1' + p_2 - p_2', \quad K''' = -p_1 - p_1' + p_2 + p_2'. \quad (66)$$

As applied to the interaction currents, the conservation laws (48), (49) can now be written in the form

$$K_\nu F_{\mu\nu}^{int} = 0, \qquad K_\mu G_\mu^{int} = 0. \quad (67)$$

It follows from considerations of relativistic invariance that if we do not take the relations (67) into account the most general forms of the operators $F_{\mu\nu}^{int}$, G_μ^{int} are

$$F_{\mu\nu}^{int} = \delta_{\mu\nu}F_1 + K_\mu'K_\nu'F_2 + K_\mu''K_\nu''F_3 + K_\mu'''K_\nu'''F_4$$
$$+ (K_\mu'K_\nu'' + K_\nu'K_\mu'')F_5 + (K_\mu'K_\nu''' + K_\nu'K_\mu''')F_6$$
$$+ (K_\mu''K_\nu''' + K_\nu''K_\mu''')F_7 + K_\mu K_\nu F_8 + (K_\mu K_\nu' + K_\nu K_\mu')F_9$$
$$+ (K_\mu K_\nu'' + K_\nu K_\mu'')F_{10} + (K_\mu K_\nu''' + K_\nu K_\mu''')F_{11}, \quad (68)$$
$$G_\mu^{int} = K_\mu f + K_\mu' f_1 + K_\mu'' f_2 + K_\mu''' f_3, \quad (69)$$

where $F_1, \ldots, F_{11}$, f, f_1, f_2, f_3 are invariant functions (form-factors), each of which depends on the six invariants (61). It follows from the conservation laws (67) that these form-factors are not all independent, but are connected by the relations

$$F_1 + K^2F_8 + (s - s')F_9 + (t - t')F_{10} + (u - u')F_{11} = 0,$$
$$(s - s')F_2 + K^2F_9 + (t - t')F_5 + (u - u')F_6 = 0,$$
$$(t - t')F_3 + K^2F_{10} + (s - s')F_5 + (u - u')F_7 = 0,$$
$$(u - u')F_4 + K^2F_{11} + (s - s')F_6 + (t - t')F_7 = 0; \quad (70)$$

$$K^2f + (s - s')f_1 + (t - t')f_2 + (u - u')f_3 = 0, \quad (71)$$

in obtaining which one can use the easily derivable relations

$$K_\mu K_\mu \equiv K_\mu^2 = 4(\varkappa_1^2 + \varkappa_2^2) - (s + s' + t + t' + u + u'),$$
$$K_\mu K_\mu' = s - s', \quad K_\mu K_\mu'' = t - t', \quad K_\mu K_\mu''' = u - u'. \quad (72)$$

By means of Eqs. (70) and (71) one can express the quantities F_8, F_9, F_{10}, F_{11} in terms of the other form-factors (as will be shown below, it is convenient to eliminate these particular form-factors, so that K^2 will be in the denominators):

$$F_9 = -[(s - s')F_2 + (t - t')F_5 + (u - u')F_6]K^{-2},$$
$$F_{10} = -[(t - t')F_3 + (s - s')F_5 + (u - u')F_7]K^{-2}, \quad (73)$$
$$F_{11} = -[(u - u')F_4 + (s - s')F_6 + (t - t')F_7]K^{-2},$$
$$F_8 = -[F_1 + (s - s')F_9 + (t - t')F_{10} + (u - u')F_{11}]K^{-2};$$
$$f = -(s - s')K^{-2}f_1 - (t - t')K^{-2}f_2 - (u - u')K^{-2}f_3. \quad (74)$$

Thus the number of independent form-factors is equal to seven for $T_{\mu\nu}^{int}(x)$ (F_1, F_2, F_3, F_4, F_5, F_6, F_7), and it is three for $j_\mu^{int}(x)$ (f_1, f_2, f_3). On the assumption of crossing symmetry for identical particles the form-factors f_1, f_2, f_3 are analytic continuations of each other. The same is true of the sets of form-factors F_2, F_3, F_4 and F_5, F_6, F_7. In this case we have three independent form-factors left for $T_{\mu\nu}^{int}(x)$ and one for $j_\mu^{int}(x)$.

To assure physically acceptable behavior of $T_{\mu\nu}(x)$ and $j_\mu(x)$ for infinitely large times and distances the form-factors must have singularities of a definite kind. Some of these singularities will be considered in Sec. 8.

7. THE DYNAMIC MOMENTS AS THE MOST CONVENIENT PHYSICAL QUANTITIES FOR THE DESCRIPTION OF RELATIVISTIC PROCESSES

Like the matrix element of the current operator, the matrix element $\langle \mathbf{p}_1\mathbf{p}_2 | T_{\mu\nu}(x) | \mathbf{p}_1'\mathbf{p}_2' \rangle$ of Eq. (59), when specified between two two-particle states, contains an enormous amount of physical information, since it exactly describes the distribution of matter at all points of space for reactions of arbitrary energy which begin with the collision of two definite particles. Since any sort of particles can be produced in the collision of two particles of sufficiently high energy, it is obvious that the prescription of the quantity $\langle \mathbf{p}_1\mathbf{p}_2 | T_{\mu\nu}(x) | \mathbf{p}_1'\mathbf{p}_2' \rangle$ contains information about all of the inert properties of these particles, i.e., about their masses, spins, moments of inertia, mean square radii of the mass distributions, and so on. In other words, the prescription of the matrix element (59) uniquely determines, if not the entire theory, at least a very large part of it.

An important advantage of the operator $T_{\mu\nu}(x)$ in comparison with other currents is its universality (all particles without exception have inertial properties). To extract this information from the matrix elements of the operators $T_{\mu\nu}(x)$, $j_\mu(x)$ we introduce into the argument a new class of physical quantities—the dynamic moments. A complete system of these moments will be considered in a separate paper. Here we shall confine ourselves to the simplest case of spinless particles.

We define the dynamic moments of the tensor $T_{\mu\nu}(x)$, which are denoted by $D^0(x_0)$, $D_i(x_0)$, $D_{ij}(x_0), \ldots$, in the following way:

$$D^0(x_0) = \int d^3x\, T_{00}(\mathbf{x}, x_0),$$
$$D_i(x_0) = \int d^3x x_i \frac{\partial T_{00}(\mathbf{x}, x_0)}{\partial x_0},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D_{i_1\ldots i_n}(x_0) = \frac{1}{n!}\int d^3x x_{i_1}\ldots x_{i_n}\frac{\partial^n T_{00}(\mathbf{x}, x_0)}{\partial x_0^n},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (75)$$

The dynamic moments $B^0(x_0)$, $B_i(x_0), \ldots$ of the current $j_\mu(x)$ are defined by the analogous relations

$$B^0(x_0) = \int d^3x j_0(\mathbf{x}, x_0),$$
$$B_i(x_0) = \int d^3x x_i \frac{\partial j_0(\mathbf{x}, x_0)}{\partial x_0},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$B_{i_1\ldots i_n}(x_0) = \frac{1}{n!}\int d^3 x x_{i_1}\ldots x_{i_n}\frac{\partial^n j_0(\mathbf{x}, x_0)}{\partial^n x_0},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (76)$$

Let us now examine the general properties of the dynamic moments defined by the relations (75) and (76), and compare them with the properties of other physical quantities used in relativistic quantum theory, which were discussed in Sec. 5. First let us study the asymptotic behavior of these quantities at infinity in time. In this case all particles become free, and the operators $T_{\mu\nu}(x)$ and $j_\mu(x)$ become sums of the one-particle operators (53) and (54). Therefore, using Eqs. (46), (47), and (55), we get for a system of N particles with the masses $\kappa_1, \ldots, \kappa_N$ and the charges $e_1, \ldots, e_N$

$$D^0 = \sum_{k=1}^{N} E^{(k)} = E,$$
$$D_i = \sum E^{(k)} v_i^{(k)} = P_i,$$
$$D_{ij} = \sum E^{(k)} v_i^{(k)} v_j^{(k)} = \sum \frac{p_i^{(k)} p_j^{(k)}}{E^{(k)}},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D_{i_1\ldots i_n} = \sum E^{(k)} v_{i_1}^{(k)}\ldots v_{i_n}^{(k)} = \sum p_{i_1}^{(k)}\ldots p_{i_n}^{(k)} (E^{(k)})^{-(n-1)}; \quad (77)$$
$$B = \sum_{k=1}^{N} e^{(k)} = e,$$
$$B_i = \sum e^{(k)} v_i^{(k)} = \sum \frac{e^{(k)} p_i^{(k)}}{E^{(k)}},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$B_{i_1\ldots i_n} = \sum e^{(k)} v_{i_1}^{(k)}\ldots v_{i_n}^{(k)} = \sum e^{(k)} p_{i_1}^{(k)}\ldots p_{i_n}^{(k)} (E^{(k)})^{-n}, \quad (78)$$

where $v_i^{(k)}$ is the velocity of the k-th particle.

It can be seen from Eqs. (77) and (78) that the dynamic moments are asymptotic integrals of the motion for arbitrary physical systems. It is also clear that by taking a sufficient number of these moments we can determine from them the momenta of any given number of particles. Therefore by means of the dynamic moments one can obtain a complete description of any physical system. Thus the dynamic moments are accessible to experimental measurement and, like the one-particle kinematic variables, they can serve as the variables of the scattering matrix.

On the other hand, it follows from the definitions (75) and (76) that the dynamic moments have a clear physical meaning not only asymptotically, but also at any finite instant of time. Therefore there are two ways in which we can go over from the values of the dynamic moments at $x_0 \to -\infty$ to their values at $x_0 \to +\infty$: by varying the time and by means of the S matrix. This makes it possible to obtain the scattering matrix directly in terms of the dynamic moments given as functions of the time, i.e., in the last analysis in terms of known matrix elements of the operators $T_{\mu\nu}(x)$, $j_\mu(x)$, by means of the relations

$$S^+D(-\infty)S = D(+\infty), \quad (79)$$

$$S^+B(-\infty)S = B(+\infty), \quad (80)$$

which are valid for all the dynamic moments. We note that since we have available an unlimited number of moments, we can use the relations (79) and (80) to get from prescribed elements (59) and (60) not only the elastic scattering cross section, but also the cross sections of all inelastic processes, together with values of the masses and charges of all particles which can arise in the reactions.

The more general formulation of the method of dynamic moments with spin properties included is more cumbersome and will be given in a separate paper. In this case we must bring into the treatment dynamic moments of the type

$$\int d^3x x_{i_1} \ldots x_{i_n} \frac{\partial^k}{\partial x_0^k} T_{00}(x),$$

where $k < n$, with terms corresponding to moments with larger numbers of time differentiations subtracted off at the beginning. Moments of this type are asymptotically equal to sums over the particles of products of the velocities and the various static moments (spin, magnetic moment, moment of inertia, mean square radius, and so on). The study of moments of this type gives the possibility of obtaining not only the spins, but also other structural characteristics of the particles produced in a reaction.

Thus the knowledge of $\langle \mathbf{p}_1\mathbf{p}_2 | T_{\mu\nu}(x) | \mathbf{p}_1'\mathbf{p}_2' \rangle$ [that is, the knowledge of the seven functions $F_1, \ldots, F_7$ of the six invariant variables (61)] for two spinless particles (for example, for two α particles) in principle contains the following information: the mass and spin spectra of all stable particles, the form-factors for the distributions of matter in these particles, and the scattering cross sections for all processes with the two original particles in the initial state. Other scattering cross sections remain unknown. It is scarcely to be expected, however, that any arbitrariness in choosing them will be possible. In this sense the operator $T_{\mu\nu}(x)$, given between two-particle states, contains the "entire" theory. It must be pointed out here that if the distributions of matter in particles and antiparticles are the same (which is by no means necessarily the case), then for the identification of particles and antiparticles one will have to consider along with the operator $T_{\mu\nu}(x)$ the operators for the electromagnetic, baryon, and lepton currents.

8. ANALYTIC PROPERTIES OF THE OPERATORS FOR THE ENERGY-MOMENTUM TENSOR AND THE CURRENTS

An important and interesting problem is the study of the analytic properties of the operators $T_{\mu\nu}(x)$, $j_\mu(x)$, $\delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y)$, and so on. Obviously there is still a great deal of complicated work to be done on this. In this section we shall give only some preliminary remarks which do not pretend to either completeness or rigor.

Various methods can be used to study the analytic properties. First, one can use the unitarity of the scattering matrix. For example, the relations (11) suggest the idea that in the variables s, s′ the tensors (59), (60) have the same singularities as the matrices S^+, S. Then, if two particles have a bound state, the residue at the corresponding pole in the s plane will be equal, apart from a constant of the nature of a coupling constant, to the operator

$$\langle \mathbf{P}, \varkappa | T_{\mu\nu}(x) | \mathbf{p}_1'\mathbf{p}_2' \rangle, \quad (81)$$

where $\mathbf{P}$ is the momentum of the compound particle and κ is its mass. Then the residue at the analogous pole in the s′ plane leads to the operator

$$\langle \mathbf{P}, \varkappa | T_{\mu\nu}(x) | \mathbf{P}', \varkappa \rangle. \quad (82)$$

Analogous expressions are obtained for the electromagnetic current. These semiintuitive arguments of course stand in need of rigorous justification. At present we can only say that these analytic properties in regard to the pole and the right-hand cut have been confirmed for a simple nonrelativistic example.[2)]

Another source of information about analytic properties is the causality condition in the form (39) or (41). Here one can use, for example, the representation of Jost, Lehmann, and Dyson.[10]

Finally, a very important source of information about analytic properties is the asymptotic properties of the various expressions when the space and time coordinates approach infinity. As an example let us consider the behavior of the dynamic moments of the two-particle tensors (59), (60), for $x_0 \to \pm\infty$. According to Eqs. (63), (64) these moments can be written in the form

$$D(x_0) = D^{(1)} + D^{(2)} + D^{int}(x_0), \quad (83)$$

$$B(x_0) = B^{(1)} + B^{(2)} + B^{int}(x_0), \quad (84)$$

where the quantities $D^{int}(x_0)$, $B^{int}(x_0)$ must go to

2)The writer expresses his gratitude to É. S. Lonskiĭ, who made these calculations.

zero for $x_0 \to -\infty$ and have finite limits for $x_0 \to +\infty$. Since, according to Eqs. (46), (47), (66), the time dependence of these quantities is given by a factor $e^{iK_0x_0}$, each of them must contain the pole factor

$$(K_0 - i\varepsilon)^{-1} \tag{85}$$

in accordance with the well known relation

$$\lim \frac{1}{2\pi i}\frac{e^{i\omega x_0}}{\omega - i\varepsilon} = \begin{cases} \delta(\omega) & x_0 \to +\infty, \\ 0 & x_0 \to -\infty. \end{cases} \tag{86}$$

This property has been verified in perturbation-theory calculations.

9. THE CONNECTION BETWEEN THE ELECTROMAGNETIC CURRENT AND THE OPERATOR FOR THE ENERGY-MOMENTUM TENSOR

The exact prescription of the energy-momentum tensor for one particle to a great extent determines the form of the electromagnetic current for the particle, so that these operators cannot be prescribed independently. As one example of this connection we shall indicate how the absolute value of the electric charge can be expressed in terms of the energy-momentum tensor. For simplicity we give the nonquantum treatment.

At a large distance R from the particle in the center-of-mass system

$$T_{00}(R) = e^2/8\pi R^4. \tag{87}$$

From this we have

$$\lim_{R\to\infty} \frac{2}{a} \int_R^{R+a} d^3x x^2 T_{00}(\mathbf{x}) = e. \tag{88}$$

In analogous ways one can get expressions for the magnetic moment, the quadrupole moment, and other static moments. The quantum-mechanical formulation is a purely technical problem.

10. THE CONNECTION BETWEEN THE HAMILTONIAN AND S-MATRIX FORMULATIONS

Since the momenta and spin projections of individual particles have meaning only asymptotically, at plus or minus infinity in time, the postulate that an S matrix exists still does not presume the existence of a Hamiltonian formulation (and in particular, the existence of an S matrix between finite times). The situation changes decidedly, however, if along with the existence of an S matrix one postulates the pseudoeuclidean character of space-time at all points. In this case the existence of an operator $T_{\mu\nu}(x)$ follows from the geometrical properties of four-space, and from the existence of $T_{\mu\nu}(x)$ there follows the existence of dynamic moments. Since the dynamic moments exist at any time and provide a complete description of the physical system, it is obvious that by taking a complete set of dynamic moments as the variables of the state vector we get a Hamiltonian description. Thus from the existence of the S matrix and the geometrical properties of space-time there follows the existence of a Hamiltonian description.

11. SUMMARY

The present paper has as its result the solution in principle of two interconnected problems.

A. The derivation of general conditions of microcovariance and microcausality in relativistic quantum theory.

B. The extraction from these conditions of information accessible to experimental verification, without bringing in quantum field theory.

To obtain concrete physical results it is necessary to solve a number of problems associated with the mathematical formalism and the physical content of the theory. At present the following problems can be indicated:

a) the parametrization of the operators $T_{\mu\nu}(x)$, $j_\mu(x)$, $\delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y)$ (see Sec. 6);

b) the construction of all possible dynamic moments for the general case of particles which have spin and spatial structure [see text after Eq. (80)];

c) the investigation of analytic properties (see Sec. 8);

d) the investigation of various causality conditions [see Eqs. (39), (41)];

e) the determination of connections between the various currents (see Sec. 9);

f) the transition to a rotating coordinate system. In particular, it is here that one can solve problems such as the effects of particle spin on the moment of inertia;

g) the treatment of the scattering of a particle in a weak external gravitational field. This process is determined by the operator $\delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y)$ as prescribed between one-particle states. Here the point of interest is of course not the actual scattering in the external field, but the fact that in this way one can try to develop a program for getting the masses, spins, and other characteristics of stable particles from a given operator $\langle \mathbf{p} | \delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y) | \mathbf{p}' \rangle$ similar to the use of $\langle \mathbf{p}_1\mathbf{p}_2 | T_{\mu\nu}(x) | \mathbf{p}_1'\mathbf{p}_2' \rangle$ in Section 7. Advantages of the use of the operator $\langle \mathbf{p} | \delta T_{\mu\nu}(x)/\delta g^{\lambda\sigma}(y) | \mathbf{p}' \rangle$ are that it depends on four (instead of six) invariants, and that in working with it one can take the causality condition into account from the very beginning.

The problems just listed are more or less understood as to mathematical formulation and methods of investigation. The solution of these problems will make it possible to carry out a number of concrete calculations. We may expect, for example, to get relations connecting the lower limit of inelastic scattering with the mean square radii of the particles.

In conclusion we indicate some questions of a more problematic character, whose study may be of interest for the theory.

1. The obtaining of two-particle operators $T_{\mu\nu}(x)$, $j_\mu(x)$ [i.e., scattering cross sections (cf. Sec. 7)] from prescribed one-particle operators for all stable particles.

2. The study of the Hamiltonian formulation (cf. Sec. 10). In this way one can in particular hope to obtain some restrictions on the quasi-local operators in the right member of the commutation relation (41).

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