DISPERSION RELATIONS IN QUANTUM ELECTRODYNAMICS

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A method is considered for writing out the dispersion relations in quantum electrodynamics. The proof is carried out in the lower orders of perturbation theory which is improved by application of the renormalization group.

1. Infrared singularities. The dispersion relations for photon-electron scattering at zero angle were first described in 1954. [1] However, subsequent application of the method of dispersion relations in quantum electrodynamics has met with a difficulty brought about by infrared singularities. The physical essence of this difficulty lies in the fact that the amplitudes of processes in which charged particles and a finite number of photons take part are equal to zero if the charged particles are scattered at non-zero angle, and are different from zero for forward scattering. Therefore, for example, the vertex functions in electrodynamics are not analytic. Also, the dependence of the scattering amplitudes on the transferred momentum is not analytic.

Cross sections of processes with an infinite number of particles (soft photons) differ from zero. The method of dispersion relations has been developed up to the present only for the amplitudes of processes with a finite number of particles. Nevertheless, we can also consider amplitudes of processes in electrodynamics with a finite number of particles if use is made of the factorization formula for the infrared singularities [2]

$$M_\lambda = e^{F_\lambda} M, \qquad (1)$$

Where M_λ is a matrix element computed with introduction of the mass $\sqrt{\lambda}$ in the photon propagation function, while the function F_λ has the form

$$F_\lambda = -\sum_{i<j} z_i a_i z_j a_j F\left((p_i a_i + p_j a_j)^2\right), \qquad (2)$$

where the summation is carried out over all charged particles, z_i is the number of the charge, $a_i = +1$ or -1 for incoming or outcoming particles, respectively, with momentum p_i. The function F is equal to [1)]

$$F\left((p'-p)^2\right) = \frac{i\alpha}{8\pi^3}\int \frac{dk}{k^2-\lambda}\left(\frac{2p'-k}{2p'k-k^2} - \frac{2p-k}{2pk-k^2}\right)^2 \qquad (3)$$

(α is the fine structure constant); it can be written in the form

$$F(t) = \frac{t}{\pi}\int_4^\infty \frac{\operatorname{Im} F(t')\,dt'}{t'(t'-t-i\varepsilon)}, \qquad (4)$$

$$\operatorname{Im} F(t) = \frac{\alpha}{4}\sqrt{\frac{t-4}{t}}\left(\frac{2t-4+\lambda}{t-4}\ln\frac{t-4+\lambda}{\lambda} - 1\right). \qquad (5)$$

Examining the coefficient of $\ln\lambda$ in F_λ, one can show that its real part is greater than zero in the physical region, vanishes for forward scattering (and also for backward electron-electron scattering), and becomes less than zero in the unphysical part of the region. If the quantity M is finite here in the limit $\lambda = 0$, this means that the matrix element M_λ in the limit $\lambda = 0$ vanishes in the physical part, is finite for forward scattering and infinite in the unphysical part of the region.

The assumption that the value of M in Eq. (1) is constant in the limit $\lambda = 0$ is not rigorously demonstrated; however, it is very likely. [2] In what follows, we shall consider the analytic properties of M for $\lambda = 0$ in several processes in the lower orders of perturbation theory that is improved by means of the renormalization group.

2. The vertex function. For a vertex with three lines, which correspond to two real charged particles and a virtual photon with square of the mass t, the value of M in the third order perturbation theory is an analytic function in the t plane with a cut from 4 to ∞. Beginning with the seventh order, diagrams arise with intermediate photons. They must give a cut from 0 to ∞. Thus, for the vertex function the value of M possesses the usual normal analytic properties.

3. Compton effect. We now consider the value of M for scattering of photons by electrons. We designate the square of the total energy of the di-

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[1)] In the system of units used, $\hbar = c = m_e = 1$; m_e is the mass of the electron. The scalar product $ab = a^0b^0 - \mathbf{a}\cdot\mathbf{b}$.

rect and crossed processes by s and u, respectively, and the square of the transferred momentum by t.

In the second-order perturbation theory, M contains the terms $M_s^{(2)}$ and $M_u^{(2)}$, which have poles at the points s = 1 and u = 1, respectively. The cited diagrams of fourth order have, among other terms, polar terms which depend on the additional magnetic moment of the electron μ'. We include these terms in $M_s^{(2)}$ and $M_u^{(2)}$, substituting in them the matrices $\gamma^n + \mu'\sigma^{nm}q_m$ for γ^n (q is the transferred momentum). The remaining terms of fourth order give the following contribution:

$$M^{(4)} = [\beta(t)\ln(1-s) + \gamma(t)]\, M_s^{(2)} + M_{sa}^{(4)} + [\beta(t)\ln(1-u) + \gamma(t)]\, M_u^{(2)} + M_{ua}^{(4)}; \tag{6}$$

$$\beta(t) = \alpha\frac{t}{\pi}\int_4^\infty \frac{t'-2}{\sqrt{t'(t'-4)}}\frac{dt'}{t'(t'-t-i\varepsilon)}, \tag{7}$$

$$\gamma(t) = -\frac{\alpha}{2}\frac{t}{\pi}\int_4^\infty \left[\frac{(t'-2)\ln t'}{\sqrt{t'(t'-4)}} - \frac{1}{2}\sqrt{\frac{t'-4}{t'}}\right]\frac{dt'}{t'(t'-t-i\varepsilon)}. \tag{8}$$

The values of $M_{sa}^{(4)}$ and $M_u^{(4)}$ (more precisely the coefficients of their spinor structures) are analytic functions of the variables s, t and u, t with branch points as singularities, and satisfy the Mandelstam representation.[3] We see that in fourth order there are terms with anomalous singularities $(1-s)^{-1}\ln(1-s)$ and $(1-u)^{-1}\ln(1-u)$.

Applying the equation of the renormalization group[4] in the variable s and choosing the constant of integration from the correspondence with perturbation theory (6), we get the result that the amplitude M close to the point s = 1 actually has a singularity of the form

$$\exp[\beta(t)\ln(1-s) + \gamma(t)]\,(M_s^{(2)} + \ldots) \tag{9}$$

and a similar singularity close to u = 1. In Eq. (9) terms of the order $[\alpha\ln(1-s)]^n$ and α^2 in perturbation theory are summed.

It is reasonable to suppose that the amplitude M has the form

$$M = \exp[\beta(t)\ln(1-s) + \gamma(t)]\, M_s^{(2)} + \exp[\beta(t)\ln(1-u) + \gamma(t)]\, M_u^{(2)} + M_a, \tag{10}$$

where β and γ are series in α, the first terms of which are represented in (7), (8), the additional magnetic moment is taken into account in $M_{s,u}^{(2)}$ and

$$M_a = M_{sa}^{(4)} + M_{ua}^{(4)} + \ldots \tag{11}$$

is an analytic function with branch points, which satisfies the Mandelstam representation.

The real part of the coefficient $\beta(t)$ in (10), (7) is less than zero in the physical region of the variable $t\,(t<0,\ t>4)$ and is larger than zero for $0<t<4$. Therefore, the value of M close to s = 1 (and similarly close to u = 1) has a singularity of the form

$$(1-s)^{-1+\beta(t)}, \tag{12}$$

which is stronger than the pole in the physical region of t.

Finally, let us consider one consequence of Eq. (10). That is, let us assume that for all physical energies one can neglect the value of the fourth order of M_a. Then we get for M an asymptote of the Regge type with exponent $-1+\beta(t)$, which satisfies the bound states of the electron and the positron. The level of these states in the nonrelativistic limit leads to the Coulomb level.

4. Electron-positron scattering. If the value of F_λ in (1) is equal to F(t) for the vertex function and the Compton effect, then it has the following form for electron-positron scattering:

$$F_\lambda = 2F(s) - 2F(u) + 2F(t), \tag{13}$$

where the variables s, u, and t have the same meaning as in the previous section.

The quantity M contains the terms $M_s^{(2)}$ and $M_t^{(2)}$ for electron-positron scattering in the second order of perturbation theory; these terms are poles for s = 0 and t = 0, respectively. As before, we take into account in them polar terms of higher orders, which depend on the additional magnetic moment.

The remaining terms of fourth order of perturbation theory give the following contribution:

$$M^{(4)} = 2(\Phi(s,t) - \Phi(u,t))\, M_t^{(2)} + M_{ta}^{(4)} + 2(\Phi(t,s) - \Phi(u,s))\, M_s^{(2)} + M_{sa}^{(4)}, \tag{14}$$

where

$$\Phi(s,t) = \varphi(s,t) - \varphi(0,t), \tag{15}$$

$$\varphi(s,t) = \frac{i\alpha}{\pi^3}\int \frac{dk\,(p'p - (p'k)(pk)/k^2)\,2qk}{k^2(k^2+2p'k)(k^2-2pk)(q^2-2qk+i\varepsilon)}, \tag{16}$$

p′ and p are the momenta of the electron and positron before (or after) the reaction, q is the transferred momentum.

The function Φ has the form

$$\Phi(s,t) = \frac{s}{\pi}\int_4^\infty \frac{\operatorname{Im}\Phi(s',t)\,ds'}{s'(s'-s-i\varepsilon)}, \tag{17}$$

$$\operatorname{Im}\Phi(s,t) = \frac{\alpha}{2}\sqrt{\frac{s-4}{s}}\left\{\frac{s-2}{s-4}\ln\frac{-t}{s-4} + \frac{1}{2}\right\}. \tag{18}$$

The functions $M_{sa}^{(4)}$ and $M_{ta}^{(4)}$ do not contain pole terms and are analytic functions of s, u and t, u, respectively, satisfying the Mandelstam representation.

One can rewrite Eq. (14) in the form

$$M^{(4)} = [(\beta(s) - \beta(u)) \ln(-t) + \varepsilon(s) - \varepsilon(u)] M_t^{(2)} + M_{ta}^{(4)} + [(\beta(t) - \beta(u)) \ln(-s) + \varepsilon(t) - \varepsilon(u)] M_s^{(2)} + M_{sa}^{(4)}, \quad (19)$$

where $\beta(t)$ is given by Eq. (7) and $\epsilon(t)$ has the form

$$\varepsilon(t) = \alpha \frac{t}{\pi} \int_4^\infty \sqrt{\frac{t'-4}{t'}} \left[\frac{t'-2}{t'-4} \ln \frac{1}{t'-4} + \frac{1}{2}\right] \frac{dt'}{t'(t'-t-i\varepsilon)}. \quad (20)$$

Repeating the discussion of the previous section, a representation of type (10) can be written for the quantity M, or the representation

$$M = \exp[(\beta(s) - \beta(u)) \ln(-t) + \varepsilon(s) - \varepsilon(u)] M_{ta} + \exp[(\beta(t) - \beta(u)) \ln(-s) + \varepsilon(t) - \varepsilon(u)] M_{sa}. \quad (21)$$

Employing the explicit form of the function (13), we can express the matrix element M_λ in the form

$$M_\lambda = \exp\{(\beta(s) - \beta(u)) \ln(-t/\lambda) + 2F(t)\} M_{ta} + \exp\{(\beta(t) - \beta(u)) \ln(-s/\lambda) + 2F(s)\} M_{sa}. \quad (22)$$

We consider the imaginary part of the exponent of the first exponential of this equation in the physical region $s > 4$, $t < 0$, $u < 0$. It is equal to

$$i \operatorname{Im} \beta(s) \ln \frac{-t}{\lambda} = i\alpha \frac{s-2}{\sqrt{s(s-4)}} \ln \frac{-t}{\lambda}. \quad (23)$$

We see that the imaginary part of the singular function in the exponent of the first exponential of (22) leads to a diverging phase:

$$\exp\left[i\alpha \frac{s-2}{\sqrt{s(s-4)}} \ln \frac{-t}{\lambda}\right] = \exp\left[i\alpha \frac{E^2+p^2}{pE} \ln \frac{2p \sin(\theta/2)}{\sqrt{\lambda}}\right] \quad (24)$$

(in the c.m. system), which is identical in the nonrelativistic limit with the diverging phase of the scattering amplitude in a Coulomb field in nonrelativistic theory.[5]

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51