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APPLICATION OF CONFORMAL MAPPING TO THE EXTRAPOLATION INTO THE UNPHYSICAL REGION OF RELATIONS OBSERVED IN THE SCATTERING OF HIGH ENERGY PARTICLES

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Conformal mapping is used to solve the problem of determining the coupling constant and spectral functions of the nucleon-nucleon scattering amplitude.

THE determination of the pion-nucleon interaction coupling constant, as well as certain other problems, involves the analytic continuation of measurable quantities into an unphysical region. It is clear fron the results of a number of authors that a reliable ex· trapolation into the unphysical region is possible onl if one deals with experimental data of rather high precision. Relatively recently (see [1]) it has been shown that this situation can be significantly improved if use is made of the conformal mapping

$$w=\left[1-\sqrt{\frac{b}{a}\left(\frac{a-x}{b+x}\right)}\right]\Big/\left[1+\sqrt{\frac{b}{a}\left(\frac{a-x}{b+x}\right)}\right], \tag{1}$$

where $x = \cos\vartheta$, and a and b are the endpoints of the segment of the real axis mapped inside the unit circle.

Below we give the results of applying this conformal mapping to the problem of determining the renormalized pion-nucleon interaction coupling constant f^2 and the spectral functions of nucleon-nucleon scattering.

DETERMINATION OF THE RENORMALIZED COUPLING CONSTANT

It may, apparently, be assumed at this time that the experimental data on nucleon-nucleon scattering are not in contradiction with the value 0.08 for the coupling constant f^2.[2] For this reason the solution of the problem of determination of f^2 makes it possible to demonstrate rather clearly the advantages resulting from going over from the $x = \cos\vartheta$ plane to the w plane, and to establish the region in which the proposed method will be applicable in practice.

To determine the coupling constant we have analyzed differential cross sections for elastic pn scattering at the energies of 90, 200, 380–400, and 630 MeV, and elastic pp scattering at the energies of 147 and 380 MeV,¹⁾ as given in the review articles.[3-5] The np scattering cross section was written in the form

$$\sigma_{np}=a_1f^4\left[\frac{1}{(x_0-x)^2}+\frac{4}{(x_0+x)^2}\right]+\frac{a_2}{x_0-x}+\frac{a_3}{x_0+x}+\sum_n A_nx^n. \tag{2}$$

A check was carried out of the order of the pole in the nucleon-nucleon scattering amplitude in the $x = \cos\vartheta$ plane at $x = \pm x_0 = \pm(1 + \mu^2/mT)$ (m and μ are the nucleon and pion masses respectively, and T is the kinetic energy of the nucleon in the laboratory frame) and the coupling constant f^2 was found with and without the branch point at $x = \pm a_0 = \pm(1 + 4\mu^2/mT)$ being taken into account.

To this end the functions

$$\sigma(x)(x_0^2-x^2)^3, \tag{3}$$

$$\sigma(x)(x_0^2-x^2)(a_0^2-x^2), \tag{4}$$

$$\sigma(x)(x_0^2-x^2)^2 \tag{5}$$

were extrapolated on the basis of expression (2) to the point $\pm x_0$. The extrapolation was carried out in the w plane. With the help of the least squares method the functions, Eqs. (3), (4), and (5) were approximated by either a power series $f(w) = \Sigma a_n w^2$ or a Legendre polynomial series $f(w) = \Sigma b_n P_n(w)$.

The results of checking the order of the pole in the scattering amplitude show that in a majority of cases the analyzed experimental data at energies of 200–630 MeV do not contradict the existence of a first order pole in the nucleon-nucleon scattering amplitude at $x = \pm x_0$, i.e., the extrapolation to the points $x = \pm x_0$ of the function, Eq. (3), gives zero within the experimental error. At 90 MeV it was not possible to establish the order of the singularity. It is possible that this is due to the fact that the

¹⁾Only the nuclear part was analyzed.

pole moves away from the edge of the physical region as the energy is lowered.

The direct method of determining the order of the pole at the energy of 630 MeV gives no answer. In that case the expression, Eq. (3), has a very complicated character and is poorly described by a power series. However if one extrapolates the logarithm of expression (3) a fully satisfactory result is obtained. It should be noted, though, that at that energy the w plane looses its advantages because the distance of the pole from the border of the physical region is bigger in the w plane than it is in the $x = \cos \vartheta$ plane (Fig. 1).

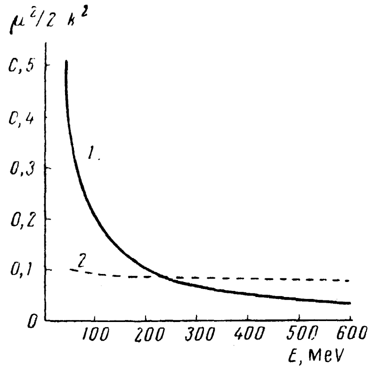

FIG. 1. $\mu^2/2k^2$, the distance of the one-meson pole from the border of the physical region, as a function of the nucleon energy.

The constant f^2 comes out close to 0.05—0.08 with an error of 10—15% (Table I). The comparison of results obtained by extrapolation in the x and w planes shows that due to, apparently, the more rapid convergence of the approximating series the results in the w plane are obtained with a somewhat smaller error at energies above 400 MeV.

DETERMINATION OF SPECTRAL FUNCTIONS

In the work of I. Ciulli, S. Ciulli and Fischer [6] it is shown that the conformal mapping (1), in which the entire $x = \cos \vartheta$ plane is mapped inside the unit circle, is quite useful for the extrapolation of the scattering amplitude into the region of the spectral functions. At that the effective jump across the cut is determined by the sum of the even part of the Fourier series into which the power series, that approximates the function $M(w)\sqrt{a_0^2 - x^2}$ being extrapolated (where M(w) is the scattering amplitude), transforms on the circle $|w| = 1$.

The effective spectral function was determined for the elements M_{SS}, M_{11}, M_{00}, M_{01} and M_{10} of the transition matrix for np and pp scattering at energies of 147, 210, and 310 MeV. To that end the imaginary parts of the matrix elements M calculated from phase shifts [2] were multiplied by $\sqrt{a_0^2 - x^2}$ and expressed in the form of a power series

$$\operatorname{Im} M \sqrt{a_0^2 - x^2} = \sum_0^N c_n w^n. \tag{6}$$

Then the sum of the series

$$\frac{1}{2i} \sum_0^N c_n \cos n\varphi$$

was calculated for $0 < \varphi < 2\pi$ (effective jump across the cut).

It was found in this way that in all cases Im M was well described by the series $\Sigma c_n w^n / \sqrt{a_0^2 - x^2}$ for a comparatively small number of terms (2)—(5). The coefficients in the approximating series are determined in a sufficiently stable manner. Increas-

Table I

Energy, MeV	Plane	A: f^2 (4)***	A: f^2 (5)****	B: f^2 (4)	B: f^2 (5)	C: f^2 (4)	C: f^2 (5)
90 (np)	w	0.079±0.01	0.06±0.01	0.074±0.02	0.06±0.015*	—	0.06±0.01
	x	—	—	—	0.086±0.002*	—	—
147 (pp)	w	0.18±0.02	0.124±0.018	—	—	0.051±0.051	0.108±0.009
	x	0.097±0.018	0.056±0.02	—	—	—	0.071±0.014
200 (np)	w	0.105±0.004	0.077±0.006	—	0.086±0.005	—	0.083±0.007
	x	—	—	—	—	—	0.061±0.005
380 (pp)	w	Converges badly	—	—	—	0.206±0.046	0.063±0.063
	x	0.109±0.018	0.062±0.020	—	—	0.0835±0.023	0.068±0.015
400 (np)	w	0.175±0.015	0.064±0.004	0.127±0.006	0.71±0.006	0.081±0.007	0.090±0.006
	w**	0.041±0.003	0.041±0.003	—	—	—	—
	x	—	—	—	0.058±0.0065	—	0.048±0.006
630 (np)	w**	0.035±0.007	0.035±0.007	—	—	—	—

*For five terms in the approximating expression $\chi^2/\overline{\chi^2} = 4.65$ in the x plane and $\chi^2/\overline{\chi^2} = 0.988$ in the w plane.

**The function that was extrapolated was $\log f(w) = \Sigma a_n w^n$.

***Result of extrapolating expression (4).

****Result of extrapolating expression (5).

Table II. Coefficients in the approximating series for $M_{11}^{np}(a_0^2 - x^2)$

	Energy, MeV		
	147	210	310
	Number of coefficients equal to 3		
c_0	—	0.18±0.02	0.228±0.027
c_1	—	1.99±0.03	1.54 ±0.04
c_2	—	8.07±0,20	3.56 ±0,177
c_3	—	—	—
c_4	—	—	—
c_5	—	—	—
	Number of coefficients equal to 4		
c_0	0.19±0.04	—0.18±0.018	0.217±0.028
c_1	2.81±0.24	2.34±0.073	2.28±0.131
c_2	14.1 ±0.53	8.09±0.20	3.63±0.177
c_3	—5.46±2.50	—3.60±0.68	—4.05±0.69
c_4	—	—	—
c_5	—	—	—
	Number of coefficients equal to 5		
c_0	0.005±0.06	—0.43±0.03	0.052±0.04
c_1	2.72±0.24	2.34±0.07	2.24±0.131
c_2	22.48±1.99	14.8±0.71	8.29±0.74
c_3	—4.51±2.51	—3.55±0.69	—3.82±0.69
c_4	—62.9±14.37	—36.3±3.66	—18.72±2.86
c_5	—	—	—
	Number of coefficients equal to 6		
c_0	—	—0.43±0.03	0.051±0.04
c_1	—	2.34±0.19	2.29±0.26
c_2	—	14.8±0.71	8.30±0.74
c_3	—	—3.68±3.90	—4.67±3.65
c_4	—	—36.3±3.66	—18.7±2.87
c_5	—	0.56±17.39	2.7±11.7

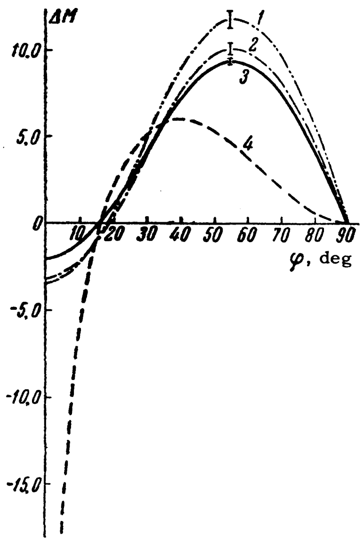

FIG. 2. The effective jump across the cut for M_{11}^{pp} as a function of the distance along the cut measured in the w plane by the polar angle φ; the curves 1, 2, 3 correspond to the jump found for $M_{11}^{pp}(a_0^2 - x^2)^{1/2}$ for energies of 147, 310 and 210 MeV respectively; the curve 4 gives the jump in M_{11}^{pp} for 210 MeV.

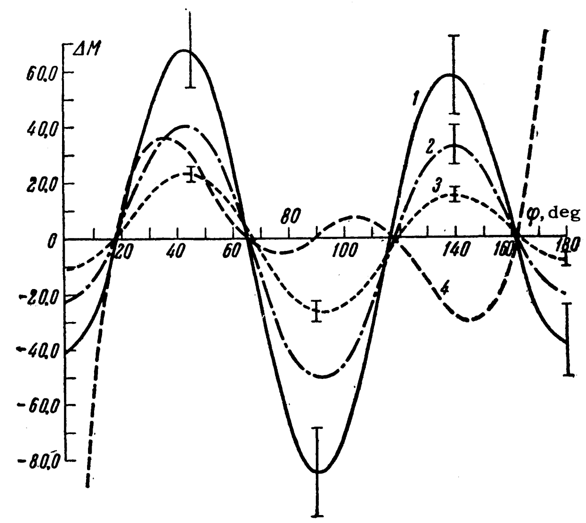

FIG. 3. The effective jump for M_{11}^{np}: curves 1, 2, 3 correspond to the jump found for $M_{11}^{np}(a_0^2 - x^2)^{1/2}$ for 147, 210 and 310 MeV respectively; curve 4 gives the jump in M_{11}^{np} for 210 MeV.

ing the number of terms affects weakly the size of the first few coefficients (Table II). The coefficients have a weak energy dependence.

Consequently the existing experimental data in the energy region of 147—310 MeV are satisfactorily described with the help of 20—25 parameters. It should be noted that a similar description is attained with the help of phase shifts and the coupling constant for 17 parameters. Therefore the representation of the matrix elements M_{ss}, M_{11}, M_{00}, M_{01}, M_{10} in the form of a power series in w is hardly worthwhile from the point of view of describing the experimental data by a minimal number of parameters.

The spectral function clearly exhibits oscillations (Figs. 2 and 3), however the details of the oscillation with the given precision of the experimental data cannot be determined (the number of terms of the series that can be determined is not sufficiently large). It is not possible to say within the errors whether the Fourier series converges on the circle $|w| = 1$, because the coefficients in the series that were determined increase with increasing number of them.

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