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THE ASYMPTOTIC VALUES OF THE AMPLITUDES OF INELASTIC PROCESSES

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A simple method is proposed for expanding any many-point amplitude in relativistic theory in terms of partial waves. The expansions obtained are used for analyzing the asymptotic behavior of inelastic processes by the Regge[1] and Gribov[2] method.

1. EXPANSION OF MANY-POINT DIAGRAMS IN HELICAL PARTIAL WAVES

LET $M_{l'm';l''m''}$ be the amplitude of the four-point diagram of Fig. 1—the transition of the particles $1, \beta$ into $2, \alpha$—where l', m' and l'', m'' are the spins and projections of the spins of particles α and β on the directions $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ of their momenta in the c.m. system of the reaction. We assume for simplicity that the spins of particles 1 and 2 (and henceforth also of all other particles, unless otherwise stipulated) are equal to zero.

According to Jacob and Wick[3] the expansion of $M_{l'm';l''m''}$ in partial waves has the form

$$M_{l'm';\,l''m''} = \sum_{LM}(2L+1) \times D^{(L)*}_{M,\,m'}(\mathbf{n}_\alpha) D^{(L)}_{M,\,m''}(\mathbf{n}_\beta)\, f^{(l',\,l'')}_{L;\,m',\,m''}(t;\, s',\, s''), \tag{1}$$

where $f^{(l',l'')}_{L;m',m''}$ denotes the helical partial amplitude; it depends on $s' = p_\alpha^2$, $s'' = p_\beta^2$, and on the energy $t = (p_\beta + p_1)^2 = (\epsilon_\beta + \epsilon_1)^2 - (\mathbf{p}_\beta + \mathbf{p}_1)^2$, while

$$D^{(L)}_{M,\,m'}(\mathbf{n}_\alpha) = D^{(L)}_{M,\,m'}(\varphi_\alpha,\, \vartheta_\alpha,\, -\varphi_\alpha) = e^{i(M-m')\varphi_\alpha} d^{(L)}_{M,\,m'}(\cos\vartheta_\alpha),$$

where $d^{(L)}_{M,0}(\cos\vartheta_\alpha) \equiv P_{LM}(\cos\vartheta_\alpha)$ for $m' = 0$ is[3,8] the associated Legendre polynomial.

We shall regard the particle α as a compound particle consisting of two particles 3 and 4, which are in their c.m.s. in a state with definite energy

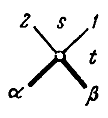

Рис. 1

$s' = (p_3 + p_4)^2$ and with definite values of angular momentum l' of their relative motion and projection m' of the momentum on the direction $\mathbf{n}_\alpha$. Then (1) will be the amplitude of the five-point diagram of Fig. 2 corresponding to the production of particles 3 and 4 in the indicated state. The amplitude $M_{\mathbf{k}';l'',m''}$ for the production of particles 3 and 4 in Fig. 2, in a state with definite momenta, i.e., with definite value of the momentum $\mathbf{k}'$ of their relative motion in their c.m.s., will be a linear combination[1)] of the quantities in (1):

$$M_{\mathbf{k}';\,l''m''} = \sum_{l'm'}(2l'+1)\, D^{(l')*}_{m',0}(\mathbf{n}')\, M_{l'm';\,l''m''}\,, \tag{2}$$

where $\mathbf{n}'$ is the direction of $\mathbf{k}'$ in the c.m.s. of the particles 3 and 4 (all particle-group c.m. systems of this type will be designated by a prime or by a double prime).

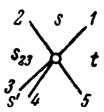

FIG. 2

We can consider in exactly the same way the case of a six-point diagram (the amplitudes for the transition of three particles into three particles). For this purpose we consider the particle β on Fig. 1, as well as α, as being a compound particle consisting of particles 5 and 6, which are (in their c.m.s.) in a state with energy $s' = (p_5 + p_6)^2$ and with values l'' and m'' of the angular momentum and its projection on the

[1)]In exactly the same way as the wave function $\psi_{\mathbf{k}'}$ of particles 3 and 4, corresponding to a definite $\mathbf{k}'$ in their c.m.s., is[3] a linear combination of the functions $\psi_{l'm'}$ corresponding to definite l', m':

$$\psi_{\mathbf{k}'} = \sum_{l'm'}(2l'+1)\, D^{(l')}_{m',0}(\mathbf{n}')\, \psi_{l'm'}$$

direction $\mathbf{n}_\beta$. In analogy with (2), the amplitude of the six-point diagram of Fig. 3, corresponding to a definite value $\mathbf{k}''$ of the momentum of the particles 5 and 6 in their c.m.s. ($\mathbf{k}'' = \mathbf{p}_5'' = -\mathbf{p}_6''$) will be a linear combination of the quantities $M_{\mathbf{k}';l''m''}$:

$$M_{\mathbf{k}',\,\mathbf{k}''} = \sum_{l''m''} (2l'' + 1)\, D^{(l'')}_{m'',\,0}(\mathbf{n}'')\, M_{\mathbf{k}';\,l''m''}, \tag{3}$$

where $\mathbf{n}''$ is the direction of $\mathbf{k}''$ in the c.m.s. of particles 5 and 6.

Denoting the partial amplitude of the four-point diagram (Fig. 4) by

$$\lambda_L(t) = f^{(0,\,0)}_{L;\,0,\,0}(t;\ m_3^2,\ m_4^2),$$

and that of the five-point diagram (Fig. 2) by

$$\varphi^{(l')}_{L;m'}(t;\ s') = f^{(l',\,0)}_{L;\,m,0}(t;\ s',\ m_5^2)$$

(we recall that the spins of all particles on Figs.

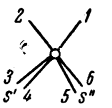

FIG. 3

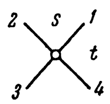

FIG. 4

4 and 2 are assumed equal to zero), and choosing the z axis of the coordinate system along $\mathbf{n}_\alpha$ in such a way that

$$D^{(L)}_{M,\,m'}(\mathbf{n}_\alpha) = \delta_{M,m'},$$

we obtain from (1), (2), and (3) the following expansions for the amplitudes M_n of four-, five-, and six-point diagrams ($n = 4, 5, 6$):

$$M_n(s_{ik}) = \sum_{L,\,m',\,m''} (2L + 1)\, d^{(L)}_{m'm''}(z)\, \chi^{(n)}_{L;\,m'm''}; \tag{4}$$

$$\chi^{(4)}_{L;\,m'm''} = \delta_{m',0}\delta_{m'',0}\lambda_L(t),$$

$$\chi^{(5)}_{L;\,m'm''} = \delta_{m'',0}e^{im'\varphi} \sum_{l'} (2l' + 1)\, P_{l',m'}(z_3')\, \varphi^{(l')}_{L;\,m'}(t;\ s'),$$

$$\chi^{(6)}_{L;\,m'm''} = e^{i(m'\varphi' - m''\varphi'')} \sum_{l',\,l''} (2l' + 1)\,(2l'' + 1)$$

$$\times\ P_{l'm'}(z_3')\, P_{l''m''}(z_5'')\, f^{(l',\,l'')}_{L;\,m'm''}(t;\ s',\ s''). \tag{5}$$

The arrangement of the momenta of the particles in the c.m.s. of the reaction is shown in Figs. 5 and 6 respectively for the case of the five- and six-point diagrams of Figs. 2 and 3. These figures indicate the angles whose cosines are denoted in (5) by z, z_3', and z_5'' (as already mentioned, a prime or two primes denotes that the corresponding angles are in the c.m.s. of particles

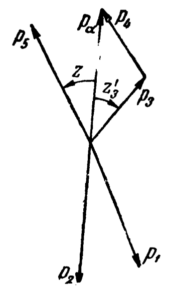

FIG. 5

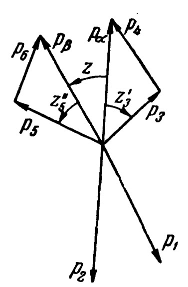

FIG. 6

3 and 4 or 5 and 6). We denote by φ the angle in Fig. 5 between the planes of the vectors $\mathbf{p}_3$, $\mathbf{p}_\alpha$, and $\mathbf{p}_5$, $\mathbf{p}_\alpha$, and by φ' and φ'' the angles in Fig. 6 between the plane $\mathbf{p}_\alpha$, $\mathbf{p}_\beta$ and the planes $\mathbf{p}_\alpha$, $\mathbf{p}_3$ and $\mathbf{p}_\beta$, $\mathbf{p}_5$, respectively.

We note that the transition from the c.m.s. of the reaction to the c.m.s. of particles 3 and 4 (in both cases—Fig. 5 and Fig. 6) is by means of a Lorentz transformation in the direction $\mathbf{n}_\alpha$, with $v/c = p_\alpha/\epsilon_\alpha$, where

$$\varepsilon_\alpha = \sqrt{p_\alpha^2 + s'} = \tfrac{1}{2}t^{-1/2}(t + s' - m_2^2)$$

is the energy of particles 3 and 4 in the c.m.s. of the reaction. It yields

$$p_3\sqrt{1 - z_3^2} = p_3'\sqrt{1 - z_3'^2}, \qquad p_3 z_3 = \frac{\varepsilon_\alpha}{\sqrt{s'}}\left(p_3' z_3' + \frac{p_\alpha}{\varepsilon_\alpha}\varepsilon_3'\right),$$

$$\varepsilon_3 = \frac{\varepsilon_\alpha}{\sqrt{s'}}\left(\varepsilon_3' + \frac{p_\alpha}{\varepsilon_\alpha}p_3' z_3'\right).$$

Here

$$\varepsilon_3' = \sqrt{p_3'^2 + m_3^2} = \tfrac{1}{2}s'^{-1/2}\,(s' + m_3^2 - m_4^2).$$

The angle $\varphi = \varphi'$ does not change under this transformation.

The invariant $s = (p_1 - p_2)^2$ is connected in all cases of Figs. 4, 2, and 3 with the cosine z by the relation $s = m_1^2 + m_2^2 - 2\epsilon_1\epsilon_2 + 2p_1p_2z$, where ϵ_1, ϵ_2 and p_1, p_2 are the energies and momenta of particles 1 and 2 in the c.m.s.; they depend on t (in the case of Fig. 4), on t and s' (for Fig. 2), or on t, s', and s'' (for Fig. 3).

The expansion (1)—(5) can be extended in trivial fashion to the case when the spins of all particles differ from zero. Writing these down for arbitrary spin values and considering any particle in (4) and (5) as a system of two or several particles[4,5], as also considering its spin as the total angular momentum of the internal motion of these particles, we can readily obtain expansions which are quite analogous to (4) and (5) for amplitudes of many-point diagrams with arbitrary particles and with an arbitrary number of lines.

The unitarity conditions for the helical partial amplitudes λ_L, $\varphi_{L;m'}$ and $f_{L;m'm''}$ have a very simple appearance. We write them out for the particularly simple case when the energy t lies below the three-particle production threshold [6,7] (i.e., in the unphysical region):

$$\lambda_L(t_+) - \lambda_L(t_-) = 2i\rho(t)\,\lambda_L(t_+)\,\lambda_L(t_-),$$

$$\varphi^{(l')}_{L;m'}(t_+;s') - \varphi^{(l')}_{L;m'}(t_-;s') = 2i\rho(t)\,\varphi^{(l')}_{L;m'}(t_+,s')\,\lambda_L(t_-),$$

$$f^{(l',l'')}_{L;m'm''}(t_+;s',s'') - f^{(l',l'')}_{L;m'm''}(t_-;s',s'')$$

$$= 2i\rho(t)\,\varphi^{(l')}_{L;m'}(t_+;s')\,\varphi^{(l'')}_{L;m'}(t_-;s''); \qquad (6)$$

here $t_\pm = t \pm i\tau$, $\tau > 0$, $\tau \to 0$, t is real, and s′ and s″ can have arbitrary complex values,

$$\rho(t) = 2t^{-1/2}p_{15}(t) = t^{-1/2}\,[t^2 - 2t(m_1^2 + m_5^2) - (m_1^2 - m_5^2)^2]^{1/2}.$$

In the appendix at the end of the article we write down for reference the next three-particle term in the unitarity condition. It has the form of an infinite sum over the momenta l_i' and m_i' of one pair of particles (out of the three). The appendix gives also in somewhat different form a derivation for the same expansions of the form (4) and (5), based on the expansion [4,5] of the eigenfunctions of the initial and final many-particle states corresponding to definite values of particle momenta, over states with definite values of angular momenta and helicities.

2. ASYMPTOTIC VALUES OF MANY-POINT AMPLITUDES

Expansions (4) and (5) enable us to construct, on the basis of the method of Regge [1] and Gribov [2], an asymptotic expression for the many-point diagrams in Figs. 2 and 3 in the physical region of the channel in which the incoming particles are 1 and 2, and those produced are 3, 4, and 5 (in the case of Fig. 2) or 3, 4, 5, and 6 (in the case of Fig. 3). Let us consider first the case when $s \to \infty$ for some value t (initially unphysical) and for fixed values of s′ and s″.

From the relation between z and s it follows that in this case $z = s/2p_1p_2 \to \infty$. The functions $d^{(L)}_{m'm''}(z)$ in (4) are determined for arbitrary complex L by the relation [8]

$$d^{(L)}_{m',m''}(z) = A^{(L)}_{m',m''}\left(\frac{z+1}{2}\right)^L\left(\frac{z-1}{z+1}\right)^{(m'-m'')/2} \times F\left(m'-L,\,-m''-L,\,m'-m''+1,\,\frac{z-1}{z+1}\right),$$

where [2)]

$$A^{(L)}_{m',m''} = \frac{i^{m'-m''}}{(m'-m'')!}\left[\frac{\Gamma(L+m'+1)\,\Gamma(L-m''+1)}{\Gamma(L-m'+1)\,\Gamma(L+m''+1)}\right]^{1/2},$$

it being assumed that a cut is drawn in the complex z plane from $-\infty$ to $+1$ and that the phases of the quantity $z \pm 1$ are equal to zero for real $z > 1$. Under this condition, the values of $d^{(L)}_{m'm''}$ on both sides of the cut are determined from the relation

$$d^{(L)}_{m',m''}(-z_\pm) = e^{\mp i(L-m'')\pi}d^{(L)}_{m',-m''}(x) = e^{\mp i(L-m')\pi}d^{(L)}_{-m',m''}(x), \qquad (7)$$

where $z_\pm = x \pm i\tau$, with x real and positive.

When $z \to \infty$ we have for $d^{(L)}_{m',m''}(z)$ a simple asymptotic expression

$$d^{(L)}_{m',m''}(z) \approx C_{L,m'}C_{L,m''}\left(\frac{z}{2}\right)^L,$$

$$C_{L,m'} = i^{m'}\left[\frac{\Gamma(2L+1)}{\Gamma(L+m'+1)\,\Gamma(L-m'+1)}\right]^{1/2}. \qquad (8)$$

To investigate the asymptotic value of the sum (4) as $z \to \infty$ we separate from (4) the parts $M^{(+)}$ and $M^{(-)}$, corresponding to summation over only even or only odd L ($M_n = M_n^{(+)} + M_n^{(-)}$). We introduce two functions $\chi^{(n,+)}_{L;m'm''}$ and $\chi^{(n,-)}_{L;m'm''}$ of the complex variable L as an analytic continuation in the region of the complex L of the functions $\chi^{(n)}_{L;m'm''}$ defined in (5) for integer even or odd L (as in the case of the elastic scattering amplitude [2], we exclude in this case the dependence of the type $(-1)^L$ of the functions $\chi^{(n)}_{L;m'm''}$ on L, a dependence which is not analytic as $L \to \infty$).

The part $M_n^{(+)}$ of the sum (4) can be written [1] in the form

$$M_n^{(+)}(z_+) = \sum_{m',m''=-\infty}^{\infty}\frac{1}{4i}\int_C dL\,\frac{2L+1}{\sin L\pi}[d^{(L)}_{m',m''}(z_+) + (-1)^{m''}d^{(L)}_{m',-m''}(-z_+)]\,\chi^{(n,+)}_{L;m',m''}, \qquad (9)$$

where the contour C encircles in the positive direction the semi-axis of the real values of L [inasmuch as $d^{(L)}_{m,-m''}(-z_+) \equiv (-1)^{m'-m''}d^{(L)}_{-m',m''}(-z_+)$, the second term in the brackets can also be written in the form $(-1)^{m'}d^{(L)}_{-m',m''}(-z_+)$]. Using (7) we can verify that the residues of the integrand

[2)]This definition is correct only when $m' \geq m''$. The values of $d^{(L)}_{m',m''}$ for $m' < m''$ are determined by the equation

$$d^{(L)}_{m',m''}(z) = (-1)^{m'-m''}d^{(L)}_{m'',m'}(z).$$

vanish for all integer L (they also vanish for all m' and m'' for which $|m'| > L$ or $|m''| > L$), and coincide for even L with the terms of the sum (4). The quantity $M_n^{(-)}$ can be written in a form which is perfectly analogous to (9), with the function $\chi^{(n,-)}_{L;m'm''}$ in place of $\chi^{(n,+)}_{L;m'm''}$ and with a minus sign in the square brackets of (9).

It is important for what follows to determine the analytic properties of the functions $\chi^{(n,\pm)}_{L;m'm''}$ (i.e., in accordance with (5), the partial amplitudes $\varphi_{L;m'}$ and $f_{L;m'm''}$) in the complex L plane. According to (4) [and by virtue of the orthogonality of the functions $d^{(L)}_{m',m''}(z)$], $\chi^{(n)}_{L;m'm''}$ and M_n are related by the formula

$$\chi^{(n)}_{L;m'm''} = \frac{1}{2}\int_{-1}^{1} M_n(z, s_{ik}) d^{(L)}_{m',m''}(z)\, dz.$$

Substituting here the dispersion relation for the many-point amplitude M_n with respect to the variable z (i.e., with respect to s)

$$M_n(z, s_{ik}) = \frac{1}{\pi}\int_{C_1(s_{ik})}^{\infty} \frac{M_n^{(1)}(z', s_{ik})}{z'-z}dz' + \frac{1}{\pi}\int_{C_2(s_{ik})}^{\infty} \frac{M_n^{(2)}(z', s_{ik})}{z'+z}dz' \qquad (10)$$

(where s_{ik} are the remaining variables, on which M_n depends, as it does also on z (or s), and the integration contours begin with some complex points $C_1(s_{ik})$ and $C_2(s_{ik})$ and then go on the real axis for large z') we obtain

$$\chi^{(n,\pm)}_{L;m',m''} = \frac{1}{\pi}\int_{C_1}^{\infty} M_n^{(1)}(z') Q^{(L)}_{m',m''}(z')\, dz' \pm \frac{1}{\pi}\int_{C_2}^{\infty} M_n^{(2)}(z')(-1)^{m''} Q^{(L)}_{m',-m''}(z')\, dz', \qquad (11)$$

where

$$Q^{(L)}_{m',m''}(z') = \frac{1}{2}\int_{-1}^{1} \frac{d^{(L)}_{m',m''}(z)\, dz}{z'-z}$$

is an analytic function defined in perfect analogy with the Legendre polynomial of the second kind (with which it coincides when $m' = m'' = 0$), while $M_n^{(1)}$ and $M_n^{(2)}$ are the absorptive parts of the many-point M_n.

Part of the integrals (12), corresponding to integration over the complex contours from C_1 or C_2 to a certain point (near $z = 1$) on the real axis, is known to be non-analytic in L near large complex L (when $L \to \pm i\infty$ the contribution of this part can increase exponentially because the function $Q^{(L)}_{m',m''}(z')$ increases for complex values of z'). The corresponding part of the integral (10) determines the so-called anomalous terms in the dispersion relation for M_n. Inasmuch as these terms are determined by integrals over contours that are not infinite but of finite length, they decrease like $1/z$ as $z \to \infty$ and are certainly of no importance in the asymptotic expression for M_n. Therefore the contribution to M_n and to $\chi^{(n,\pm)}_{L;m',m''}$ from these terms can be disregarded. As a result, the analysis of the analytical properties of the right half of (11) in the L plane can be made in perfect analogy with what was done by Gribov[2] for the case of the partial amplitudes $\lambda_L(t)$ of four-point diagrams.[3)] It follows then from (11) that the part of $\chi^{(n)}_{L;m',m''}$ which is essential for the asymptotic value of M_n is (a) an analytic function of L in the right half plane and (b) decreases rapidly as $L \to \infty$. Therefore the contour of integration C in (9) can be deformed into a contour parallel to the imaginary axis.

The asymptotic expression for the integral (9), corresponding to $z \to \infty$, is determined after transforming the contour of the extreme right singularity in L of the function $\chi^{(n,+)}_{L;m'm''}$. If this singularity is the pole

$$\frac{\pi}{2}(2L+1)\chi^{(n,+)}_{L;m'm''} \approx \frac{R^{(n)}_{m',-m''}}{L-\alpha} \text{ as } L \to \alpha, \qquad (12)$$

the position α of which does not depend on m' and m'', then it follows from (8) and (9) that as $z \to \infty$ we have

$$M_n^{(+)}(z_+) \approx \left(\frac{z}{2}\right)^{\alpha} I_\alpha \sum_{m',m''} C_{\alpha,m'} C_{\alpha,m''} R^{(n)}_{m',m''}, \qquad (13)$$

where $I_\alpha = i - \cot(\pi\alpha/2)$.

The asymptotic value of the part $M^{(-)}_{(n)}$ will have exactly the same form, but with different values of $R^{(n)}_{m',m''}$ and α. We shall henceforth, for the sake of being definite (and to simplify the exposition) assume that Re α has the largest value in the term $M_n^{(+)}$ (vacuum pole). Therefore as $z \to \infty$ we can neglect the part $M_n^{(-)}$ and $M_n(z) \approx M_n^{(+)}(z)$.

Solving the unitarity conditions (6) with respect to the amplitudes

$$\lambda_L(t_+), \quad \varphi^{(l')}_{L;m'}(t_+, s'), \quad f^{(l',l'')}_{L;m'm''}(t_+; s', s''),$$

we can readily verify that they all (and consequently, in accordance with (5), also $\chi^{(n+)}_{L;m'm''}$) actually have a common pole[4)] of the type (12).

[3)]The author is grateful to V. N. Gribov for many valuable remarks concerning this part of the work.

[4)]At a value $L = \alpha(t)$ for which $1 - 2i\rho(t)\lambda_L(t_-) = 0$.

In addition (as shown by an analysis of (6) in perfect analogy with that carried out by Gribov and Pomeranchuk[9]) the residues of all the amplitudes factor out; when $L \to \alpha(t)$

$$\frac{\pi}{2}(2L+1)\lambda_L(t) \approx \frac{u^2(t)}{L-\alpha(t)},$$

$$\frac{\pi}{2}(2L+1)\varphi^{(l')}_{L;m'}(t;s') \approx \frac{u(t)\, v_{l'm'}(t,s')}{L-\alpha(t)},$$

$$\frac{\pi}{2}(2L+1) f^{(l',l'')}_{L;m'm''}(t;s',s'') \approx \frac{v_{l'm'}(t,s')\, v_{l''m''}(t,s'')}{L-\alpha(t)}, \qquad (14)$$

where $u(t)$ is real and the $v_{l'm'}(t, s')$ are generally speaking complex.

We note that, as follows from the invariance of the theory under space reflections, the amplitudes $\varphi^{(l')}_{L;m''}$ and consequently also $v_{l'm'}$, should not change when m' is replaced by $-m'$ [see relation (43) on page 417 of the article by Jacob and Wick[3]]

Substituting (14) in (5) and determining the values of $R^{(n)}_{m',-m''}$, we obtain from (13) the following asymptotic values of the many-point diagrams (4) as $s \to \infty$:

$$M_4 \approx g^2(t)\, I(t)\, s^{\alpha(t)}, \qquad M_5 \approx g(t)\, G(t,\mathbf{k}')\, I(t)\, s^{\alpha(t)},$$

$$M_6 \approx G(t,\mathbf{k}')\, G(t,\mathbf{k}'')\, I(t)\, s^{\alpha(t)}, \qquad (15)$$

where*

$$I(t) = i - \operatorname{ctg}\left(\frac{\pi}{2}\alpha(t)\right), \qquad g(t) = [2p_1(t)]^{-\alpha(t)} C_{\alpha,0}\, u(t),$$

$$G(t,\mathbf{k}') = [2p_2(t,s')]^{-\alpha(t)} \sum_{l',m'} (2l'+1)$$

$$\times\, C_{\alpha,m'} v_{l',m'}(t,s')\, P_{l',m'}(z'_3)\, e^{im'\varphi}. \qquad (16)$$

The amplitudes (15) have the form of a contribution of the pole diagrams of Fig. 7, in which the factor $I(t)\, s^{\alpha(t)}$ corresponds to the virtual particle (the "reggion"), $g(t)$ to the triple node, and $G(t, k')$ to the quadruple one. The difference from the ordinary pole diagrams with scalar particles in the intermediate state is connected with the fact that the conservation laws relate the projection m' of the "reggion" spin with the azimuthal angle φ of the vector k' (on Figs. 5 or 6). As a result, the vertex G —the four-point diagram

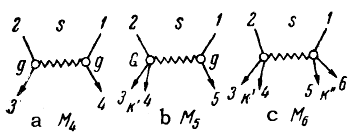

FIG. 7

*ctg = cot.

(16)—depends not only on the two variables (s' and s_{23} for diagrams 7b and c) and the mass t of the "reggion", but also on the "extra" variable, the angle φ. This angle is measured from the direction p_5 (for Fig. 5), i.e., it is connected with the momentum configuration at the other vertex in diagrams of the type 7b and c.

All the results can be readily extended to the case of the many-point diagram of Fig. 8a, namely the amplitude of the process in which two groups of n' and n'' particles (two showers) are produced. The asymptotic value of this amplitude for $s \to \infty$ and for fixed s', s'', and t will have a form (Fig. 8b) which is perfectly analogous to (15):

$$M_n = G_{n'}(t,\xi')\, G_{n''}(t,\xi'')\, I(t)\, s^{\alpha(t)}, \qquad (15')$$

where $n = n' + n'' + 2$ is the total number of vertex lines of Fig. 8a, and ξ' and ξ'' are variables characterizing the states of the particles in both groups (showers) in their c.m.s. (these include, of course, also the squares of the total energies s' and s'' of the particles of each group).

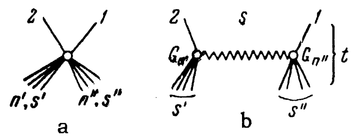

FIG. 8

Formulas (15) and (15′) determine the asymptotic behavior as $s \to \infty$ of "almost" elastic or shower processes, for which the small-angle scattering of fast particles in the c.m.s. of the reaction is accompanied by an excitation of their internal state. As a result of the excitation, the squares of the particle masses become equal to s' and s''; the particles disintegrate and produce showers (which are the narrower, the smaller the ratio s'/s for one shower and s''/s for the other). "Quasi-elastic" processes are usually defined[10] as different elastic collisions, in which the almost elastic (small-angle) scattering of the particles is accompanied by the production of one or more slow particles in the c.m.s. of the reaction. In "quasielastic" and many other—truly inelastic—processes, quantities of the type s' and s'' increase with increasing s (but do not remain constant as $s \to \infty$).

Let us determine the form of this type of asymptotic expression for the amplitudes M_5, M_6, and M_n. We start with the case of the five-point M_5 (Fig. 2). We consider first a case where for fixed values of t and s' we have not only $s \to \infty$, but also $s_{23} \to \infty$. From Fig. 5 and from the formulas for the Lorentz transformation indicated in Sec. 1, it follows that

$$s_{23} = m_2^2 + m_3^2 + 2\varepsilon_3'^{s'-1/2}(\varepsilon_\alpha\varepsilon_2 + p_\alpha p_2) + 2\sqrt{t/s'}\,p_2 p_3' z_3',$$

where all the quantities pertain to the t-channel (and are functions of t and s′, see Sec. 1). Therefore when $s_{23} \to \infty$ we have $z_3' \approx \sqrt{s'/t}\, s_{23}/2p_2p_3'$ $\to \infty$.

We are interested in the physical region of the channel, in which the incoming particles on Fig. 2 are particles 4 and 5, and the produced particles are 1, 2, and 3. In this region (or near it) the invariant s_{35} should be a quantity of the same order as s_{45}, s, and s_{23}. Determining s_{35} with the aid of Fig. 5, we can readily note that this can occur only if the azimuthal angle φ has a fixed unphysical (pure imaginary) value, such that

$$e^{i\varphi} = \left(\frac{\varepsilon_\alpha - p_\alpha}{\varepsilon_\alpha + p_\alpha}\right)^{1/2}.$$

In the region $s \to \infty$, M_5 is determined by (15), and we can obtain from (16) the asymptotic value of $G(t, \mathbf{k}')$ corresponding to $s_{23} \to \infty$, i.e., z_3' $\to \infty$. For this purpose we transform the sum (16) into a Sommerfeld-Watson integral of the form (9), and assume that the extreme right singularity of $v_{l'm'}$ in the l' plane is a pole. Near this pole, the function $v_{l'm'}(t, s')$ can be obtained with the aid of the unitarity conditions in the s′-channel in a form perfectly analogous to (14):

$$\frac{\pi}{2}(2l' + 1)v_{l'm'}(t, s') \approx \frac{u(s')\, w_{l'm'}(t, s')}{l' - \alpha(s')},$$

where both $u(s')$ and $\alpha(s')$ are the same functions as in (14) (we assume for simplicity that the conservation laws allow the existence of a vacuum pole $\alpha(s')$ in the s′ channel, too).

We then obtain from (16) as $z_3' \to \infty$

$$G(t, \mathbf{k}') = g(s')\,\gamma(s', t)\, I_{\alpha'}\, s_{23}^{\alpha'},$$

where $\alpha = \alpha(s')$ and

$$\gamma(s', t) = (2p_2)^{-\alpha(t)}\left(\sqrt{\frac{s'}{t}}\frac{p_1}{2p_2}\right)^{\alpha(s')} \times \sum_{m'=-\infty}^{\infty} v_{\alpha'm'}(t, s')\, C_{\alpha'm'}\, C_{\alpha m'} e^{im'\varphi},$$

with $\alpha = \alpha(t)$, while for $e^{i\varphi}$ it is necessary to substitute here the value indicated above.

We then obtain from (15) for the amplitude M_5

$$M_5 \approx g(s_{34})\,\gamma(s_{34}, t)\, g(t)\, I(s_{34})\, s_{23}^{\alpha(s_{34})} I(t)\, s^{\alpha(t)},$$

which corresponds to the pole diagram of Fig. 9 with two "reggions." For the s-channel (in which the incoming particles are 1 and 2) the analogous

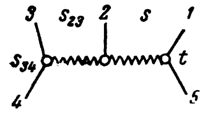

FIG. 9

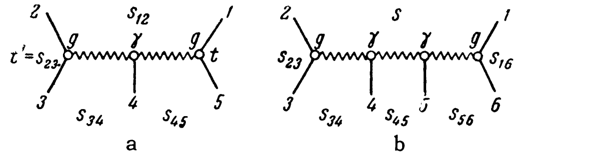

FIG. 10

asymptotic expression has the form (Fig. 10a):

$$M_5 \approx g(t')\,\gamma(t', t)\, g(t)\, I(t')\, s'^{\alpha(t')} I(t)\, s_{45}^{\alpha(t)} \qquad (17)$$

and corresponds to the case when we have $s \to \infty$, $s' \to \infty$, and $s_{45} \to \infty$ for small fixed values of t and $t' = s_{23}$.

In exactly the same way we can obtain the asymptotic form of the six-point amplitude M_6 (see Fig. 3) corresponding to the case when for $s \to \infty$ the invariants $t = s_{12}$, s_{23}, and s_{16} are fixed and small, while $s_{34} \to \infty$, $s_{45} \to \infty$, and $s_{56} \to \infty$. According to Fig. 10b we obtain in this case for M_6

$$M_6 \approx g(s_{23})\,\gamma(s_{23}, t)\,\gamma(t, s_{16})\, g(s_{16}) \times I(s_{23})\, s_{34}^{\alpha(s_{23})} I(t)\, s_{45}^{\alpha(t)} I(s_{16})\, s_{56}^{\alpha(s_{16})}.$$

In analogy with (17), the many-point amplitude M_n, which is the amplitude for the production of two showers with n′ and n″ particles and a group of ν particles with low energy in the c.m.s., will correspond to the diagram of Fig. 11 and to a value

$$M_n = G_{n'}(t', \xi')\,\Gamma_\nu(t', \eta, t'')\, G_{n''}(t'', \xi'')\cdot I(t')\, s'^{\alpha(t')}\cdot I(t'')\, s''^{\alpha(t'')}.$$

Here $n = n' + n'' + \nu + 2$, and η are variables characterizing the state of the group of ν particles in their c.m.s., the remaining notation being clear from Fig. 11.

We hope to study in the future the asymptotic values of the cross sections of various inelastic processes. There are many interesting questions here, such as the most probable momentum configurations of the particles produced in the inelastic processes, and the dependence on s of the total

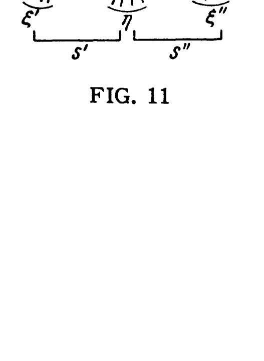

FIG. 11

probabilities of production of a given number of particles as $s \to \infty$.

We consider below only the simplest consequences, for almost-elastic or "shower" type of processes.

3. ASYMPTOTIC CROSS SECTIONS OF ALMOST-ELASTIC, "SHOWER" PROCESSES

From (15) and (15′) we can draw several conclusions, which can be verified experimentally.

1. The s and t dependence of the differential cross sections of processes of the "shower" type as $s \to \infty$ is determined by the same Regge pole[5] as the dependence of the cross sections for elastic scattering:

$$d\sigma_4\,(12 \to 34) \approx g^4\,(t)\; s^{2[\alpha(t)-1]}\; dt,$$

$$d\sigma_5\,(12 \to 345) \approx g^2\,(t)\,|\,G\,(t, \mathbf{k}')\,|^2\; s^{2[\alpha(t)-1]}\; dt\; d\tau', \qquad (18)$$

$$d\sigma_6\,(12 \to 3456) \approx |\,G\,(t, \mathbf{k}')\,|^2 |\,G\,(t, \mathbf{k}'')\,|^2 s^{2[\alpha(t)-1]}\;\; dt\; d\tau'\; d\tau'', \;\ldots;$$

$$d\sigma_n\,(12 \to n' + n'') \approx |\,G_{n'}\,(t, \xi')\,|^2\,|\,G_{n''}\,(t, \xi'')\,|^2\; s^{2[\alpha(t)-1]} dt\; d\xi'\; d\xi''. \qquad (18')$$

Here

$$d\tau' = 2k' s'^{-1/2}\, ds'\; d\mathbf{n}'/4\pi, \;\; d\tau''$$

$d\tau''$ is defined analogously; the factor

$$|\,I\,(t)\,|^2 = 1 + \mathrm{ctg}^2\,(\pi\alpha\,(t)/2),$$

which is of no importance to what follows, is left out everywhere.

The value of $\alpha(t)$ decreases with increasing $(-t)$, and when $(-t)$ is small $\alpha(t) - 1 \approx t\alpha_0'$, where $\alpha_0' = (d\alpha/dt)_{t=0}$ is positive and of the order of $1/M^2$ if M is of the order of the particle mass.

If $s \to \infty$ for fixed s' and s'', then

$$(-t) \;\; \approx {}^1/_2\, s(1 - \cos\vartheta_\alpha) \approx s\vartheta_\alpha^2/4,$$

where ϑ_α is deflection of the summary momentum $\mathbf{p}_\alpha$ of the group of produced particles (in an acute cone in the c.m.s. of the reaction, a cone which is the more acute the larger s'/s) from the initial direction. Therefore the factor

$$(s/M^2)^{2[\alpha(t)-1]} \approx \exp\,[\tfrac{1}{2}\,\alpha_0'\, s\, \vartheta_\alpha^2\, \ln\,(s/M^2)],$$

together with the cross sections (18) and (18′), decreases rapidly with increasing ϑ_α; the only probable values of ϑ_α are of the order $M/\sqrt{s \ln(s/M^2)}$. Substituting this factor in (18) and (18′), we conclude that differential cross sections for the production of one or two groups of particles with given total masses (energies) s' and s'', integrated over ϑ_α^2, (i.e., over t), decrease when $s \to \infty$ like $1/\ln(s/M^2)$, i.e., like the cross sections for elastic scattering[2].

2. It follows from (18) and (18′) that when $s \to \infty$ the distribution over k' of the particles produced in the shower (for example, of particles 3 and 4 in Figs. 7b and c) does not depend on the manner in which this group was produced (on the particle causing it). In particular, the mass spectrum of the produced particles is determined by the function

$$\eta\,(t; s') = \frac{2k'}{\sqrt{s'}} \int |\,G\,(t, \mathbf{k}')\,|^2 \frac{d\mathbf{n}'}{4\pi},$$

and depends on the momentum transfer t, and not on the kind and the number of particles entering into the second vertex of Figs. 7b and c).

3. In the region $s \to \infty$ the cross sections for the almost-elastic "shower" type of process are interrelated by a whole set of equations of the type derived by Gribov, Pomeranchuk[9], and Gell-Mann[11] for processes with two particles in the final state (some of these relations are mentioned in the article by Gribov, Ioffe, Rudik, and Pomeranchuk[12]). All the cross sections contained in these relations can be directly measured for inelastic processes.

Let us consider, for example, pion production in πN and NN collisions. The corresponding cross sections (12a and b) will be denoted by $\sigma_\pi(t, \mathbf{k}')$ and $\sigma_N(t, \mathbf{k}')$ (their dependence on s is not indicated, it being implied that the value of s is the same everywhere). Let $\sigma_{\pi N}(t)$ and $\sigma_{NN}(t)$ be the differential cross sections for πN and NN scattering (Figs. 12d and e), and let $\sigma_N(t; \mathbf{k}', \mathbf{k}'')$ be the cross section of the process $N + N \to (N + \pi)' + (N + \pi)''$ (Fig. 12c). Writing all these cross sections, in accordance with Fig. 12, in the form (15) we obtain, as $s \to \infty$, the following relations between them:

$$\sigma_\pi\,(t, \mathbf{k}') = \frac{\sigma_{\pi N}\,(t)}{\sigma_{NN}\,(t)}\,\sigma_N\,(t, \mathbf{k}'),$$

$$\sigma_{NN}\,(t)\,\sigma_N\,(t;\; \mathbf{k}',\; \mathbf{k}'') = \sigma_N\,(t, \mathbf{k}')\,\sigma_N\,(t,\; \mathbf{k}'').$$

These relations, as well as many others for reactions with different particles, can be directly checked by experiment.

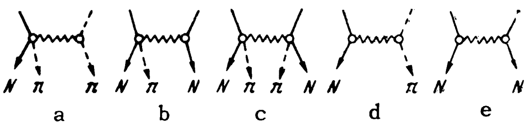

FIG. 12

[5] Naturally, under the condition that this is permitted by the laws of conservation of isotopic spin, parity, strangeness, etc.

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APPENDIX

EXPANSION IN STATES WITH DEFINITE MOMENTA AND HELICITIES

We present another method[5], different from that considered in Sec. 1, for constructing expansions (4) and (5). We write the amplitudes M_4, M_5, and M_6 in the form

$$M_4 = \langle \mathbf{p}_\alpha | \hat{T} | \mathbf{p}_\beta \rangle,$$

$$M_5 = \langle \mathbf{p}_\alpha \mathbf{k}' | \hat{T} | \mathbf{p}_\beta \rangle, \qquad M_6 = \langle \mathbf{p}_\alpha \mathbf{k}' | \hat{T} | \mathbf{p}_\beta \mathbf{k}'' \rangle, \qquad \text{(a)}$$

where $\hat{T}$ is the T-matrix and $|\mathbf{p}_\beta\rangle$, $|\mathbf{p}_\beta \mathbf{k}''\rangle$, and $|\mathbf{p}_\alpha\rangle$, $|\mathbf{p}_\alpha \mathbf{k}'\rangle$ [with $\langle \mathbf{p}_\alpha | = (|\mathbf{p}_\alpha\rangle)^*$, $\langle \mathbf{p}_\alpha \mathbf{k}'| = (|\mathbf{p}_\alpha \mathbf{k}'\rangle)^*$] are the initial and final states of the system of two or three particles (on Figs. 4, 2, and 3), corresponding to definite values of their momenta. A system of three particles (for example, the three particles 2, 3, and 4 in the final state on Figs. 2 and 3) is characterized by a summary momentum $\mathbf{p}_\alpha = \mathbf{p}_3 + \mathbf{p}_4$ of two particles (in the c.m.s. of the three particles, i.e., $\mathbf{p}_\alpha = -\mathbf{p}_2$) and a momentum $\mathbf{k}'$ of relative motion of the same two particles in their c.m.s.

We denote the eigenstate of the system of two particles (for example, particles α and 2 in Fig. 1), in the case when the spin of one of them differs from zero, by $|\mathbf{p}_\alpha; l'm'\rangle$; here l' is the spin of particle α and m' is its projection on $\mathbf{p}_\alpha$. When $l' = m' = 0$, the state $|\mathbf{p}_\alpha; l'm'\rangle$ will be denoted by $|\mathbf{p}_\alpha\rangle$. Precisely similar symbols are used to describe the initial states ($\mathbf{p}_\beta = \mathbf{p}_5 + \mathbf{p}_6$ and $\mathbf{k}'' = \mathbf{p}_5'' = -\mathbf{p}_6''$). According to Jacob and Wick[5]

$$|\mathbf{p}_\alpha;\ l'm'\rangle = \sum_{LM} \sqrt{2L+1}\, D^{(L)}_{M,m'}(\mathbf{n}_\alpha)\, |LM;\ l'm'\rangle, \qquad \text{(b)}$$

where $|LM; l'm'\rangle$ is the state of particles 2 and α, corresponding to definite l' and m' and to definite values of the total momentum L of these particles and its projection M on the z axis (the direction of which is chosen arbitrarily).

As in Sec. 1, we consider the particle α to be compound and consist of "spinless" particles 3 and 4. Then $|\mathbf{p}_\alpha; l'm'\rangle$ describes a state of the particles 2, 3, and 4 such that particles 3 and 4 in their c.m.s. are in a state with definitive values of angular momentum l' of their relative motion, and its projection m' on $\mathbf{p}_\alpha$. The state $|\mathbf{p}_\alpha; \mathbf{k}'\rangle$ will, in analogy with (b), be a linear combination of the functions $|\mathbf{p}_\alpha; l'm'\rangle$:

$$|\mathbf{p}_\alpha;\ \mathbf{k}'\rangle = \sum_{l'm'} (2l'+1)\, D^{(l')}_{m',0}(\mathbf{n}')\, |\mathbf{p}_\alpha;\ l'm'\rangle \qquad \text{(c)}$$

(for convenience in normalization we write in (c) $2l' + 1$ in place of $\sqrt{2l'+1}$).

Putting in (b) $l' = m' = 0$, or substituting (b) in (c), we obtain the following expansions for the eigenfunctions of the final states in (a):

$$|\mathbf{p}_\alpha\rangle = \sum_{LM} \sqrt{2L+1}\, D^{(L)}_{M,0}(\mathbf{n}_\alpha)\, |LM\rangle,$$

$$|\mathbf{p}_\alpha,\ \mathbf{k}'\rangle = \sum_{LM,\, l'm'} \sqrt{2L+1}$$

$$\times\ (2l'+1)\, D^{(L)}_{M,m}(\mathbf{n}_\alpha)\, D^{(l')}_{m',0}(\mathbf{n}')\, |LM,\ l'm'\rangle. \qquad \text{(d)}$$

Writing down the wave functions of the initial states in (a) in the same form, substituting in (a), and recognizing that the T-matrix is diagonal in L, we obtain

$$M_4 = \sum_{LM} (2L+1)\, D^{(L)*}_{M,0}(\mathbf{n}_\alpha)\, D^{(L)}_{M,0}(\mathbf{n}_\beta)\, \lambda_L(t),$$

$$M_5 = \sum_{LM,\, l'm'} (2L+1)(2l'+1)\, D^{(L)*}_{M,m'}(\mathbf{n}_\alpha)$$

$$\times\ D^{(L)}_{M,0}(\mathbf{n}_\beta)\, D^{(l')*}_{m',0}(\mathbf{n}')\, \varphi^{(l')}_{L;m'}(t;\ s'),$$

$$M_6 = \sum_{LM,\, l'm',\, l''m''} \varkappa(L,\ l',\ l'')\, D^{(L)*}_{M,m'}(\mathbf{n}_\alpha)\, D^{(L)}_{M,m''}(\mathbf{n}_\beta)$$

$$\times\ D^{(l')*}_{m',0}(\mathbf{n}')\, D^{(l'')}_{m'',0}(\mathbf{n}'')\, f^{(l',\, l'')}_{L;m'm''}(t;\ s',\ s''), \qquad \text{(e)}$$

where

$$\varkappa(L,\ l',\ l'') = (2L+1)(2l'+1)(2l''+1),$$

$$\lambda_L(t) = \langle LM | \hat{T} | LM \rangle,$$

$$\varphi^{(l')}_{L;m'}(t;\ s') = \langle LM,\ l'm' | \hat{T} | LM \rangle,$$

$$f^{(l',\, l'')}_{L;m'm''}(t;\ s',\ s'') = \langle LM,\ l'm' | \hat{T} | LM,\ l''m'' \rangle$$

are the partial many-point amplitudes. Choosing the z axis along n (i.e., putting $D^{(L)}_{M,m'}(\mathbf{n}_\alpha) = \delta_{M,m'}$), we obtain directly the expansions (4) and (5).

The expansions (b), (c), and (d) can be generalized in trivial fashion to the case when the spins of all the particles differ from zero. Then, for example, the eigenfunction $|\mathbf{p}_\alpha; \mathbf{k}_x'; l_x', m_x'\rangle$ of the system of three particles 2, 3, and x, where l_x' is the spin of the x particle and m_x' is its projection on the direction $\mathbf{k}_x'$ of its momentum in the c.m. system of particles 3 and x, can be written in perfect analogy with (d):

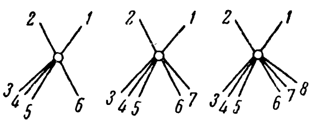
$$|\mathbf{p}_\alpha;\mathbf{k}'_x;\, l'_x m'_x\rangle = \sum_{LM,\, l'm'} \sqrt{2L+1}\,(2l'+1)\, D^{(L)}_{M,\, m'}(\mathbf{n}_\alpha) \times D^{(l')}_{m',\, m'_x}(\mathbf{n}'_x)\,|LM;\, l'm';\, l'_x m'_x\rangle.$$

Regarding x as consisting of two particles 4 and 5, we obtain in analogy with (d) the following expansion for the eigenstate $|\mathbf{p}_\alpha;\mathbf{k}'_x, \mathbf{q}'\rangle$ of the system of four particles 2, 3, 4, and 5 ($\mathbf{q}'$ is the momentum of particles 4 and 5 in their c.m.s.):

$$|\mathbf{p}_\alpha,\, \mathbf{k}'_x,\, \mathbf{q}'\rangle = \sum_{LM,\, l'm',\, l'_x m'_x} \frac{\varkappa(L,\, l',\, l'_x)}{\sqrt{2L+1}}\, D^{(L)}_{M,\, m'}(\mathbf{n}_\alpha)\, D^{(l')}_{m',\, m'_x}(\mathbf{n}'_x) \times D^{(l')}_{m',\, 0}(\mathbf{n}')\,|LM;\, l'm';\, l'_x m'_x\rangle.$$

Using this equation, we readily obtain expansions perfectly similar to (e) and (4), (5) for amplitudes of six-, seven-, and eight-point diagrams (Figs. 13a, b, c):

$$M'_6 = \langle \mathbf{p}_\alpha,\, \mathbf{k}'_x,\, \mathbf{q}' \,|\, \hat{T} \,|\, \mathbf{p}_\beta\rangle, \qquad M_7 = \langle \mathbf{p}_\alpha,\, \mathbf{k}'_x,\, \mathbf{q}' \,|\, \hat{T} \,|\, \mathbf{p}_\beta,\, \mathbf{k}''\rangle,$$
$$M_8 = \langle \mathbf{p}_\alpha,\, \mathbf{k}'_x,\, \mathbf{q}' \,|\, \hat{T} \,|\, \mathbf{p}_\beta,\, \mathbf{k}''_y,\, \mathbf{q}''\rangle.$$



a b c

FIG. 13

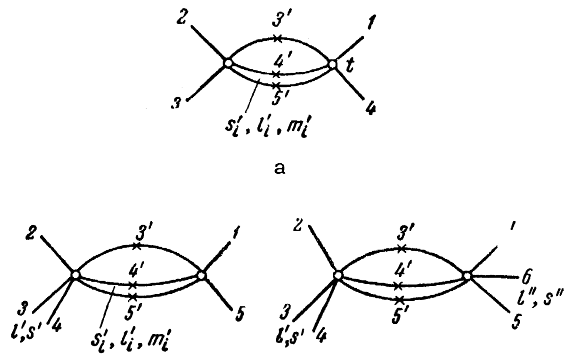
It is clear that expansions can be obtained in this manner for the amplitudes of arbitrary multi-point diagrams.

In conclusion let us dwell briefly on the unitarity conditions for the helical partial amplitudes. They can be obtained from the unitarity conditions for M_4, M_5, and M_6 and from the expansions (e), or else directly, using eigenfunctions of the type $|LM, l'm'\rangle$, $|LM\rangle$ and calculating different matrix elements from the operator equation $T - T^+ = iTT^+$. When taking two-particle states into account, the values given in the right half of (6) arise. When three-particles states are taken into account, it is necessary to add to the right half of (6) terms of the type

$$2i\rho(t) \sum_{l'_i,\, m'_i} (2l'_i + 1) \int\limits_{(m_5+m_4)^2}^{(\sqrt{t}-m_3)^2} B\,(t,\, s'_i)\, \frac{2k'_i}{\sqrt{s'_i}}\, ds'_i,$$

which correspond to the diagrams of Figs. 14a, b, and c. In accordance with these diagrams, the respective values of B for the first, second, and third lines of equations (6) are

$$\varphi^{(l'_i)}_{L;\, m'_i}(t_+,\, s'_{i+})\, \varphi^{(l'_i)}_{L;\, m'_i}(t_-;\, s'_{i-}),$$
$$f^{(l',\, l'_i)}_{L;\, m',\, m'_i}(t_+,\, s',\, s'_{i+})\, \varphi^{(l'_i)}_{L;\, m'_i}(t_-,\, s'_{i-}),$$
$$f^{(l',\, l'_i)}_{L;\, m',\, m'_i}(t_+,\, s',\, s'_{i+})\, f^{(l'_i,\, l'')}_{L;\, m'_i,\, m''}(t_-,\, s'_{i-},\, s''),$$



a

b c

FIG. 14

where $s'_{i\pm} = s'_i \pm i\tau$. Here, as in (6), s′ and s″ can have arbitrary complex values.

Note added in proof (December 8, 1962). Cook and Lee[13], in articles published after this paper was completed, also obtained expressions in the form (4) and (5), and unitarity conditions as written out in (6) and in the appendix.

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