THE REGGE POLE TRAJECTORIES FOR WEAK COUPLING

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The Regge pole trajectories are calculated for scattering by a Yukawa potential (or a superposition of such potentials) for the case of weak coupling. The motion of the poles exhibits the following qualitative peculiarities: 1) the poles oscillate about negative integer points for energies corresponding to the left cut of the partial wave amplitudes, 2) the poles collide and move off into the complex plane already at negative energies, 3) there is an accumulation of poles at the point $l = -1/2$ at threshold energy, 4) the trajectories do not close. Certain peculiarities arising when the coupling is stronger are also discussed.

UNTIL recently, in the study of the analytic properties of partial wave amplitudes as a function of the angular momentum one's attention was concentrated on the singularity that lay farthest to the right in the l plane. [1] These singularities govern the behavior of the main terms in the asymptotic expansion of the amplitude at large momentum transfers. However if one thinks of the "reggeization" hypothesis as forming the basis for the construction of a theory of strong interactions, [2] then the trajectories of all Regge poles become significant.

In Sec. 1 we determine the pole trajectories for the scattering by a Yukawa potential (or a superposition of such potentials) for the case of weak coupling. We establish a connection between poles that lie, analogous to the case of the Coulomb interaction, at negative integer points for infinite energy and poles that accumulate at the point $l = 1/2$ for threshold energy. [3] Some of the properties of the pole trajectories should, apparently, survive also in a strong coupling theory. This, in particular, should be true of the pole osciallations, the moving off into the complex plane for negative energies, the accumulation for threshold energy and the open nature of the trajectories.

In Sec. 2 a discussion of the results is given together with a comparison with the results of numerical computations carried out by Ahmadzadeh et al. [4]

The connection between moving Regge poles and the perturbation theory amplitude at large momentum transfers and the situation that arises in field theory will be discussed in a subsequent paper.

1. THE REGGE POLE TRAJECTORY FOR A YUKAWA POTENTIAL

It is convenient to write the equation determining the pole trajectory in terms of the exact radial wave function. This function satisfies the following integral equation:

$$\psi_l(r) = j_l(kr) + \frac{1}{k\cos\pi l}\int_0^r [j_l(kr)\, j_{-l-1}(kr') - j_{-l-1}(kr)\, j_l(kr')]\, U(r')\, \psi_l(r')\, dr',$$

$$j_l(x) = \sqrt{\pi x/2}\, J_{l+1/2}(x),\quad U(r) = 2mV(r) = 2ma r^{-1} e^{-\mu r}, \quad (1)$$

where J is the Bessel function.

The behavior of the wave function at small r is determined by the free term. For large r the function $\psi_l(r)$ contains incoming and outgoing waves. Making use of Eq. (1) at large r one easily obtains the following expression for the S matrix in terms of the wave function:

$$S_l = e^{2i\delta_l} = \left[1 + \frac{1}{ik}\int_0^\infty h_l^{(2)}(kr)\, U(r)\, \psi_l(r)\, dr\right] \times \left[1 - \frac{1}{ik}\int_0^\infty h_l^{(1)}(kr)\, U(r)\, \psi_l(r)\, dr\right]^{-1}. \quad (2)$$

The functions $h_l^{(1,2)}$ are related to the Hankel functions by:

$$h_l^{(1,2)}(x) = \sqrt{\pi x/2}\, H_{l+1/2}^{(1,2)}(x).$$

As has been shown by a number of authors [1,5] the only singularities of S_l in the l plane consist of moving poles, whose position is determined by the zeros of the denominator in Eq. (2). It is easy to rewrite the equation for the pole trajectory by expressing $h_l^{(1)}$ in terms of the Bessel functions j_l and j_{-l-1}:

$$\frac{1}{k}\int_0^\infty j_l(kr)\, U(r)\, \psi_l(r)\, dr = \frac{ie^{i\pi l}}{k}\int_0^\infty j_{-l-1}(kr)\, U(r)\, \psi_l(r)\, dr + ie^{i\pi l}\cos\pi l. \quad (3)$$

In the framework of perturbation theory the wave function $\psi_l(r)$ may be found by integration of Eq. (1). In lowest approximation in the potential one obtains the following equation for the pole trajectory:

$$\frac{\alpha m}{k} Q_l\left(1+\frac{\mu^2}{2k^2}\right) = ie^{i\pi l}\frac{\alpha m}{k} R_l\left(1+\frac{\mu^2}{2k^2}\right) + ie^{i\pi l}\cos \pi l;$$

$$Q_l\left(1+\frac{\mu^2}{2k^2}\right) = 2\int_0^\infty j_l^2(kr)\frac{e^{-\mu r}}{r}dr, \qquad (4)$$

$$R_l\left(1+\frac{\mu^2}{2k^2}\right) = 2\int_0^\infty j_l(kr)\, j_{-l-1}(kr)\frac{e^{-\mu r}}{r}dr. \qquad (5)$$

$Q_l(z)$ is the Legendre function of the second kind. In the complex l plane Q_l has poles at negative integer points while R_l is an entire function. For $R_l(1+\mu^2/2k^2)$ the point $k^2 = 0$ is a root-type branch point, and the point $k^2 = -\mu^2/4$ is a logarithmic branch point. The analytic properties of Q_l as a function of its argument are well known.

For $k^2 < 0$ Eq. (4) becomes an equation for $l(k^2)$ with real coefficients. For $k^2 < -\mu^2/4$ the functions Q_l and R_l acquire an additional complex part. However on the physical sheet ($k = i\kappa$, $\kappa > 0$) this complex part cancels out. Consequently in the whole $k^2 < 0$ region the poles are either real or form complex conjugate pairs.

For a superposition of Yukawa potentials

$$U(r) = 2mV(r) = 2m\sum_s \alpha_s r^{-1} e^{-\mu_s r}$$

Eq. (4) must be replaced by

$$\sum_s \frac{\alpha_s m}{k} Q_l\left(1+\frac{\mu_s^2}{2k^2}\right) = ie^{i\pi l}\left[\sum_s \frac{\alpha_s m}{k} R_l\left(1+\frac{\mu_s^2}{2k^2}\right) + \cos \pi l\right]. \qquad (4a)$$

Let us follow in detail the motion of the poles as a function of k^2. Equation (4) contains terms linear in α and independent of α. For large negative k^2 the small factor $\alpha m/k$ can be compensated in only one way, namely by the large value of Q_l which arises in the neighborhood of negative integer points $l = -n-1$, where the Legendre function of the second kind has simple poles with residues equal to the Legendre polynomials P_n. Near such points Eq. (4) takes the form

$$\frac{\alpha m}{k}\frac{P_n(1+\mu^2/2k^2)}{l+n+1} = i, \qquad (6)$$

so that the trajectory is described by the expression

$$l_n = -n-1+\frac{\alpha m}{ik}P_n\left(1+\frac{\mu^2}{2k^2}\right), \qquad k = i\varkappa, \quad \varkappa > 0. \qquad (7)$$

It is clear from its derivation that Eq. (7) is valid only for l_n sufficiently close to $-n-1$, which is certainly true for sufficiently small $\alpha m/\mu$ and finite n up to small k^2/μ^2. For $k^2 = -\infty$ all the poles lie at negative integer points. As k^2 starts to increase they move along the real axis oscillating about integer points, remaining close to them at distances very small compared to unity. At $k^2 = -\mu^2/4$ the argument of the Legendre polynomials becomes equal to -1 after which all the poles move monotonically along the real axis. For $\alpha > 0$ (repulsion) all the poles with even n go to the left, all those with odd n go to the right and, consequently, the even poles approach their left neighbors. In particular the zeroth pole approaches the first one. For $\alpha < 0$ (attraction) the direction of motion of the poles changes so that the even poles approach their right neighbors. The zeroth pole turns out to be a special case in the sense that it does not approach any other pole. As k^2 increases it moves monotonically to the right and, as can be seen from Eq. (4), for $k^2 = 0$ comes out to the right of $l = -1/2$ by the amount $-\alpha m/\mu$.[1]

Equation (7) ceases to be valid when the distance between poles becomes small. Until the end of the region of applicability of this formula is reached we follow the poles moving monotonically towards each other with a velocity that increases with increasing k^2 and that grows exponentially with the number n. It therefore seems that as k^2 continues to increase the poles collide, and, possibly move off into the complex plane. This will be indeed shown to follow from Eq. (4) somewhat later. We note right now that poles with arbitrarily large numbers should collide for k^2 arbitrarily close to $-\mu^2/4$. Thus, in crossing the point $-\mu^2/4$ an infinite number of poles goes off into the complex plane. Such a drastic change in the singularity structure of the partial wave amplitude in l appears natural since the point $k^2 = -\mu^2/4$ corresponds to the beginning of the left cut of the partial amplitude.

For very small k^2 Eq. (7) becomes invalid. In the region $|k^2/\mu^2| \ll 1$ one may make use of the asymptotic form of the functions $Q_l(1+\mu^2/2k^2)$ and $R_l(1+\mu^2/2k^2)$ for large arguments. It is relevant to note that the region of applicability of the resultant equation overlaps with the region of applicability of Eq. (7). This has to do with the fact that Eq. (7) remains valid for smaller k^2 the smaller the quantity $\alpha m/\mu$. In this way we can study continuously the pole trajectory.

For l not half-integer the function $R_l(1+\mu^2/2k^2)$ goes like k/μ for $k^2/\mu^2 \to 0$. Therefore the corresponding term in Eq. (4) is small: $\sim \alpha m/\mu$. Under these same conditions the function $Q_l(1+\mu^2/2k^2)$ goes like $(k^2/\mu^2)^{l+1}$. For $\mathrm{Re}\, l < -1/2$ this term in Eq. (4) increases with decreasing $|k^2|$. Therefore Eq. (4) can now be satisfied at some distance from the poles of Q_l. In that region Eq. (4) becomes

$$\frac{\alpha m}{k}\left(-\frac{ie^{-i\pi l}}{\sin \pi l}\right)\frac{\sqrt{\pi}\,\Gamma(-l-{}^1/_2)}{\Gamma(-l)}\left(\frac{k^2}{\mu^2}\right)^{l+1}=1. \quad (8)$$

Equation (8) is not applicable near negative half-integer points. As is shown in the Appendix, the length of the segment of the trajectory that falls out of Eq. (8) as the pole approaches these points is of the order of $\alpha m/\mu$, i.e., always small.

Let us follow the motion of the poles in the region $|k^2/\mu^2| \ll 1$ according to Eq. (8). For $k^2 = -\kappa^2 < 0$ it may be rewritten in the form

$$\xi_l\left(\frac{\kappa^2}{\mu^2}\right)^{l+{}^1/_2}=-\sin \pi l, \qquad \xi_l=-\frac{\alpha m}{\mu}\frac{\sqrt{\pi}\,\Gamma(-l-{}^1/_2)}{\Gamma(-l)}. \quad (9)$$

Except for the neighborhood of the point $l = -1/2$ the quantity ξ_l is a slowly varying function of l which, for simplicity, we may take as constant. This does not change the trajectory qualitatively. For $\kappa^2 \sim \mu^2$ the roots of Eq. (9) lie near negative integer points (zeros of the sine function) in agreement with Eq. (7). As κ^2 decreases neighboring roots of (9) move towards each other, collide and go off into the complex plane becoming each other's complex conjugate. The position of the point at which they leave the real axis may be determined by solving simultaneously Eq. (9) and the equation obtained by differentiating both sides of Eq. (9) with respect to l.

Since the departure of the poles into the complex plane is connected with the coincidence of two roots of Eq. (9) it is clear that the trajectories of both poles have at that point a root type branch point. The point of coincidence of the poles is not a singular point of the amplitude since it lies to the right of the left cut. That is possible only in the case when the trajectories of both the colliding poles are branches of one and the same analytic function.

When the poles are sufficiently far removed from the real axis Eq. (9) can be simplified by keeping on the right side only the large exponential. Then, upon taking logarithms, Eq. (9) becomes

$$i\pi(l+{}^1/_2)-(l+{}^1/_2)\tau+\ln 2\xi_l=-2\pi i p. \quad (10)$$

This equation is written for poles moving in the upper half plane; p is a positive integer, $\tau = \ln(\mu^2/\kappa^2)$. From here it is easy to see that the pole describes a trajectory that is nearly circular with the equation

$$(\mathrm{Re}\, l+p+{}^1/_2)^2+(\mathrm{Im}\, l-(2\pi)^{-1}\ln 2\xi_l)^2=p^2 + (2\pi)^{-2}\ln^2 2\xi_l. \quad (11)$$

Equation (11) refers to the attractive case ($\xi_l > 0$), in the case of repulsion one must make the substitution $p \to p + 1/2$ and $\xi_l \to -\xi_l$.

By following the motion away from the real axis it is easy to establish the connection between p and the limiting position of the pole when $k^2 = -\infty$, $l_n = -n - 1$. Namely the trajectory with the number p corresponds to the colliding pair of poles in which the even pole has $n = 2p$ both in the case of attraction and in the case of repulsion.

As $\kappa^2 \to 0$ all poles move towards the point $l = -1/2$. From the nature of their motion in the vicinity of this point it is clear that they represent the Gribov and Pomeranchuk poles [3] accumulating at the threshold energy. Thus in the case of the Yukawa potential the Gribov and Pomeranchuk poles turn out to be the same poles that at $k^2 = -\infty$ lie at the negative integer points.

For $k^2 > 0$ the poles diverge into the upper and lower half planes without changing the sign of Im l in going through $k^2 = 0$. We consider first poles lying in the upper half plane. It can be shown from Eq. (4) that they move to the right of the line Re $l = -1/2$ by the small amount $\sim \alpha m/\mu$ and then return into the half plane Re $l < -1/2$. Thereafter their motion is again described by Eq. (8). Far from the real axis we may again replace the sine by the exponential and, upon taking logarithms, arrive at the equation

$$-(l+{}^1/_2)\tau'+\ln 2\xi_l=-2\pi i p, \qquad \tau'=\ln(\mu^2/k^2). \quad (12)$$

Here p is the same as in Eq. (10).

In contrast to the case of negative energies, where the motion of the poles in the upper half-plane followed the circles, Eq. (11), now we find in the approximation ξ_l = const that the trajectories are given by straight lines going off into the second quadrant of the l plane:

$$2\pi p\,(\mathrm{Re}\, l+{}^1/_2)=\ln 2\xi_l \cdot \mathrm{Im}\, l. \quad (13)$$

Equation (13) is written for the case of attraction; in the case of repulsion one must, as before, make the substitutions $p \to p + 1/2$ and $\xi_l \to -\xi_l$.

According to Eq. (13) as k^2 increases the poles go off in the complex plane moving away from the real axis. For $k^2/\mu^2 \sim 1$ they move away from zero by the large distance $\sim \ln \xi_l$. In order to study their further motion one may utilize in the basic Eq. (4) the asymptotic behavior of the functions Q_l and R_l for large indices. The region of applicability of the resultant equation overlaps the region of applicability of Eqs. (12) and (13) so that again we can follow the poles continuously.

For large l in the second quadrant Q_l grows exponentially with increasing l, whereas $e^{i\pi l} R_l \sim 1/l$. Consequently also here the quantity R_l may be ignored and on replacing the cosine by the growing exponential we arrive at the equation

$$-\frac{\alpha m}{k}\sqrt{-2\pi/l\,\mathrm{sh}\,\eta}\,e^{-(l+1/2)\eta}=1,\quad \mathrm{ch}\,\eta=1+\mu^2/2k^2. \quad (14)^*$$

For $k^2/\mu^2 \gg 1$ Eq. (14) simplifies and upon taking of logarithms becomes

$$\ln(-\alpha m\sqrt{2\pi}/\mu)+{}^1/_2\ln(\mu/k)$$
$$-{}^1/_2\ln(-l)-(l+{}^1/_2)\mu/k=-2\pi ip. \quad (15)$$

The pole number p is the same as in Eqs. (10) and (12).

It is easy to show from Eq. (15) that the following equalities are satisfied asymptotically by the pole trajectory:

$$\mathrm{Im}\,l=2\pi p\frac{k}{\mu},\qquad \mathrm{Re}\,l=\frac{k}{\mu}\ln\left|\frac{\alpha m}{\mu}\sqrt{\frac{2\pi}{l}\frac{k}{\mu}}\right|. \quad (16)$$

Thus the pole goes off to infinity and both $|\mathrm{Re}\,l|$ and $\mathrm{Im}\,l$ increase without bound.

Equations (15) and (16) refer to the case of attraction; for the case of repulsion one must, as before, make the substitutions $p \to p + 1/2$, $\alpha \to -\alpha$.

Let us pass now to the poles that lie in the lower half plane for small k^2. For negative k^2 they were complex conjugates of the poles in the upper half plane, for $k^2 > 0$ their trajectories acquire an entirely different character. It is easy to show from Eq. (4) that these poles remain at all times to the left of the line $\mathrm{Re}\,l = -1/2$, in agreement with the results of Regge.[1] After leaving the point $l = -1/2$ by a distance larger than $\alpha m/\mu$ their motion is again described by Eq. (8). Replacing in the lower half-plane $\sin\pi l$ by the growing exponential we arrive upon taking of logarithms at the equation

$$\ln 2\xi_l-\tau'(l+{}^1/_2)-2\pi i(l+{}^1/_2)=2\pi ip. \quad (17)$$

For negative energies ($\tau' = \tau - i\pi$) Eq. (17) is the same as the complex conjugate of Eq. (10), $p > 0$. Therefore the trajectory with the number p corresponds now, as before, to a colliding pair of poles wherein the even pole has the number $n = 2p$ ($l_n = -n - 1$ for $k^2 = -\infty$). For repulsion, as before, one must make the substitution $p \to p + 1/2$, $\xi_l \to -\xi_l$. To Eq. (17) corresponds motion on a circle:

$$(\mathrm{Re}\,l+p/2+{}^1/_2)^2+(\mathrm{Im}\,l+(4\pi)^{-1}\ln 2\xi_l)^2$$
$$=p^2/4+(4\pi)^{-2}\ln^2 2\xi_l. \quad (18)$$

It is easy to see upon comparison of Eqs. (18) and (11) that for positive k^2 the circle in the lower half plane has a radius half as large as for $k^2 < 0$. On approaching the real axis one must instead of Eq. (18) use directly Eq. (8).

Let us consider first the simpler case of repulsion ($\alpha > 0$). Then Eq. (8) has no real solutions and the pole cannot fall on the real axis. However already for small k^2/μ^2 the poles approach the neighborhood of negative integer points. Thereafter their motion is described by Eq. (7), so that as $k^2 \to +\infty$ the poles approach from below negative integer points. The pole with the number p, referring to the pair which for $k^2 = -\infty$ was at the points $-2p - 1$ and $-2p - 2$, now goes to the point $-p - 1$.

In the case of attraction ($\alpha < 0$) the poles, as can be seen from Eq. (8), cross the real axis at negative half-integer points. The pole with number p crosses the axis at the point $-p - 1/2$ at the energy

$$\frac{k_p^2}{\mu^2}=\left[-\frac{\alpha m}{\mu}\frac{\sqrt{\pi}\,\Gamma(p)}{\Gamma(p+{}^1/_2)}\right]^{1/p}. \quad (19)$$

It is relevant that these points lie in the region of applicability of Eq. (8). For the first few numbers the values (19) for k_p^2 agree in the weak coupling limit with the exact values.[4]

In the vicinity of the crossing points Eq. (17) should be replaced by the more exact equation in which both exponentials of $\sin\pi l$ are taken into account. Introducing $l_1 = l + p + 1/2$, which is the distance from the point of crossing, we obtain for l_1 the equation:

$$\ln 2\xi_l+\tau' p-\tau' l_1-2\pi i l_1-\ln(1+e^{-2\pi i l_1})=0. \quad (20)$$

It is easy to show from Eq. (20) that the imaginary part of l_1 remains small during the motion in the upper half plane. The real part becomes negative so that the pole moves to the left in the vicinity of the real axis. With further increase of k^2 the term $\tau' p$ becomes unimportant and the motion of all poles becomes practically the same. The large quantity $\ln 2\xi_l$ is now compensated by the large quantity $\ln(\cos\pi l_1)$ when l_1 approaches the vicinity of $-1/2$. Thereafter the motion of the poles is again described by Eq. (7) and the pole approaches from above the point $l = -p - 1$ as $k^2 \to +\infty$.

2. DISCUSSION OF RESULTS

Let us summarize the nature of the pole trajectories found above. For $k^2 = -\infty$ all the poles lie at negative integer points. For $-\infty < k^2 < -\mu^2/4$ (left cut) the poles oscillate about their limit positions. Thereafter they start to move monotonically, collide in pairs and leave the real axis becoming each other's complex conjugates. This does not apply to the right-most pole in the case of attraction. For $k^2 \to 0$ all poles approach the point

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*sh = sinh; ch = cosh.

$l = -1/2$. For $k^2 > 0$ they keep the sign of their imaginary parts unchanged. The poles that lie in the upper half plane move slightly to the right of the line $\mathrm{Re}\, l = -1/2$, then return into the left half plane and go off to infinity in the second quadrant of the l plane.

The poles that lie in the lower half-plane move on a trajectory that is nearly circular and whose radius is half as large as for $k^2 < 0$. In the case of repulsion they tend to the negative integer points and reach them from below for $k^2 \to +\infty$. In the case of attraction they cross the real axis at negative half-integer points and for $k^2 \to +\infty$ reach from above the nearest integer point to the left. The zeroth pole in the case of attraction returns to its point of departure through the upper half-plane. Thus all the trajectories, except for that of the zeroth pole in the case of attraction, are open. Schematically the form of the trajectories is shown in Fig. 1 ($k^2 < 0$) and Fig. 2 ($k^2 > 0$). Figure 1 corresponds to attraction. In Fig. 2 the dashed lines correspond to repulsion.

It is of interest to compare our results with the results of numerical calculations. Ahmadzadeh et al [4] have carried out the corresponding calculation for positive energies for various values of the coupling constant. Weak coupling was considered for the case of attraction ($-2\alpha m/\mu = A = 0.05$). The

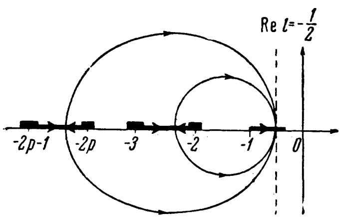

FIG. 1. The pole trajectories for $k^2 < 0$. Motion in the direction of the arrow corresponds to k^2 increasing from large negative values to zero. In the blackened regions near integer values $-l$ the trajectories oscillate. The case of attraction is shown. For repulsion the picture looks analogously except that the colliding poles start out from the points $-2p$ and $-2p+1$.

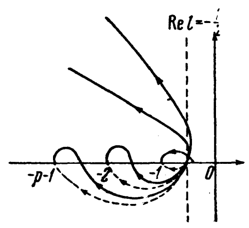

FIG. 2. The pole trajectories for $k^2 > 0$. Motion in the direction of the arrow corresponds to k^2 increasing from 0 to $+\infty$. The solid curves correspond to attraction, the dashed curves in the lower half plane to repulsion. The form of the curves that go off to infinity in the second quadrant is qualitatively the same for repulsion as for attraction.

trajectories of poles that are situated at negative integer points for $k^2 = +\infty$ were followed. The nature of the first trajectories fully corresponds to the picture described above (Fig. 2).

Some of the more distant trajectories calculated in [4] show a different behavior. Namely for $k^2 = 0$ some poles are situated on the negative real axis to the left of the point $l = -1/2$. On comparing the results of the computations for various values of the coupling constant Ahmadzadeh et al [4] arrive at the conclusion that the position of these poles at zero energy tends to $-\infty$ as $A = -2m\alpha/\mu \to 0$. It is therefore natural that these poles do not appear in the lowest order of perturbation theory.

It is curious that such poles appear already in the next order of perturbation theory. Indeed, it can be shown that accurate to terms of order $-(\alpha m/\mu)^2$ Eq. (9) takes on the form

$$\frac{\alpha m}{\mu}\frac{\sqrt{\pi}\,\Gamma(-l-{}^1\!/_2)}{\Gamma(-l)}\left(\frac{\varkappa^2}{\mu^2}\right)^{l+{}^1\!/_2}\frac{1}{\sin\pi l}$$

$$\times\left[1-\frac{\alpha m}{\mu}\frac{1}{l+{}^1\!/_2}\left(\frac{1}{2^{2l+1}}-1\right)\right]=1. \qquad (21)$$

For $\kappa^2 \to 0$ the large quantity $(\kappa^2/\mu^2)^{l+1/2}$ ($l + 1/2 < 0$) can now be compensated by the vanishing of the expression in the square brackets. This gives for the position of the pole for $k^2 = 0$ the equation

$$\frac{\alpha m}{\mu}\frac{1}{l_0+{}^1\!/_2}\left(2^{-(2l_0+1)}-1\right)=1. \qquad (22)$$

By direct calculation of the next higher approximation for small k^2 one can convince oneself that the resultant corrections are small. For the case $-2\alpha m/\mu = A = 0.05$ [4] Eq. (22) gives rise to the correct value $l_0 \approx -4.15$. As the coupling constant is decreased l_0 goes off logarithmically to $-\infty$. More distant poles of this type are, apparently, outside the region of applicability of the second order approximation. It is interesting to note that Eq. (22) has a solution only for the case of attraction ($\alpha < 0$). Therefore, in the case of repulsion such poles are either completely absent or are, at the very least, considerably farther away on the negative axis.

As can be seen from the numerical results of Ahmadzadeh et al [4] for stronger coupling the picture, including the additional poles described above, differs little qualitatively from that obtained in this work.

It may be noted that for not too small negative α some of the trajectories that start for $k^2 = +\infty$ from points with integer $-l$ cross the real axis not at the nearest half-integer value, but at some other half-integer or integer value $-l$ smaller than the starting one. If $-\alpha$ is sufficiently large then a few of the trajectories that start for $k^2 = +\infty$ from the smallest integer $-l$, cross for certain values of $k^2 > 0$ the axis $\mathrm{Re}\, l = -1/2$ in the upper half plane and behave thereafter in the "normal" manner for $k^2 = 0$ they enter the real axis from above and with increasing $-k^2$ move along the axis to the left returning to the point where the trajectory started for $k^2 = +\infty$. If such a trajectory crosses an integer nonnegative l one has the usual bound state. If for some energy the trajectory crosses a positive half-integer value $l = N - 1/2$, then another, "compensating," trajectory should at that same energy cross the point $l = -N - 1/2$, and the residues divided by the derivatives $l'(k^2)$ should coincide at both poles. If that were not the case the amplitude would have an unphysical singularity.[5] It is possible that all compensating trajectories are "normal," independently of whether for $k^2 = 0$ they enter the real axis to the right or to the left of the point $l = -1/2$.

The poles accumulating at the point $l = -1/2$ from the upper half plane were not found in [4]. This is quite natural since these poles, in the limit as $k^2 \to +\infty$, go off to infinity while in [4] only those poles were studied which for $k^2 \to \infty$ approached negative integer points.

The oscillations in the l plane for values of k^2 corresponding to the left cut represent a significant qualitative peculiarity of the trajectories that were here obtained. These oscillations give rise to difficulties in the discussion of the singularities in the k^2 plane as a function of l. To the real point l lying within the range of oscillation of a certain pole there correspond in the k^2 plane several poles on the left cut.

As the point l moves to the right along the real axis the poles in the k^2 plane collide in pairs and go off into the complex plane. At that they must remain on the physical sheet since moving poles cannot disappear through the left cut.[6] However as the point l moves further along the real axis to the right the poles in the k^2 plane should finally leave the physical sheet through the right cut. If the point $l = -1/2$ is crossed strictly along the real axis then, as it follows from the equations of Gribov and Pomeranchuk,[3] the poles cannot leave the physical sheet through the point $k^2 = 0$. (In the neighborhood of $l = -1/2$ the position of the poles is determined by the relation $k^2/\mu^2 \sim -\exp[-2\pi i p/(l + 1/2)]$.)

In the case of attraction the poles may leave through points with positive k^2 on crossing through half-integer negative $l \neq -1/2$, as is apparent from our results. However in the case of repulsion this possibility is certainly absent. In that case it is to be expected that the poles leave the physical sheet at infinity. One may suppose that this occurs on crossing integer negative l. This variant is also possible in the case of attraction. Apparently the oscillations problem is related to the open nature of the trajectories.

It should be noted that the open nature of the trajectories and the oscillations could be explained if it is assumed that the poles can collide in the l plane for complex k^2. At that the trajectory of each pole would have complex branch points, and separate trajectories would represent different branches of the same analytic function.

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APPENDIX

Equation (8) is inapplicable in the immediate neighborhood of negative half-integer l. This has to do with the fact that the main terms of the asymptotic forms of R_l and Q_l for small k^2/μ^2 vanish as these points are approached. It is easy to determine the distance Δl from the half-integer point at which the main asymptotic term still exceeds the terms that do not vanish as $\Delta l \to 0$. For the function $R_l(1 + \mu^2/2k^2)$ the corresponding condition has the form

$$\Delta l \gg (k^2/\mu^2)^{-l-1/2}. \tag{A.1}$$

As the pole approaches the half-integer point Δl decreases while the quantity $(k^2/\mu^2)^{-l-1/2}$ remains approximately constant and proportional to, according to Eq. (8), $\alpha m/\mu$. It is therefore clear that Eq. (8) remains valid for the pole trajectory down to

$$\Delta l \sim \alpha m/\mu. \tag{A.2}$$

The analogous requirement for $Q_l(1 + \mu^2/2k^2)$ gives rise to the even weaker condition $\Delta l \sim (\alpha m/\mu)^2$.

The above discussion does not apply to the neighborhood of the point $l = -1/2$. Here the functions R_l and Q_l behave like k/μ and $(k^2/\mu^2)^{\Delta l + 1/2}$ respectively. Therefore the condition for the applicability of Eq. (8) reduces to

$$(k^2/\mu^2)^{\Delta l} \gg 1 \qquad (\mathrm{Re}\, l < -1/2).$$

On approaching the point $l = -1/2$ from the left the quantity Δl is determined from the equation

$$\left(\frac{k^2}{\mu^2}\right)^{\Delta l} \frac{\alpha m}{\mu} \frac{1}{-\Delta l} = 1 .$$

From here it is clear that the length of the segment of the trajectory that must be left out is determined as before by the condition (A.2).

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