THEORY OF SECOND ORDER EQUATIONS FOR FERMIONS

V. S. VANYASHIN

Dnepropetrovsk State University

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It is suggested that within the framework of the theory of second order equations for fermions there exists a universal connection between chirality and electric charge. More precisely, the components of any spinor field of positive chirality ($\psi_+ = 1/2(1+\gamma_5)\psi$) are always electrically charged and are invariant under an electromagnetic gauge transformation, while the components of negative chirality ($\psi_- = 1/2(1-\gamma_5)\psi$) are always electrically neutral. The universal weak interaction is of the type which changes the chirality, this being manifested in its axial vector structure as well as in the charged nature of the weak currents. A discussion is also given of the form assumed in the theory under investigation by the Lagrangian describing the interaction with a pseudoscalar field.

1. INTRODUCTION

SECOND order equations for fermions were first discussed by Markov [1] as long ago as 1936. In 1958 they were utilized by Gell-Mann and Feynman to establish the axial vector nature of the weak interaction [2].

A basic question which arises in utilizing a second order equation in place of the Dirac equation is the problem of the interpretation of the additional degree of freedom. A second order equation admits twice as many solutions compared to the Dirac equation. These solutions can be characterized in addition to the usual quantum numbers also by the eigenvalues of the operator γ_5 (the "chirality" operator using Eddington's terminology [3]). Chirality, which in this case, in contrast to the case of the Dirac equation, is a conserved quantity, appears at first glance to be a "superfluous" quantum number. Therefore, it was proposed [4] to consider as "physical" solutions only one-half of the components of ψ selected by the projection operator $1/2(1+\gamma_5)$. The two-component formalism in the papers by Feynman and Gell-Mann [2], Brown [5] and Tonin [6] is based essentially on this point of view.

The other point of view, according to which all the components of ψ are physical, but only the components of positive chirality have an electromagnetic interaction, has without doubt some heuristic value. Marx has shown [7,8] that the parts of the wave function characterized by positive and negative chirality can be placed in correspondence with the charged and neutral components of the (e, ν) and (p, n) doublets. In order to obtain the complete picture it is necessary to include in the general scheme also the muon and the hyperons. This was carried out, but encountered definite difficulties (cf., reference [8]). It will be shown below that the difficulty associated with the Λ and Σ hyperons is to some extent surmountable. Also there appear to be at the present time no difficulties in connection with including the muon in the above scheme. The muon is described by a single wave function together with its neutrino support for the existence of which is given by the latest experiments [9].

2. THE LAGRANGIAN AND THE S-MATRIX FOR THE ELECTROMAGNETIC INTERACTION

The Lagrangian for the free spinor field satisfying a second order equation can be written in the following form [1,10]:

$$\mathcal{L} = \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x^m}\gamma^m\gamma^n\frac{\partial\psi}{\partial x^n} - m\bar{\psi}\psi. \qquad (1)$$

In this expression the rest mass is the same for all the components of ψ. If necessary, the inequality of the bare masses of the components characterized by different chirality can be taken into account in the Lagrangian with the aid of the projection operators $a = 1/2(1+\gamma_5)$ and $\bar{a} = 1/2(1-\gamma_5)$.

The commutation relations are of the form

$$[\psi(x), \bar{\psi}(y)]_+ = -imD(x-y). \qquad (2)$$

The Lagrangian for the electromagnetic interaction is obtained from (1) by the substitution $\partial_n \rightarrow \partial_n - ieA_n a$:

$$\mathcal{L}_e = \frac{ie}{m}\bar{\psi}a\hat{A}\gamma^n\frac{\partial\psi}{\partial x^n} - \frac{ie}{m}\frac{\partial\bar{\psi}}{\partial x^n}\gamma^n\hat{A}a\psi + \frac{e^2}{m}A_n^2\bar{\psi}a\psi. \quad (3)$$

As is always true in the case of gradient coupling, the S-matrix is defined in terms of the T*-exponential, and the contractions of the field operators have the following form

$$\langle T(\psi(x)\bar{\psi}(y))\rangle_0 = -imD^c(x-y), \quad (4a)$$

$$\langle T^*(\psi'(x)\bar{\psi}(y))\rangle_0 = \langle T(\psi(x)\bar{\psi}'(y))\rangle_0 = \gamma^n\frac{\partial}{\partial x^n}D^c(x-y). \quad (4b)$$

Both here and later the following notation is used

$$\psi' = \frac{i}{m}\gamma^n\frac{\partial\psi}{\partial x^n}, \qquad \bar{\psi}' = -\frac{i}{m}\frac{\partial\bar{\psi}}{\partial x^n}\gamma^n.$$

The term containing e^2 in the Lagrangian (3) can be omitted if at the same time the T*-product of two derivatives is redefined:

$$\langle T^*(\psi'(x)\bar{\psi}'(y))\rangle_0 = -imD^c(x-y). \quad (4c)$$

In the usual definition the right hand side of (4c) would also contain a δ-function, but the terms in the S-matrix arising from it are exactly cancelled by the terms arising from the term containing e^2 in the Lagrangian.

We now add a few words with respect to the gauge transformation. In going over from A_n to $A_n + \partial_n\Lambda$ the spinor wave functions transform in the following manner:

$$\psi \to \exp(iea\Lambda)\psi, \qquad \bar{\psi} \to \bar{\psi}\exp(-iea\Lambda). \quad (5)$$

The transformations for ψ and $\bar{\psi}$ are not related to one another by means of the operation of Dirac conjugation: an additional change of sign of γ_5 occurs. Such a "lack of consistency" guarantees the invariance of the commutation relations (2) and does not lead to a contradiction, since in the theory utilizing second order equations the spinor $\bar{\psi}$ is not obtained as a result of the operation of Hermitian conjugation [4,10] of the spinor ψ. This property is a general property of transformations containing the matrix γ_5, and this can be seen below in the course of investigating the baryon gauge transformations and isospin rotations.

3. NUCLEONS. BARYON GAUGE TRANSFORMATION AND THE GROUP OF ISOSPIN ROTATIONS

If we utilize the spinor ψ for the simultaneous description of the proton and the neutron, then for the baryon gauge transformation we must take the transformation

$$\psi \to e^{i\gamma_5\lambda}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\gamma_5\lambda}. \quad (6)$$

The transformation (6) leaves invariant the Lagrangian (1) and the commutation relations (2) and defines the conserved baryon current

$$I_B^n = \bar{\psi}\gamma_5\gamma^n\psi' + \bar{\psi}'\gamma^n\gamma_5\psi. \quad (7)$$

We introduce the charge-conjugate spinors

$$\psi^c = C\psi, \quad \bar{\psi}^c = -\bar{\psi}C^{-1};$$

$$C = -C^T = (C^{-1})^*, \; C^{-1}\gamma^n C = -\gamma^{nT}. \quad (8)$$

The group of isospin rotations is represented by the linear transformation of ψ and ψ'^c realized by the unimodular matrix:

$$\begin{pmatrix}\psi\\ \psi'^c\end{pmatrix} \to \begin{pmatrix}\alpha & \beta\gamma_5\\ -\beta^*\gamma_5 & \alpha^*\end{pmatrix}\begin{pmatrix}\psi\\ \psi'^c\end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (9a)$$

If we utilize the notation for the operation of Hermitian conjugation, then this transformation can be written in the form of the Pauli transformation [11]:

$$\psi \to \alpha\psi + \beta\gamma_5 C(\psi^+\gamma^0), \quad (10)$$

where $\psi^+ = \eta\psi^*\eta$ is the spinor which is Hermitian conjugate to ψ, while η is the metric operator introduced in [10].

The transformation of the conjugate spinors $\bar{\psi}$ and $\bar{\psi}'^c$ is accomplished by the inverse matrix

$$(\bar{\psi} \;\; \bar{\psi}'^c) \to (\bar{\psi} \;\; \bar{\psi}'^c)\begin{pmatrix}\alpha^* & -\beta\gamma_5\\ \beta^*\gamma_5 & \alpha\end{pmatrix}, \quad (9b)$$

which guarantees the invariance of the commutation relations.

In order that the transformation (9a), (9b) should mix states of the same baryon charge, the baryon gauge transformation was chosen to be of the form (6). We shall retain the simple phase transformation

$$\psi \to e^{i\lambda}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\lambda} \quad (11)$$

as the lepton gauge transformation.

The transformation (9a), (9b) enables us to determine the conserved isospin current with components

$$T_3^n = {}^1/_2\,(\bar{\psi}\gamma^n\psi' + \bar{\psi}'\gamma^n\psi), \quad (12a)$$

$$T_+^n = T_1^n + iT_2^n = {}^1/_2\,(\bar{\psi}\gamma_5\gamma^n\psi^c + \bar{\psi}'\gamma^n\gamma_5\psi'^c), \quad (12b)$$

$$T_-^n = T_1^n - iT_2^n = {}^1/_2\,(\bar{\psi}^c\gamma^n\gamma_5\psi + \bar{\psi}'^c\gamma_5\gamma^n\psi'). \quad (12c)$$

Since the electric charge current has the form

$$I_e^n = \bar{\psi}a\gamma^n\psi' + \bar{\psi}'\gamma^n a\psi, \quad (13)$$

then the electric charge, the third isospin component and the baryon charge of the ψ-field are related by the expression $Q = T_3 + 1/2\,B$ which does in fact hold for nucleons.

4. NUCLEONS. INTERACTION WITH π MESONS

In a number of articles [5,7] it has been asserted that a second order equation for fermions does not permit one to describe an interaction with a pseudoscalar field. If such an assertion were correct, then it would cast doubt on the justification for using a second order equation instead of the Dirac equation, so that, therefore, we will examine it in greater detail.

In Brown's paper [5] the requirement of the absence of derivatives of ψ has been imposed on the Lagrangian for the interaction with the pseudoscalar field. In the theory under consideration such a requirement is too strong, since it is not satisfied even by the Lagrangian for the electromagnetic interaction. In the paper by Marx [7] a wave function is used which, although satisfying a second order equation, was chosen to obey the Dirac commutation relations. The requirement of the absence of derivatives in the Lagrangian, which in this case is needed for the renormalizability of the theory, indeed does not allow us to describe an interaction with a pseudoscalar field.

In order to describe an interaction with π mesons we must use a field operator which satisfies the commutation relations involving the D-, and not the S-function, and we must permit the appearance of derivatives in the Lagrangian. Taking into account the form of the isospin transformations (9a), (9b), we form a Lorentz-invariant isovector with components

$$B_3 = \bar{\psi}\psi - \bar{\psi}'^c\psi'^c, \quad B_+ = B_1 + iB_2 = 2\bar{\psi}\gamma_5\psi'^c,$$

$$B_- = B_1 - iB_2 = 2\bar{\psi}'^c\gamma_5\psi. \tag{14}$$

The isotopically invariant Lagrangian for the interaction with π mesons can now be written in the form

$$\mathcal{L}_\pi = ig\varphi_i B_i = ig\varphi_0\left(\bar{\psi}\psi - \frac{1}{m^2}\frac{\partial\bar{\psi}}{\partial x^m}\gamma^m\gamma^n\frac{\partial\psi}{\partial x^n}\right)$$

$$+\sqrt{2}g\varphi\bar{\psi}C^{-1}\gamma_5\frac{1}{m}\gamma^n\frac{\partial\psi}{\partial x^n} + \sqrt{2}g\varphi^*\frac{1}{m}\frac{\partial\bar{\psi}}{\partial x^n}\gamma^n\gamma_5 C\bar{\psi}. \tag{15}$$

Since the Lagrangian (15) contains terms only of the first order in g, then in constructing the S-matrix we must utilize the contraction rules (4a)—(4c). The S-matrix constructed in this manner reproduces the results of the usual theory with pseudoscalar coupling. The simplest way of verifying this is to form with the aid of the function ψ and of its first derivative $\psi' \equiv im^{-1}\hat{\partial}\psi$ the proton and the neutron wave functions satisfying the Dirac equation:

$$\psi_p = a\psi + \bar{a}\psi', \quad \psi_n = \bar{a}\psi^c + a\psi'^c,$$

$$\bar{\psi}_p = \bar{\psi}a + \bar{\psi}'\bar{a}, \quad \bar{\psi}_n = \bar{\psi}^c\bar{a} + \bar{\psi}'^c a. \tag{16}$$

Here we have defined $\bar{\psi}_{p,n} = \psi^+_{p,n}\gamma^0$ and $\bar{\psi} = \psi^*\gamma^0$, so that in the formulas for the conjugate functions an additional change of sign of γ_5 has been introduced.

From (2) the usual commutation relations for ψ_p and ψ_n follow:

$$[\psi_p(x), \bar{\psi}_p(y)]_+ = [\psi_n(x), \bar{\psi}_n(y)]_+ = -iS(x-y). \tag{17}$$

The interaction Lagrangian (15) expressed in terms of ψ_p and ψ_n also assumes the usual form

$$\mathcal{L}_\pi = ig(\varphi_0\bar{\psi}_p\gamma_5\psi_p - \varphi_0\bar{\psi}_n\gamma_5\psi_n + \sqrt{2}\varphi\bar{\psi}_n\gamma_5\psi_p$$

$$+\sqrt{2}\varphi^*\bar{\psi}_p\gamma_5\psi_n). \tag{18}$$

Moreover, the contraction rules following from (4a), (4c)

$$\langle T(\psi_p(x)\bar{\psi}_p(y))\rangle_0 = \langle T(\psi_n(x)\bar{\psi}_n(y))\rangle_0$$

$$= -iS^c(x-y) \tag{19}$$

enable us to draw a conclusion with regard to the equivalence of the theory under consideration and the theory utilizing the Dirac equation. An analogous equivalence also exists in the case of the electromagnetic interaction.

5. OTHER BARYONS

We now proceed to investigate the other baryons.

The Ξ-particle differs from the nucleon by the fact that its electrically positively charged component carries a negative baryon charge. Therefore, the baryon gauge transformation for ψ_Ξ should be chosen in the form which differs by the sign of the phase from (6), and the previous form for the electromagnetic gauge transformation (5) and for the isospin rotations (9a), (9b) should be retained.

Such a choice leads to the relation $Q = T_3 - 1/2\,B$ which is in fact satisfied by the Ξ-field. The wave functions satisfying the Dirac equation are defined in terms of ψ_Ξ by

$$\Xi^0 = \bar{a}\psi_\Xi + a\psi'_\Xi, \quad \Xi^- = a\psi^c_\Xi + \bar{a}\psi'^c_\Xi. \tag{20}$$

For the description of the Λ and Σ hyperons we shall use the often used representation of the Λ isosinglet and the Σ isotriplet by the two isodoublets

$$N_2 = \begin{pmatrix} \Sigma^+ \\ (\Lambda - \Sigma^0)/\sqrt{2} \end{pmatrix}, \quad N_3 = \begin{pmatrix} (\Lambda + \Sigma^0)/\sqrt{2} \\ \Sigma^- \end{pmatrix},$$

which transform jointly under isospin rotations. The corresponding wave functions ψ_2 and ψ_3 transform under the baryon gauge transformation like ψ and ψ_Ξ, while the electromagnetic gauge transformation has, as before, the form (5).

The isospin transformations are defined by the formula

$$\begin{pmatrix} \bar{a}\psi_3 & a\psi_2 \\ \bar{a}\psi_3'^c & a\psi_2'^c \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \bar{a}\psi_3 & a\psi_2 \\ \bar{a}\psi_3'^c & a\psi_2'^c \end{pmatrix} \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix}, \quad (21)$$

while the isospin components have the form

$$T_3^n = \sum_{i=2,3} (\bar{\psi}_i'\gamma^n a\psi_i + \bar{\psi}_i a\gamma^n\psi_i'), \quad (22a)$$

$$T_-^n = \sqrt{2}\Big[\tfrac{1}{2}\sum_{i=2,3} (\bar{\psi}_i^c\gamma_5\gamma^n\psi_i + \bar{\psi}_i'^c\gamma^n\gamma_5\psi_i') + \bar{\psi}_2\gamma^n a\psi_3 + \bar{\psi}_3\gamma^n a\psi_2 + \bar{\psi}_2' a\gamma^n\psi_3' + \bar{\psi}_3' a\gamma^n\psi_2'\Big]. \quad (22b)$$

In this case the relation $Q = T_3$ holds. The transition to the Dirac wave functions is accomplished with the aid of the following formulas

$$\Sigma^+ = a\psi_2 + \bar{a}\psi_2', \qquad \Sigma^- = a\psi_3^c + \bar{a}\psi_3'^c,$$

$$\Sigma^0 = \bar{a}(\psi_3 - \psi_2^c)/\sqrt{2} + a(\psi_3' - \psi_2'^c)/\sqrt{2},$$

$$\Lambda = \bar{a}(\psi_3 + \psi_2^c)/\sqrt{2} + a(\psi_3' + \psi_2'^c)/\sqrt{2}. \quad (23)$$

6. THE WEAK INTERACTION

Thus, we see that by utilizing second order equations for the description of fermions one can develop a formalism in which the electromagnetic interaction appears as a universal interaction for the components of positive chirality of arbitrary spinor fields. In turn, if we assume that a fundamental property of the universal weak interaction is the obligatory change of chirality, then from this property will follow both the axial vector structure of the interaction and also the charged nature of weak currents.

The weak current of "bare" baryons which conserves strangeness can be defined as that part of the projection of the isotopic vector T_-^n which does not contain derivatives (cf., (12c), (22b)). On going over to Dirac wave functions by means of formulas (16), (20), and (23) we can see that such a definition leads to the V – A-interaction in the case of the transitions $n \rightarrow p$, $\Sigma^0 \rightarrow \Sigma^+$ and to the V + A interaction in the case of the transitions $\Sigma^- \rightarrow \Sigma^0$, $\Xi^- \rightarrow \Xi^0$:

$$J^n = \bar{\psi}_n\gamma^n a\psi_p + 2\bar{\Sigma}^0\gamma^n a\Sigma^+ + 2\bar{\Sigma}^- a\gamma^n\Sigma^0 + \bar{\Xi}^- a\gamma^n\Xi^0. \quad (24)$$

The same form of the current also follows from the following rule: the wave functions for all the particles are taken with a factor a [2] with the one additional requirement that the "fundamental" baryons are always taken to be the baryons of positive electric charge. Unfortunately, at the present time it is not known which form of interaction is realized in the β decay of hyperons.

7. A POSSIBLE RELATION BETWEEN THE NUMBERS OF LEPTONS AND BARYONS IN THE UNIVERSE

In conclusion we shall make one remark which appears to be highly probable even though it cannot be proven rigorously. Since in the part of the universe with which we are familiar the electric charges of the baryons and the leptons neutralize one another, a comparison of the electromagnetic (5), the baryon (6) and the lepton (11) gauge transformations leads us to think that for the total baryon and lepton charge in the universe the relation B – L = 0 holds. In this case there exists in the universe an excess of ν_e over anti-ν_e. The magnitude of this excess is equal to the number of neutrons bound in nuclei ($\approx$ 15% of the number of protons).

Such an excess of neutrinos is also predicted by the theory which regards the atoms of hydrogen as the raw material for the synthesis of different elements [12]. We note that the hypothesis of the existence of antiworlds leads to the equality in the numbers of neutrinos and antineutrinos, while theories of the origin of elements from neutrons lead to an excess of antineutrinos.

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