# *NONLINEAR THEORY OF THE INTERACTION OF A MONOENERGETIC BEAM WITH A PLASMA*

V. D. SHAPIRO

Physico-technical Institute, Academy of Sciences, Ukr.S.S.R.

Submitted to JETP editor August 3, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 613-625 (February, 1963)

The nonlinear theory of oscillations excited by the interaction of a monoenergetic beam with a plasma has been investigated. The oscillation amplitude at saturation and the growth of the thermal energy of the beam and plasma resulting from the interaction are determined.

1. It is well known that under certain conditions the interaction of a plasma and a beam of charged particles can lead to instabilities. The amplitude of the oscillations associated with this instability has been investigated in the linear approximation by Akhiezer and Faĭnberg[1] and Bohm and Gross.[2] This approximation holds only at the beginning of the process and, for this reason, cannot be used to determine the ultimate amplitude of the oscillations or the ultimate state of the beam-plasma system.

A number of recent papers have been concerned with the changes in the distribution function and various mean quantities (mean velocity and temperature) resulting from the "feedback" effect of the oscillations.[3-7] If the distribution function is "smeared out" $v_{T_1} \gg \gamma_k/k$ ($v_{T_1}$ is the thermal velocity of the beam, $\gamma_k$ is the growth rate of the k-th unstable mode, $|v - v_{ph}| \sim \gamma_k/k$ is the velocity range in which the beam particles exchange energy with this mode), the distortion of the distribution function resulting from feedback leads to eventual saturation at rather low amplitudes, in which case the nonlinear interaction between modes is unimportant. This circumstance yields the possibility of formulating a quasilinear theory for the instabilities arising in this case.[5,7]

In the present paper we investigate the interaction between a plasma and an initially monoenergetic low-density beam in the nonlinear approximation. For this case the inequality $v_{T_1} \ll \gamma_k/k$ is satisfied at the initial stage of the process for the most unstable part of the spectrum so that all the beam particles exchange energy with each mode. The assumption of a small beam density made in the present work is most important as it allows us to follow the development of the instability to its conclusion. We also assume that the initial state of the beam-plasma system is uniform, that the oscillations are one-dimensional, and that binary collisions in the plasma and beam can be neglected (it is assumed that the relaxation time of the collective interaction is small compared to the mean time between collisions).

The instability resulting from the interaction of a plasma and a monoenergetic beam can be described as follows. The feedback effect of the oscillations retards the beam but increases its temperature. In the initial stage of the process (low oscillation amplitude) the beam remains cold and the hydrodynamic description given in Sec. 2 applies. The change in beam parameters during this stage does not lead to any important change in the dispersion relation and the oscillation amplitudes grow exponentially with time, as in the usual linear theory. Later on, however, changes in oscillation amplitude cause changes in the beam parameters so that the nonlinear interaction between modes results in saturation over a rather narrow range of wave phase velocities $|v_{ph} - u_0| \lesssim (N_1/N_2)^{1/3}u_0$; at this point the total oscillations energy is of order $(N_1/N_2)^{1/3}N_1mu_0^2$ ($u_0$ is the initial beam velocity, $N_1$ and $N_2$ are the beam and plasma densities). This stage of the process is treated in Sec. 3.

The change in mean beam velocity causes a broadening of the spectrum and an increase in total oscillation energy. It is important that for a low-density beam the criterion for applicability of the quasilinear approximation $v_{T_1} \gg \gamma_k/k$ is satisfied for values of the oscillation energy small compared with the energy of the directed beam motion. The quasilinear stage of the beam-plasma interaction is treated in Sec. 4. The beam particle velocities diffuse to values of the order of the thermal velocity of the plasma and a plateau appears in the distribution function. In the process the beam loses an appreciable part of its energy of directed motion (up to 75%); this energy is converted into energy associated with the oscillations and the thermal energy of the beam and plasma particles.

2. In analyzing the change of the original state of the beam and plasma resulting from the feedback effect of the oscillations it is convenient to write the distribution function as a sum of two parts $f = f_0 + f_1$, where $f_1(t, x, v)$ is the oscillating function in time and space that describes the oscillations and $f_0(t, v)$ is the "background" on which the oscillations develop. The equation that describes the change in $f_0$ resulting from the oscillations can be obtained from a kinetic equation in the Vlasov form by averaging over distances large compared with the oscillation wavelength:

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\left\langle E\frac{\partial f_1}{\partial v}\right\rangle = 0, \qquad f_0 = \langle f\rangle \tag{1}$$

(the brackets denote averages and $E$ is the electric field associated with the oscillations). In the initial stage, in which the beam is still monoenergetic, we need not determine the time behavior of the distribution function itself $f_0$, but rather the time behavior of the mean velocity and temperature, i.e., the first and second moments of the distribution function. (It is these quantities that appear in the dispersion equation and in the criteria that define the monoenergetic properties of the beam).

To compute the mean velocity and temperature of the $\alpha$-component (i.e., the beam or plasma) we make use of the following equations which are obtained from the original equation (1):

$$\frac{d}{dt}(mN_\alpha u_\alpha) = \int mv\frac{\partial f_{0\alpha}}{\partial t}dv = \langle E\rho_\alpha\rangle,$$

$$\frac{d}{dt}\left(\frac{N_\alpha T_\alpha}{2}\right) = \int\frac{m}{2}(v-u_\alpha)^2\frac{\partial f_{0\alpha}}{\partial t}dv = -e\left\langle E\int f_{1\alpha}(v-u_\alpha)\,dv\right\rangle$$
$$= \langle EJ_\alpha\rangle - u_\alpha\langle E\rho_\alpha\rangle; \tag{2}$$

where $\rho_\alpha$ and $J_\alpha$ are the perturbation in charge and current density caused by the oscillations.

We assume that the oscillations are linear and determine $\rho_\alpha$ and $J_\alpha$ from the linearized equations for $f_{1\alpha}$. Writing $E$ and $f_{1\alpha}$ in the form

$$E = \frac{1}{2}\left\{\sum_k E_k(t)\exp\left[i(kx - \omega_k^r t)\right] + \text{c.c.}\right\},$$

$$f_{1\alpha} = \frac{1}{2}\left\{\sum_k f_{k\alpha}(t, v)\exp\left[i(kx - \omega_k^r t)\right] + \text{c.c.}\right\},$$

we find that the moments $\xi_{n\alpha} = \int f_{k\alpha}(v - u_\alpha)^n$ are given by the equations (cf. [4]):

$$\hat{D}\xi_{0\alpha} = -ik\xi_{1\alpha},$$
$$\hat{D}\xi_{1\alpha} = -\frac{eN_\alpha}{m}E_k - ik\xi_{2\alpha} - \frac{du_\alpha}{dt}\xi_{0\alpha},$$
$$\hat{D}\xi_{2\alpha} = -ik\xi_{3\alpha} - 2\frac{du_\alpha}{dt}\xi_{1\alpha},$$
$$\hat{D}\xi_{3\alpha} = -\frac{3eN_\alpha T_\alpha}{m^2}\dot{E}_k - 3\frac{du_\alpha}{dt}\xi_{2\alpha}. \tag{3}$$

Here we use the notation $\hat{D} = d/dt + i(ku_\alpha - \omega_k^r)$.

If the following condition is satisfied:[1)]

$$|k^2T_\alpha/m(ku_\alpha - \omega_k)^2| \ll 1, \qquad |k\delta u_\alpha/(ku_\alpha - \omega_k)| \ll 1 \tag{4}$$

(where $\delta u_\alpha = u_\alpha - u_{0\alpha}$ is the change in mean velocity resulting from the feedback effect) the solution of (3) is

$$\xi_{0\alpha} = -\frac{k\xi_{1\alpha}}{ku_{0\alpha} - \omega_k}, \qquad \xi_{1\alpha} = -\frac{eN_\alpha}{m}\frac{E_k}{i(ku_{0\alpha} - \omega_k)}, \tag{5}$$

where $\omega_k = \omega_k^r + i\gamma_k$, $\gamma_k$ is the growth rate for the k-th mode, $dE_k/dt = \gamma_k E_k$, $\omega_k^r = -\omega_k^r$ for the unstable modes, for which $\gamma_{-k} = \gamma_k > 0$.

Substituting $\xi_{0\alpha}$ from (5) in the Poisson equation $ikE_k = -4\pi e\sum_\alpha \xi_{0\alpha}$ we obtain the dispersion equation for $\omega_k$ in the linear theory

$$1 = \frac{\omega_2^2}{\omega_k^2} + \frac{\omega_1^2}{(\omega_k - ku_0)^2}$$

$$\left(\omega_\alpha^2 = \frac{4\pi e^2 N_\alpha}{m}, \qquad u_{01} = u_0, \qquad u_{02} = 0\right).$$

The solution of this equation has been obtained in [1]. The maximum growth rate occurs for $ku_0 = \omega_2$, in which case $\gamma_{max} = \sqrt{3}\,2^{-4/3}\omega_1^{2/3}\omega_2^{1/3}$; the corresponding "detuning" $ku_0 - \omega_k^r = 2^{-4/3}\omega_1^{2/3}\omega_2^{1/3}$, that is to say, $ku_0 - \omega_k^r \sim \gamma_k$ in the instability arising from the interaction between a plasma and a monoenergetic beam. This is to be contrasted with the "weak" instability that occurs in the interaction between a plasma and a smeared out beam, in which case $|\omega - ku_0| \sim kv_{T_1} \gg \gamma$. The width of the most unstable part of the spectrum (for which $\gamma \sim \gamma_{max}$) is $\Delta k = (\omega_2/u_0)(N_1/N_2)^{1/3}$. Thus, in the linear stage those modes are excited whose phase velocities are close to the initial beam velocity $v_{ph} \approx u_0$; these lie in the range $\Delta v_{ph} = (\partial v_{ph}/\partial k)\Delta k \sim u_0(N_1/N_2)^{1/3}$.

Substituting (5) in (2) we obtain the following equations for the mean velocity and temperature of the beam:

$$\frac{du_1}{dt} = -\frac{2e^2}{m^2}\sum_k\frac{k(ku_0 - \omega_k^r)}{[(ku_0 - \omega_k^r)^2 + \gamma_k^2]^2}\gamma_k|E_k|^2,$$

$$\frac{dT_1}{dt} = \frac{2e^2}{m}\sum_k\frac{\gamma_k|E_k|^2}{(ku_0 - \omega_k^r)^2 + \gamma_k^2}. \tag{6}$$

---

1)In (3) we have formed a chain of equations (having set $\xi_4 = 0$) that corresponds to neglecting terms of higher order in $|k\delta u/(ku_0 - \omega)|$ and $|k^2T/[m(ku_0 - \omega)^2]|$ compared with those that appear in (3).

Thus, in the initial stage the temperature and retardation of the beam grow exponentially together with the amplitude ($|E_k|^2 \sim e^{2\gamma kt}$). In (6) we replace the factor multiplying the exponential by the value corresponding to the most unstable mode ($ku_0 = \omega_2$); then integrating (6) with respect to t we have

$$N_1 m u_0 \delta u_1 = -\frac{1}{4\pi}\sum_k |E_k|^2, \qquad \frac{N_1 T_1}{2} = \frac{1}{4\pi}\left(\frac{N_1}{2N_2}\right)^{1/3}\sum_k |E_k|^2 \tag{7}$$

[it is assumed that $T_1(t = 0) = 0$].

In similar fashion we obtain the following equations for the mean velocity and temperature of the plasma:

$$\frac{du_2}{dt} = \frac{2e^2}{m^2}\sum_k \frac{k}{\omega_k^{r3}}\gamma_k |E_k|^2, \qquad \frac{dT_2}{dt} = \frac{2e^2}{m}\sum_k \frac{\gamma_k |E_k|^2}{\omega_k^{r2}}.$$

Thus

$$\frac{N_2 m u_2^2}{2} \approx \frac{1}{8\pi}\sum_k |E_k|^2 \cdot \frac{1}{4\pi N_2 m u_0^2}\sum_k |E_k|^2.$$

$$\frac{N_2 \delta T_2}{2} = \frac{1}{8\pi}\sum_k |E_k|^2 \qquad (\delta T_2 = T_2 - T_2(t = 0)). \tag{8}$$

The increased temperatures of the beam and plasma can be explained as follows. At the initial time the electrons are uniformly distributed with respect to phases of modes resulting from fluctuations and the magnitudes of the fields acting on the electrons are different. The phase spread means that the energy obtained by the electrons from the modes is different for different particles; a temperature increase is obtained when averages are taken over phase. In the linear approximation $J_\alpha = \rho_\alpha u_\alpha - eN_\alpha V_\alpha$ ($V_\alpha$ is the velocity with which the particles oscillate) and we have from (2)

$$\frac{d}{dt}\left(\frac{N_\alpha T_\alpha}{2}\right) = -eN_\alpha \langle EV_\alpha\rangle = \frac{d}{dt}\left\langle N_\alpha \frac{mV_\alpha^2}{2}\right\rangle,$$

that is to say, the increment in the thermal energy of the beam and plasma derives from the kinetic energy of the oscillations.

The energy lost by the beam in the initial stage is distributed equally between the potential energy of the oscillations and the thermal energy of motion in the plasma. A small part of the energy lost by the beam ($\sim (N_1/N_2)^{1/3}$) goes into increasing the thermal energy of the beam. However, since $N_1 \ll N_2$ the beam temperature remains high compared with the plasma temperature.

We note that (8) describes the plasma up to the very end of the process, where $\sum_k |E_k|^2 \sim N_1 m u_0^2$; on the other hand it is easy to show, by substituting $u_2$ and $\delta T_2$ in (4), that the analogous relations for the beam (7) apply only for amplitudes $\sum_k |E_k|^2 \sim (N_1/N_2)^{1/3}/N_1 m u_0^2$ small compared with the beam energy [beyond these amplitudes $\delta u_1 \sim u_0 (N_1/N_2)^{1/3}$ and $k^2 T_1/m\gamma^2 \sim 1$ and the beam conditions specified in (4) are violated]. It will be shown below that the nonlinear interaction between modes becomes important at these amplitudes.

3. To analyze the nonlinear interaction between modes we use the following system of equations:

$$\frac{\partial f_k}{\partial t} + ikvf_k - \frac{e}{m}E_k\frac{\partial f_0}{\partial v} - \frac{e}{m}\sum_q{}' E_{k-q}\frac{\partial f_q}{\partial v}\exp\left[-i(\omega_{k-q}^r + \omega_q^r - \omega_k^r)t\right],$$

$$ikE_k = -4\pi e\int f_k dv. \tag{9}$$

(The primes over the summation signs mean that we omit terms characterized by $q = 0$, which are included in $f_0$; $f_0$ is the distribution function for the beam and plasma which changes because of the oscillations). In the linear approximation the solution of (9) is

$$f_k^{(1)} = \frac{e}{m}\frac{E_k \partial f_0/\partial v}{i(kv - \omega_k)}. \tag{10}$$

Taking account of second-order terms in the oscillation amplitude in (9) results in the appearance of frequencies $\omega_k \approx 2\omega_2$ and $\omega_k \approx 0$ in the spectrum. The next (third) approximation leads to nonlinear terms with frequencies $\omega_k \approx \omega_2$. Taking account of these terms leads to a change in the time dependence of the oscillation amplitude $E_k$ as compared with the linear theory. Some rather complicated calculations yield the following equation for $E_k(t)$ in this approximation:

$$\frac{dE_k}{dt} = \gamma_k E_k + i\sum_{q,\varkappa}{}' H(k,q,\varkappa)E_{k-q}E_{q-\varkappa}E_\varkappa \times \exp\left[-i(\omega_{k-q}^r + \omega_{q-\varkappa}^r + \omega_\varkappa^r - \omega_k^r)t\right];$$

$$H(k,q,\varkappa) = \frac{4\pi e^4}{m^3 k}\frac{\omega_k^r - \omega_{k-q} - \omega_{q-\varkappa} - \omega_\varkappa}{\varepsilon(\omega_{k-q} + \omega_{q-\varkappa} + \omega_\varkappa; k)} \times \left\{\int \frac{dv}{\omega_{k-q} + \omega_{q-\varkappa} + \omega_\varkappa - kv}\frac{\partial}{\partial v}\left[\frac{1}{\omega_{q-\varkappa} + \omega_\varkappa - qv}\frac{\partial}{\partial v}\left(\frac{1}{\omega_\varkappa - \varkappa v}\frac{\partial f_0}{\partial v}\right)\right] - \frac{4\pi e^2}{mq\varepsilon(\omega_{q-\varkappa} + \omega_\varkappa; q)}\int \frac{dv}{\omega_{q-\varkappa} + \omega_\varkappa - qv}\frac{\partial}{\partial v} \times \left(\frac{1}{\omega_\varkappa - \varkappa v}\frac{\partial f_0}{\partial v}\right)\int \frac{dv}{\omega_{k-q} + \omega_{q-\varkappa} + \omega_\varkappa - kv}\frac{\partial}{\partial v} \times \left[\left(\frac{1}{\omega_{q-\varkappa} + \omega_\varkappa - qv} + \frac{1}{\omega_{k-q} - (k-q)v}\right)\frac{\partial f_0}{\partial v}\right]\right\},$$

$$\varepsilon(\omega_k, k) = 1 + \frac{4\pi e^2}{mk}\int \frac{dv}{\omega_k - kv}\frac{\partial f_0}{\partial v}, \tag{11}$$

where $\epsilon(\omega_k, k)$ is the longitudinal dielectric constant. Because the beam velocity is approximately the same as the phase velocity of the unstable waves $|v_{ph} - u_0| \lesssim u_0(N_1/N_2)^{1/3}$, when a plasma interacts with a low-density monoenergetic beam the nonlinearity appears in the beam much earlier than in the plasma. Hence, the plasma can be analyzed in the linear approximation. Keeping only the largest terms associated with the beam in (11) we have

$$\frac{dE_k}{dt} = \gamma_k^{(0)} E_k + i \frac{e^2}{m^2\omega_1^2\omega_2^2} \sum_{q,\varkappa} \frac{\omega_{k-q} + \omega_{q-\varkappa} + \omega_\varkappa - \omega_k'}{\varepsilon(\omega_{k-q} + \omega_{q-\varkappa} + \omega_\varkappa, k)}$$

$$\times E_{k-q}E_{q-\varkappa}E_\varkappa \exp[-i(\omega_{k-q}^r + \omega_{q-\varkappa}^r + \omega_\varkappa^r - \omega_k^r)t]$$

$$\times \frac{1}{(\eta_{k-q} + \eta_{q-\varkappa} + \eta_\varkappa)^2(\eta_{q-\varkappa} + \eta_\varkappa)\eta_\varkappa} \left[\frac{2k\varkappa}{(\eta_{k-q} + \eta_{q-\varkappa} + \eta_\varkappa)\eta_\varkappa}\right.$$

$$+ \frac{6k^2}{(\eta_{k-q} + \eta_{q-\varkappa} + \eta_\varkappa)^2} + \frac{4kq}{(\eta_{k-q} + \eta_{q-\varkappa} + \eta_\varkappa)(\eta_{q-\varkappa} + \eta_\varkappa)}$$

$$\left. + \frac{2q^2}{(\eta_{q-\varkappa} + \eta_\varkappa)^2} + \frac{q\varkappa}{(\eta_{q-\varkappa} + \eta_\varkappa)\eta_\varkappa}\right],$$

$$\eta_k = (\omega_k - ku_0)/\omega_1^{2/3}\omega_2^{1/3}. \tag{12}$$

In (12) we include terms characterized by $q = 0$ in the summation; these terms correspond to the change in growth rate resulting from the change in beam parameters. In the present case these terms are of the same order as the $q \neq 0$ terms, which correspond to the change in growth rate associated with the nonlinearity of the oscillations. Then $\gamma_k^{(0)}$ is the growth rate in the linear stage; $\gamma_k^{(0)} \sim \gamma_{max} = \sqrt{3}\, 2^{-4/3}\omega_2(N_1/N_2)^{1/3}$ over a rather narrow range of phase velocities

$$|v_{ph} - u_0| \lesssim u_0(N_1/N_2)^{1/3}.$$

Using (12) we consider the saturation of the oscillations in this portion of the spectrum $k = k_0 + n\delta k$; $k_0 = \omega_2/u_0$ is the most unstable mode in the spectrum and $\delta k$ is the spacing between two neighboring modes, where $|n\delta k| \ll k_0$.

Equation (12) is extremely complicated. Since we shall be only interested in an order-of-magnitude estimate of the saturation amplitude for this stage, in (12) we replace the kernel by its value for the most unstable mode. In place of (12) we then obtain a simple equation for the amplitudes $E_n$:

$$\frac{dE_n}{dt} = \gamma_{max}E_n - \frac{33}{16\sqrt{3}}\frac{\omega_2}{N_1mu_0^2}(1 - i4\sqrt{3})\sum_{n_1n_2} E^*_{n_1+n_2-n}E_{n_1}E_{n_2}. \tag{13}$$

We now introduce the dimensionless amplitudes $a_n = E_n[(N_1/N_2)^{1/3} \times N_1mu_0^2]^{-1/2}$ in (12) and carry out a Fourier transformation $a(\xi) = \Sigma a_n e^{in\xi}$ [in this case the summation in (12) disappears], thereby obtaining the following equation for $(t, \xi)$:

$$\partial a/\partial t = \gamma_{max}(a - \alpha a^2 a^*),$$

$$\alpha = \tfrac{11}{8} 2^{1/3}(1 - 4\sqrt{3}\,i) = \alpha' + i\alpha''. \tag{14}$$

Thus, $\frac{\partial}{\partial t}|a|^2 = \gamma_{max}(|a|^2 - a'|a|^4)$.

A similar equation has been used earlier by Landau and Lifshitz for determination of the amplitudes of turbulent motion in hydrodynamics. [8] Solving this equation we have

$$|a|^2 = |a_0|^2 e^{2\gamma_{max}t}[1 + \alpha'|a_0|^2(e^{2\gamma_{max}t} - 1)]^{-1}.$$

For small values of t the amplitudes grow exponentially with time; when t satisfies the condition

$$|a_0|^2 e^{2\gamma_{max}t} \sim \sum_n |E_n|^2/N_1mu_0^2(N_1/N_2)^{1/3} \sim 1,$$

the nonlinear mode interaction retards the exponential growth and saturates the amplitudes.

In computing $\Sigma|E_n|^2$ we assume that a strong longitudinal magnetic field is applied to the system, in which case the mode spectrum becomes one-dimensional. Assuming that the initial oscillations in the plasma are due to fluctuations $|E_k^0|^2/8\pi \sim T_2^{(0)}$ and using the results obtained earlier in [4], we obtain the following expression for $\Sigma|E_n|^2$ in the linear stage:

$$\sum_n |E_n|^2 = 0{,}02\pi\frac{\omega_2^3}{u_0^3}T_2^{(0)}\left(\frac{N_2}{N_1}\right)^{1/6}\exp\left\{\frac{\sqrt{3}}{2^{1/3}}\left(\frac{N_1}{N_2}\right)^{1/3}\tau\right\}\tau^{-3/2},$$

$$\tau = \omega_2 t. \tag{15}$$

It is assumed that $\tau \gg (N_2/N_1)^{1/3}$. Then the time required for the onset of saturation is given by

$$t_{sat} \approx \frac{2^{1/3}}{\sqrt{3}}\frac{1}{\omega_2}\left(\frac{N_2}{N_1}\right)^{1/3}\ln\left[\frac{N_1u_0^3}{\omega_2^3}\frac{mu_0^2}{T_2^{(0)}}\right].$$

The saturation amplitude, obtained from the relation

$$\sum_n |a_n|^2 = \frac{1}{2\pi}\int|a(\xi)|^2 d\xi,$$

is given by the following expression when $t \to \infty$:

$$\sum_n |E_n|^2 = \frac{1}{\alpha'}N_1mu_0^2\left(\frac{N_1}{N_2}\right)^{1/3}. \tag{16}$$

It must be pointed out that (14) has been obtained by perturbation theory and, strictly speaking, applies only when $|a|^2 \ll 1$. When $|a|^2 \sim 1$ the equation for $|a|^2$ is of the form $d|a|^2/dt = f(|a|^2)$; in (14) we have retained only the first two terms in the expansion of $f(|a|^2)$ in $|a^2|$. If this expansion converges the roots of the equation $f(|a|^2) = 0$, which determine the saturation amplitude, are of the same order of magnitude as the roots of $d|a|^2/dt = 0$ as determined from (14).[2)] This approximation is fully adequate in the present case because, as will be shown below, the contribution of the portion of the spectrum given here represents a small part of the total energy of the oscillations $[\sim (N_1/N_2)^{1/3}]$.

Thus, the analysis of the nonlinearity carried out in this section can be used to trace the saturation of the modes produced in the linear stage, for which $|v_{ph} - u_0| \lesssim (N_1/N_2)^{1/3} u_0$. The total energy of these modes is given by (16). Inasmuch as the excitation of the modes is due to the energy of the directed motion of the beam, when

$$\sum_k |E_k|^2 \sim N_1 m u_0^2 (N_1/N_2)^{1/3}$$

we have

$$\delta u \sim u_0 (N_1/N_2)^{1/3} \sim \Delta v_{ph}$$

The change in mean velocity of the beam by an amount of the order of the width of the excited mode spectrum $\Delta v_{ph}$ means that neighboring portions of the spectrum are excited, causing an increase in the total energy of the oscillations and the thermal energy in the beam. When $\Sigma |E_k|^2 \sim N_1 m u_0^2 \times (N_1/N_2)^{1/3}$, it follows from (7) that $kv_{T_1}/\gamma_k \sim 1$.

As the energy associated with the oscillations becomes still greater the distribution function of the beam becomes smeared out to such an extent $(kv_{T_1}/\gamma_k \gg 1)$ that the quasilinear approximation applies. In the quasilinear approximation we take account of the distortion of the distribution function due to the feedback effect of the oscillations, but do not take account of the nonlinear interaction between modes.

4. To derive the equations that describe the change in $f_0$ resulting from the feedback effect of the oscillations we start with (1). In (1) we expand $f_1$ and $E$ in plane waves and use the linear theory (10) to relate the amplitudes of $f_k$ and $E_k$; taking averages and going to the limit $\gamma \to 0$ $(\gamma \ll \omega)$ we obtain

$$\frac{\partial f_0}{\partial t} = \frac{2\pi e^2}{m^2} \sum_{k>0} |E_k|^2 \frac{\partial}{\partial v}\left(\delta(kv - \omega_k^r) \frac{\partial f_0}{\partial v}\right). \qquad (17)$$

In the quasilinear theory we neglect the interaction between modes so that the equation describing the time variation of the energy in the k-th mode is of the form

$$\frac{d}{dt}|E_k|^2 = 2\gamma_k |E_k|^2, \qquad \gamma_k = \frac{\pi}{2} v_{ph}^2 \frac{\omega_k^r}{N_2} \frac{\partial f_0}{\partial v_{ph}}, \qquad (18)$$

where $\gamma_k$ is the growth rate in the quasilinear stage. The frequency $\omega_k^r$ is given by

$$\omega_k^r = \omega_2 (1 + \tfrac{3}{2} k^2 \lambda_D^2) \approx \omega_2,$$

where $\lambda_D = \sqrt{T_2/4\pi N_2 e^2}$ is the Debye radius in the plasma. The initial distribution function in the quasilinear stage is taken to be the function $f_0^{(0)}(v)$ arising in the preceding stage. This function is shown in Fig. 1 by the dashed lines. At the beginning of the quasilinear stage the mean velocity of the beam $u_1^{(0)} \approx u_0$ and the thermal velocity $v_{T_1}^{(0)} \sim u_0 (N_1/N_2)^{1/3}$. The exact form of the distribution function $f_0^{(0)}$ is unimportant in what follows since this function is used only to determine the amplitude of the oscillations over a narrow portion of the spectrum $|v_{ph} - u_0| \sim u_0 (N_1/N_2)^{1/3}$ whose contribution in the total oscillation energy is approximately $(N_1/N_2)^{1/3}$.

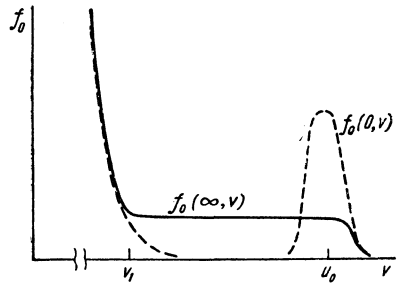


FIG. 1. Change in the distribution function resulting from the development of an instability. The dashed lines show the distribution function at the beginning of the quasilinear stage. The peak at $v \approx u_0$ corresponds to the beam. The solid curve is the distribution function that is established at the end of process.

The derivative $\partial f_0^{(0)}/\partial v_{ph} > 0$ for velocities in the range $|v_{ph} - u_0| \lesssim u_0 (N_1/N_2)^{1/3}$; modes with phase velocities in this range are excited in the beginning of the quasilinear stage and

$$\gamma^{(0)} \sim \omega_k^r \frac{N_1}{N_2} \frac{v_{ph}^2}{(\Delta v_{ph})^2} \sim \omega_2 \left(\frac{N_1}{N_2}\right)^{1/3}$$

is of the same order of magnitude as the growth rate for a monoenergetic beam interacting with a plasma. The diffusion of beam particles in velocity space caused by the oscillations causes the width of the spectrum to grow. The stable state is one for which $v > 0$ and $\partial f_0/\partial v \leq 0$. Hence, as a result of the feedback effect of the oscillations the particles in the beam diffuse to a velocity of the order of the thermal velocity of the plasma and a plateau develops in the distribution function

[2)]Similarly, the relations in (7) apply, strictly speaking, only when $\sum_k |E_k|^2 \ll N_1 m u_0^2 (N_1/N_2)^{1/3}$, but give the proper order of magnitude when extrapolated to the region $\sum_k |E_k|^2 \sim N_1 m u_0^2 \times (N_1/N_2)^{1/3}$.

(the distribution function established at the end of the process $f_0(\infty, v)$ is also shown in Fig. 1). If the thermal velocity of the plasma is small compared with the directed velocity of the beam the beam velocity is changed considerably $\delta u \sim u_0$ as a result of diffusion and the energy of the oscillations excited in the quasilinear stage will be of the same order as the initial energy of the beam.[3)]

Carrying out the integration over k in (17) we have

$$\frac{\partial f_0}{\partial t} = \frac{e^2}{m^2}\frac{\partial}{\partial v}\left(\frac{|E_k|^2}{v}\frac{\partial f_0}{\partial v}\right) \qquad \left(k = \frac{\omega_2}{v}\right). \tag{19}$$

Substituting in (19) the quantity $|E_k|^2 \partial f_0/\partial v$ from (18) and integrating over v and t we obtain the following relation, which gives the mode spectrum in terms of the change in the distribution function (this integral of (17) and (18) has been obtained earlier in [5] and [7]):

$$|E_k|^2 - |E_k^0|^2 = \frac{\pi m^2}{N_2 e^2} v^3 \omega_2 \int\limits_{v_1}^{v} dv\, (f_0(t, v) - f_0^{(0)}(v)). \tag{20}$$

The quantity $|E_k^0|^2$ is the mode spectrum at the beginning of the quasilinear stage that has been considered in Sec. 3.

The height of the plateau in the distribution function $f_0^\infty$ is determined from the conservation of the number of particles:

$$f_0^\infty (v_2 - v_1) = N_1 + \int\limits_{v_1}^{\infty} f_0^{\mathrm{pl}} dv \approx N_1, \tag{21}$$

where $v_1$ and $v_2$ are the lower and upper spectrum limits. When $t \to \infty$ the quantities $v_1$ and $v_2$ are obtained from the equations[4)]

$$f_0^\infty = f_0^{\mathrm{pl}}(v_1) = \frac{N_2}{(2\pi T_2/m)^{1/2}} \exp\left(-\frac{mv_1^2}{2T_2}\right), \qquad f_0^\infty = f_0^0(v_2). \tag{22}$$

It follows from (21) and (22) that

$$v_2 - u_0 \sim u_0\left(\frac{N_1}{N_2}\right)^{1/3}, \qquad v_1 \approx \sqrt{\frac{T_2}{2m}\ln\left\{\frac{N_2}{N_1}\frac{u_0}{(2\pi T_2/m)^{1/2}}\right\}}.$$

The quantity $f_0(\infty, v) \approx N_1/(u_0 - v_1)$ for $v_1 < v < v_2$; $f_0(v)$ does not change greatly outside this range. In order to determine the steady-state spectrum we must go to the limit $t \to \infty$ in (18).

[3)]The author's attention has been directed to this feature by A. A. Vedenov and E. P. Velikhov.

[4)]It is assumed that the plasma distribution function is not changed greatly in the interaction with the beam. For this condition to be satisfied we require that $N_2 T_2 \gg N_1 m u_0^2$. The latter condition, however, is not necessary since the change in the form of the distribution function $f_0^{\mathrm{pl}}(v)$ does not change the results greatly as long as $v_1 \sim v_{T_2} \ll u_0$.

Then, substituting $f_0(\infty, v)$ in (20) and noting that $|v - u_0| \lesssim u_0(N_1/N_2)^{1/3}$ everywhere outside of a small portion of the spectrum we can neglect $f_0^0$ compared with $f_0^\infty$ and $|E_k^0|^2$ compared with $|E_k^\infty|^2$ thereby obtaining the following relation for the amplitudes of the steady-state spectrum:

$$|E_k^\infty|^2 = 4\pi^2 N_1 m \frac{\omega_2^2}{k^3}\frac{\omega_2/k - v_1}{u_0 - v_1},$$

$$\frac{\omega_2}{v_2} < k \leqslant \frac{\omega_2}{v_1} = \frac{\sqrt{2}}{\lambda_D}\ln^{-1}\left[\frac{N_2 u_0}{N_1(2\pi T_2/m)^{1/2}}\right]. \tag{23}$$

The amplitudes and the range of phase velocities $|v - u_0| \lesssim u_0(N_1/N_2)^{1/3}$ are characterized by the following inequality:

$$|E_k^\infty|^2 < |E_k^0|^2 + 4\pi^2 N_1 m \frac{\omega_2^2}{k^3}\frac{\omega_2/k - v_1}{u_0 - v_1}. \tag{24}$$

By means of this inequality it can be shown easily that the contribution of this portion of the spectrum in the total oscillation energy is approximately $(N_1/N_2)^{1/3}$, which is negligibly small. The mode spectrum arising in the interaction of a beam with a plasma is shown in Fig. 2. The dotted portion shows that part of the spectrum in which an important role is played by the nonlinear interaction of the modes and whose amplitudes have been determined to within an order of magnitude.

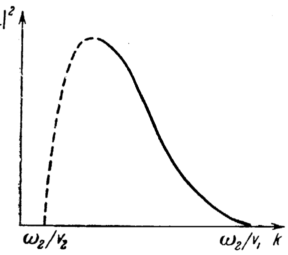


FIG. 2. Spectral density of oscillation energy arising in the interaction of a beam with a plasma.

The potential energy of the oscillations is given by the integral

$$W = \frac{2}{2\pi}\int\limits_{v_1}^{v_2} dv \left|\frac{dk}{dv}\right| \frac{|E_k|^2}{8\pi} \approx \frac{N_1 m}{2}\int\limits_{v_1}^{u_0} dv\, v\frac{v - v_1}{u_0 - v_1}$$
$$= \frac{N_1 m}{6}(u_0 - v_1)\left(u_0 + \frac{v_1}{2}\right) \approx \frac{N_1 m u_0^2}{6}. \tag{25}$$

(The factor 2 in the numerator arises because we must take account of the two regions $k > 0$ and $k < 0$ in the integration over k). Using (8) we obtain the change in the energy of the thermal and directed motion in the plasma:

$$\tfrac{1}{2} N_2\, \delta T_2 = W \approx \tfrac{1}{6} N_1 m u_0^2, \qquad \tfrac{1}{2} N_2 m u_2^2 = W N_1/3N_2. \tag{26}$$

The mean velocity in the beam as $t \to \infty$ is

$$u_\infty = \frac{1}{N_1}\int\limits_{v_1}^{\infty} f_0(\infty, v)\, v dv = \frac{u_0 + v_1}{2}.$$

Thus, the thermal energy of the beam as $t \to \infty$ is

$$\frac{N_1T_1}{2} = \int_{v_1}^{\infty} f_0(\infty, v)\frac{m}{2}\left(v - \frac{u_0+v_1}{2}\right)^2 dv$$
$$= \frac{N_1m(u_0-v_1)^2}{24} \approx \frac{N_1mu_0^2}{24}, \quad (27)$$

while the loss of energy associated with the directed motion by the beam is

$$\delta\varepsilon = \tfrac{1}{2}N_1mu_0^2 - \tfrac{1}{2}N_1mu_\infty^2$$
$$= \tfrac{3}{8}N_1m(u_0-v_1)(u_0+v_1/3) \approx \tfrac{3}{8}N_1mu_0^2.$$

It is evident that the energy of directed motion lost by the beam is equal to the energy that goes into oscillations and into increasing the beam and plasma temperatures.

The order-of-magnitude of the time required to establish the stationary spectrum (23) is

$$t_R \sim \frac{1}{\gamma^{(1)}}\ln\frac{W}{W_0}, \qquad \gamma^{(1)} \approx \omega_2\frac{N_1}{N_2}\frac{v^2}{(\Delta v)^2} \sim \omega_2\frac{N_1}{N_2};$$

where $\gamma^{(1)}$ is the growth rate for the highly smeared beam distribution function, $W \sim N_1mu_0^2$ is the final energy of the oscillations, and $W_0 \sim T_2/\lambda_D^3$ is the initial energy of the oscillations (the thermal noise energy in the plasma).

5. In order for the approximation used in this work to apply the nonlinear mode interaction must be negligibly small in the relaxation of the beam to the final quasilinear stage.[5)] The second term in (11) describes the change in amplitude due to mode interactions. We note, first, that in the quasilinear stage the nonlinearity in the beam is much weaker than in the earlier stages since the particle distribution function has been smeared out. In (11) we now separate the nonlinear terms associated with oscillations of beam particles and calculate the ratio of these terms to the main terms in (11) $\gamma_k E_k$; this ratio is of order

$$\beta = \sum_k |E_k|^2/N_2mu_0^2(\Delta v/u_0)^4,$$

where $\Delta v$ is the width of the spectrum, i.e., the width of the distribution function. In the quasilinear stage we have $\Delta v \sim u_0$, $\sum_k |E_k|^2 \sim N_1mu_0^2$ and $\beta \sim N_1/N_2 \ll 1$.

Now, using (11) we estimate the time required for the nonlinear mode interaction to become important, causing the decay of the stationary spectrum (23). The summation in (11) contains terms in phase with the linear term $\gamma_k E_k$. These terms correspond to $\kappa = k$ and $\kappa = q-k$. These "coherent" terms only change the dispersion relation, i.e., they change the frequency and growth rate because of the nonlinearity; the contribution of these terms is treated separately.

From (11) we then have

$$\frac{d^{\text{coh}}}{dt}|E_k|^2 = |E_k|^2\sum_q{}' |E_q|^2 \,\text{Im}\{H(k, k+q, k)$$
$$+ H(k, k+q, q)\}. \quad (28)$$

When $k\lambda_0 \ll 1$, the largest contribution to $\text{Im}\{H(k, k+q, k) + H(k, k+q, q)\}$ comes from terms for which $\omega_k \approx -\omega_q$. Taking this feature into account, after some simple calculations we obtain

$$\text{Im}\{H(k, k+q, k) + H(k, k+q, q)\} = \frac{4\pi e^4}{m^3}\frac{1}{\partial\varepsilon(\omega_k, k)/\partial\omega_k}$$
$$\times \text{Im}\left\{\int\frac{(k+q)(\partial f_0/\partial v)\,dv}{(\omega_k - kv)^3(\omega_q - qv)(\omega_k + \omega_q - (k+q)v)}\right.$$
$$+ \int\frac{2k(\partial f_0/\partial v)\,dv}{(\omega_k - kv)^4(\omega_q - qv)} + \frac{4\pi e^2}{m\varepsilon(\omega_k + \omega_q, k+q)}$$
$$\times \left.\left[\int\frac{(\partial f_0/\partial v)\,dv}{(\omega_k - kv)(\omega_q - qv)(\omega_k + \omega_q - (k+q)v)}\right]^2\right\}$$
$$\approx \frac{9e^2}{2m^2}\frac{\omega_2}{\omega_k^2\omega_q^2}\frac{k^2q^2}{(k+q)^2}\,\text{Im}\frac{1}{\varepsilon(\omega_k + \omega_q; k+q)}.$$

Here,

$$\varepsilon(\omega_k + \omega_q, k+q)$$
$$\approx \frac{1}{(k+q)^2\lambda_D^2} - \pi i\frac{4\pi e^2|k+q|}{m(k+q)^3}\left.\frac{\partial f_0}{\partial v}\right|_{v=(\omega_k+\omega_q)/(k+q)}$$
$$= \frac{1}{(k+q)^2\lambda_D^2}\left[1 + \frac{3}{2}\sqrt{\frac{\pi}{2}}\,i(k-q)\frac{|k+q|}{k+q}\lambda_D\right],$$
$$(\omega_k + \omega_q \approx \frac{3}{2}(k^2 - q^2)\lambda_D^2). \quad (29)$$

Thus, finally we have

$$\text{Im}\{H(k, k+q, k) + H(k, k+q, q)\}$$
$$\approx -\frac{27}{4}\frac{e^2}{m^2\omega_2^3}k^2q^2(k-q)\frac{|k+q|}{k+q}\lambda_D^3.$$

Substituting the steady-state amplitude from (23) in (28) we obtain the following approximate relation:

---

[5)]Particle trapping is also neglected in the quasilinear approximation. In the case treated here, a wave packet with a random distribution of phases, we require that the width of the packet $\Delta v$ be somewhat greater than the velocity with which a captured particle would move in the potential well $e\varphi_0$: $\Delta v \gg \sqrt{/e\varphi_0|m}$;[5] $\varphi_0$ is the potential, $\varphi_0 \sim \sqrt{\langle\varphi^2\rangle} = \sum_k |E_k|^2/k^2)^{1/2}$. Substituting $|E_k|^2$ from (23) we have

$$\frac{e\varphi_0}{m(\Delta v)^2} \sim \frac{e\sqrt{N_1mu_0^2u_0^2/\omega_2^2}}{mu_0^2} \sim \left(\frac{N_1}{N_2}\right)^{1/2} \ll 1.$$

$$\frac{d^{\text{coh}}}{dt}|E_k|^2 \sim k^3\lambda_D^3\gamma_k^{(1)}|E_k|^2 \ll \gamma_k^{(1)}|E_k|^2, \qquad (30)$$

since $k^3\lambda_D^3 \sim (v_{T_2}/u_0)^3 \ll 1$.

To determine the change of amplitude due to the incoherent terms we make use of a method similar to that used in quantum mechanical perturbation theory:[7]

$$\frac{d^{\text{incoh}}}{dt}|E_k|^2 = |E_k|^2 \sum_q (|E_q|^4 \mathcal{H}(k,q) - |E_k|^2|E_q|^2\mathcal{H}(q,k)) \left|\frac{d\omega_k}{dk} - \frac{d\omega_q}{dq}\right|^{-1},$$

$$\mathcal{H}(k,q) = H(k, k+q, k) + H(k, k+q, q) + H(k, k-q, k) + H(k, k-q, -q) + H(k, 0, q) + H(k, 0, -q) \sim e^2k^4\lambda_D^2/m^2\omega_2^3. \qquad (31)$$

Substituting $|E_q|^2$ from (23) in (28) we have

$$\frac{d^{\text{incoh}}}{dt}|E_k|^2 \sim \frac{N_1}{N_2}k^2\lambda_D^2\gamma_k^{(1)}|E_k|^2 \ll \gamma_k^{(1)}|E_k|^2, \qquad (32)$$

that is to say, the time required for the coherent and incoherent interactions of modes given by (23) to become important is several orders of magnitude greater than the growth time for the oscillations; hence, the nonlinear mode interaction can be neglected in the analysis of the establishment of the mode spectrum.

Thus, in the absence of binary collisions the interaction of a monoenergetic low-density beam with a plasma causes the beam to lose a large part of its energy of directed motion; this is approximately $\frac{3}{8}N_1mu_0^2$. One part of this energy goes into the energy of the electric field $\frac{1}{6}N_1mu_0^2$ while the other goes into increasing the thermal energy of the beam $\frac{1}{24}N_1mu_0^2$ and the plasma $\frac{1}{6}N_1mu_0^2$. The energy of directed motion lost by the plasma in the interaction with the beam is small: $N_2mu_2^2 \sim (N_1/N_2)N_1mu_0^2$.

6. Experimental investigations of the distribution function of a beam of electrons interacting with a plasma have been carried out by Karchenko, Faĭnberg et al[9]. In the experiments reported by these workers waves were amplified in propagating along the beam and measurements were made of the distribution function, averaged over a time interval large compared with the oscillation period. For a small interaction length, in which case the instability cannot develop and the beam loses a small (approximately 1%) part of its energy, the distribution function is smeared out slightly remaining bell-shaped. On the other hand, the distribution function exhibits a plateau when the interaction length is large.

In the experiments reported by Karchenko et al[9] the beam energy in the region of instability was characterized by values ranging from 2500 to 3500 eV, i.e., the beam distribution function was characterized by the parameters $u_0 \approx 3 \times 10^9$ cm/sec, $v_{T_1}/u_0 \lesssim 0.1$. The energy distribution of the beam particles was smeared out strongly in the region of the instability whereas the velocity distribution function for the beam particles remained constant in the velocity range from $1.2 \times 10^9$ cm/sec to $3.2 \times 10^9$ cm/sec, corresponding to energies from 500 to 3500 eV (lower energies were not investigated). Consequently, the beam lost at least 50% of the energy of directed motion, which was converted into oscillations and thermal energy. These experimental data are in good agreement with the results of the present work. A more detailed comparison of theory and experiment would be difficult since there are no experimental data concerning the spectra of longitudinal oscillations excited by the beam (the dependence of the oscillation amplitude on phase velocity) nor precise measurements of plasma temperature.

In conclusion the author wishes to thank Ya. B. Faĭnberg for proposing this subject and for guidance, A. I. Akhiezer for discussion of the results, A. A. Vedenov and E. P. Velikhov for valuable comments, and V. I. Shevchenko for assistance.

[1] A. I. Akhiezer and Ya. B. Faĭnberg, DAN SSSR **64**, 555 (1949); JETP **21**, 1262 (1951).

[2] D. Bohm and E. Gross, Phys. Rev. **75**, 1851 (1949).

[3] V. D. Shapiro, Izv. Vuzov, Radiofizika **4**, 867 (1961).

[4] V. D. Shapiro and V. I. Shevchenko, JETP **42**, 1515 (1962), Soviet Phys. JETP **15**, 1053 (1962).

[5] Vedenov, Velikhov, and Sagdeev, Report No. 199, Conference on Plasma Physics, Salzburg, 1961.

[6] Yu. A. Romanov and G. F. Filippov, JETP **40**, 123 (1961), Soviet Phys. JETP **13**, 87 (1961).

[7] W. E. Drummond and D. Pines, Report No. 134, Conference on Plasma Physics, Salzburg, 1961.

[8] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred, (Mechanics of Continuous Media), Gostekhizdat, 1953.

[9] Kharchenko, Faĭnberg, Nikolaev, Kornilov, Lutsenko, and Pedenko, Report No. 230, Conference on Plasma Physics, Salzburg, 1961.

Translated by H. Lashinsky
101