NEWTONIAN EQUATIONS OF MOTION AND THE HARMONIC COORDINATE CONDITIONS IN THE THEORY OF GRAVITATION

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The relation between the Newtonian equations of motion and the harmonic coordinate conditions in Einstein's theory of gravitation is investigated. It is proved that if the metric tensor can be represented as a power series, then zero-order harmonic coordinate conditions must be applied in order to deduce the Newtonian equations of motion. It is also proved that in the Infeld method for deducing the Newtonian equations of motion from the gravitational field equations coordinate conditions are employed which not only contain zero order harmonic conditions but are even stronger than these conditions.

1. INTRODUCTION

INFELD[1] has used a model for the sources of a gravitational field in conjunction with several other assumptions to deduce from the gravitational field equations both the Newtonian and the post-Newtonian equations of motion of particles. He gave a simple and condensed derivation of the equations of motion that had previously been derived by Einstein, Infeld, and Hoffman.[2-4]

The Newtonian equations of motion have also been derived by Fock,[5] using a different model of the gravitational field. In a continuation of Fock's work the post-Newtonian equations have been derived by Petrova[6] and Papapetrou.[7] The harmonic coordinate conditions have an essential role in the method of Fock, who declared that they are necessary for deriving the equations of motion. Infeld states, on the contrary, that neither the harmonic conditions nor any other coordinate conditions are involved in deriving the equations of motion.

It is shown in the present paper that zero-order harmonic conditions are necessary and sufficient for deriving the Newtonian equations of motion of particles from the gravitational field equations. It is also shown that in Infeld's method of deriving the Newtonian equations coordinate conditions stronger than the zero-order harmonic conditions are assumed.

In a paper being prepared for publication Wojewoda has proved that harmonic conditions of zero, first, and second orders are necessary and sufficient for deriving the post-Newtonian equations of motion of particles.

2. POSTULATES OF INFELD'S METHOD

In Infeld's method the symmetric tensor $g_{\mu\nu}$ characterizing the gravitational field and the coordinates x^α are related by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where

$$x^0 = ct \tag{2}$$

with Greek indices running over 0, 1, 2, 3 and with repeated indices being summed from 0 to 3. To distinguish the time coordinate we assume Hilbert's conditions

$$g_{00} > 0, \qquad \begin{vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{vmatrix} < 0,$$

$$\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} > 0, \qquad g = \det(g_{\mu\nu}) < 0. \tag{3}$$

The metric tensor $g_{\mu\nu}$ is determined from the gravitational equation

$$R^{\alpha\beta} - \tfrac{1}{2}Rg^{\alpha\beta} = -8\pi T^{\alpha\beta}, \tag{4}$$

where $R^{\alpha\beta}$ is the second-rank curvature tensor, R is the curvature invariant, and $T^{\alpha\beta}$ is the mass tensor.

In Infeld's papers particles are regarded as singularities of the field. The coordinates of the A-th singularity are denoted by ξ^μ_A. These coordinates are functions of an orbitrary parameter λ. For each singularity a "proper time" is introduced:

$$ds_A = \int_{\Omega_A} \delta(x^\mu - \xi^\mu_A)(g_{\alpha\beta}d\xi^\alpha_A d\xi^\beta_A)^{1/2}d^4x, \tag{5}$$

where integration is performed over small four-dimensional regions surrounding the singularities. The "good" four-dimensional function $\delta(x^\mu - \xi^\mu)$ possesses all the properties of Dirac's δ function plus the property that the function

$$f(\xi^\mu) = \int \delta(x^\mu - \xi^\mu) f(x^\mu)\, d^4x$$

is non-singular even when $f(x^\mu)$ possesses singularities of the type r^{-p}, $p = 1, 2, \ldots$.

The first postulate of Infeld's method consists in assuming a mass tensor of the form

$$\sqrt{-g}\, T^{\alpha\beta} = c^{-2} \int k \sum_A \mu_A \delta(x^\sigma - \xi_A^\sigma) \frac{d\xi_A^\alpha}{ds_A} \frac{d\xi_A^\beta}{ds_A} ds_A. \quad \text{(I)}$$

According to the second postulate, the metric tensor can be expanded in a power series:

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + c^{-1} g_{\alpha\beta}^{(1)} + c^{-2} g_{\alpha\beta}^{(2)} + c^{-3} g_{\alpha\beta}^{(3)} + \ldots \quad \text{(II)}$$

According to the third postulate the quantities

$$\partial\Phi/\partial t, \qquad \partial\Phi/\partial x^m \quad \text{(III)}$$

are of zero order if Φ is of zero order. The fourth postulate gives the behavior of the metric tensor at infinity:

$$(g_{\alpha\beta})_\infty = \eta_{\alpha\beta}, \qquad \lim_{r\to\infty}(g_{\mu\nu} - \eta_{\mu\nu}) = \lim_{r\to\infty}(M/r), \quad \text{(IV)}$$

where M is a constant and

$$r^2 = \delta_{ij} x^i x^j, \qquad \eta_{ab} = -\delta_{ab}, \qquad \eta_{00} = 1, \qquad \eta_{0m} = 0$$

with Latin indices running over 1, 2, 3 and with repeated indices being summed from 1 to 3. The second equation of IV insures the Newtonian character of the solutions of the field equations; this is also among the postulates of Infeld's method. According to the fifth and last postulate we have

$$g_{\alpha\beta}^{(0)} = \eta_{\alpha\beta}. \quad \text{(V)}$$

We shall show that these relations do not follow from the correspondence principle.

It follows from the field equations (4) that

$$\nabla_\alpha T^{\alpha\beta} = 0, \quad (6)$$

where ∇_α is the covariant derivative with the metric tensor $g_{\mu\nu}$. Substituting (I) in (6), we obtain

$$\frac{d^2\xi_A^\alpha}{ds_A^2} + \Gamma_{A\mu\nu}^\alpha \frac{d\xi_A^\mu}{ds_A} \frac{d\xi_A^\nu}{ds_A} = 0. \quad (7)$$

Solving the field equations (4) in conjunction with the postulates (I)—(V), the Newtonian equations of motion are derived uniquely from (7).

3. EXACT STATEMENT OF THE PROBLEM

In Fock's method the functions $g_{\mu\nu}^*$ characterizing the gravitational field are related to the coordinates $x^{*\alpha}$ by

$$ds^2 = g_{\mu\nu}^* dx^{*\mu} dx^{*\nu}, \quad (8)$$

where

$$x^{*0} = c^{-1} x^0, \qquad x^{*m} = x^m. \quad (9)$$

From (1), (2), (8), and (9) we obtain

$$g_{00}^* = c^2 g_{00}, \qquad g_{0n}^* = c g_{0n}, \qquad g_{mn}^* = g_{mn}. \quad (10)$$

The coordinates $x^{*\alpha}$ are harmonic when

$$\Gamma^\alpha = -\sqrt{-g^*}\, \partial(\sqrt{-g^*} g^{*\alpha\beta}) / \partial x^\beta = 0. \quad (11)$$

The substitution of (10) in (11) gives

$$\Gamma^0 = -\sqrt{-g}\, [c^{-2} \partial(\sqrt{-g}\, g^{00}) / \partial t + c^{-1} \partial(\sqrt{-g}\, g^{0m}) / \partial x^m] = 0, \quad (12)$$

$$\Gamma^n = -\sqrt{-g}\, [c^{-1} \partial(\sqrt{-g}\, g^{0n}) / \partial t + \partial(\sqrt{-g}\, g^{nm}) / \partial x^m] = 0. \quad (13)$$

Hence for the zeroth approximation we obtain

$$\partial(\sqrt{-g^{(0)}} g^{(0)mn} / \partial x^n = 0. \quad \text{(A)}$$

When the harmonic conditions are written in finite form the choice of variables is not significant. To estimate the degree of smallness of the harmonic conditions we must use "natural" coordinates which are defined without c.

We know that Newton's equations for N particles are

$$\mu_A\, d^2\xi_A^m/dt^2 + F_A^m = 0, \quad (14)$$

where

$$F_A^m = -\sum_{B=1,\, B\neq A} k \frac{\partial}{\partial \xi_A^m} \frac{\mu_A \mu_B}{r_{AB}},$$

$$(r_{AB})^2 = \delta_{ij} (\xi_A^i - \xi_B^i)(\xi_A^j - \xi_B^j).$$

The constants μ_A denote the masses of the particles and k is the gravitational constant.

It is our task to prove the following theorems:

(a) To derive the equations of motion (14) from the field equations (4) assuming (I)—(IV), the zero-order harmonic conditions (A) are necessary and sufficient.

(b) The conditions (V) are stronger than the zero-order harmonic conditions (A) and are not necessary for deriving the equations of motion (14) from the field equations (4) on the basis of the postulates (I)—(IV).

4. PROOF OF THEOREM (a)

To prove theorem (a) we must obtain the most general functions $g_{\alpha\beta}^{(0)}$, $g_{0\alpha}^{(1)}$, and $g_{00}^{(2)}$ satisfying the field equations (4) and the postulates (I)—(IV). We

know that by means of the transformations

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \tag{15}$$

it is always possible to fulfill the conditions

$$\partial\,(\sqrt{-g}\,g^{\alpha\beta}) / \partial x^\beta = 0. \tag{16}$$

Thus the most general solutions of (4) assuming (I)—(IV) are obtained by means of (15) from the most general solutions of (4) assuming (I)—(IV) and (16). We introduce the notation

$$(g^{(0)}_{\alpha\beta}) = \begin{pmatrix} -e^{2\Phi}\gamma_{mn} + e^{-2\Phi} f_m f_n & e^{-2\Phi} f_m \\ e^{-2\Phi} f_n & e^{-2\Phi} \end{pmatrix},$$

$$f_{mn} = D_m f_n - D_n f_m, \tag{17}$$

where D_m is the coordinate derivative with the metric tensor γ_{mn}. In this notation the field equations (4) give in the zeroth approximation

$$P_{mn} - \tfrac{1}{2} P\gamma_{mn} = U_{mn} - Q_{mn}, \tag{18}$$

$$\gamma^{mn} D_m D_n \Phi = \tfrac{1}{4} e^{-4\Phi} f_{ij} f^{ij}, \tag{19}$$

$$D_m\,(e^{-4\Phi} f^{mn}) = 0, \tag{20}$$

where

$$U^{ij} = e^{-4\Phi}\left(\tfrac{1}{2}\gamma^{k\ell}\gamma^{si}\gamma^{ab} - \tfrac{1}{4}\gamma^{ks}\gamma^{ab}\gamma^{ij}\right) f_{ka} f_{sb}, \tag{21}$$

$$Q^{ij} = (2\gamma^{a\ell}\gamma^{bj} - \gamma^{ab}\gamma^{ij}) D_a\Phi D_b\Phi \tag{22}$$

and P_{mn} is the second-rank curvature tensor corresponding to the tensor γ_{mn} and P is the curvature invariant.

For (16) in zeroth order we obtain

$$\partial\,(\sqrt{\gamma}\,\gamma^{mn}) / \partial x^m = 0, \qquad \partial\,(\sqrt{\gamma} f^m) / \partial x^m = 0, \tag{23}$$

where

$$\gamma = \det(\gamma_{mn}).$$

It follows from (13) and (17) that the form

$$dl^2 = \gamma_{mn} dx^m dx^n \tag{24}$$

is positive definite. Then

$$e^{-4\Phi} f_{mn} f^{mn} \geqslant 0. \tag{25}$$

Since the equations (19) are elliptic, it follows from (25) and the conditions (IV) at infinity that

$$\Phi = 0. \tag{26}$$

Equations (18) and (19) now assume the simpler forms

$$P_{ij} = \tfrac{1}{2}\gamma^{ab} f_{ai} f_{bj}, \tag{27}$$

$$D_i f^{ij} = 0. \tag{28}$$

Equations (25) and (27) show that the form

$$dP = P_{mn} dx^m dx^n \tag{29}$$

is positive definite. Expanding (28) and considering the second equation in (23), we obtain

$$\gamma^{mn} D_m D_n f_k = P_{mk} f^m. \tag{30}$$

The general solution of these equations satisfying the conditions (IV) at infinity has the form (Chapter 2 of [8])

$$f_m = 0. \tag{31}$$

From (27) and (31) we obtain

$$P_{mn} = 0. \tag{32}$$

We know [9] that these equations are necessary and sufficient conditions for the Euclidean character of three-dimensional space; we thus have

$$\gamma_{mn} = \delta_{ij}\frac{\partial a^i}{\partial x^m}\frac{\partial a^j}{\partial x^n}. \tag{33}$$

The substitution of (33) in (23) gives

$$\gamma^{mn}\partial^2 a^k / \partial x^m\,\partial x^n = 0. \tag{34}$$

Considering the conditions (IV) at infinity, we obtain from (34)

$$\gamma_{mn} = \delta_{mn}. \tag{35}$$

The final solution of (4) and (16) in zeroth order is

$$g^{(0)}_{\alpha\beta} = \eta_{\alpha\beta}. \tag{36}$$

In first and second approximations (4) and (16) give

$$\delta^{mn}\,\partial^2 g^{(1)\,0k} / \partial x^m \partial x^n = 0, \tag{37}$$

$$\delta^{mn}\,\partial^2 g^{(2)\,00} / \partial x^m\,\partial x^n = -8\pi\sum_A k\mu_A \delta\,(x^k - \xi^k_A). \tag{38}$$

The general solutions of these equations satisfying the conditions (IV) at infinity are

$$g^{(1)\,0n} = 0, \qquad g^{(2)\,00} = -\varphi, \tag{39}$$

where

$$\varphi = -2k\sum_A \mu_A / |\,\xi^m_A - x^m\,|. \tag{40}$$

Collecting the results with the requisite accuracy, we have

$$g_{00} = 1 + c^{-2}\varphi + O\,(c^{-3}),$$
$$g_{0n} = O\,(c^{-2}),$$
$$g_{mn} = -\delta_{mn} + O\,(c^{-1}). \tag{41}$$

With the proper change of notation, from (15) and (41) we obtain the most general solutions of (4) subject to (I)—(IV):

$$g_{00} = 1 + 2c^{-1}\frac{\partial S^{(1)0}}{\partial t} + c^{-2}\Big(\varphi + 2\frac{\partial S^{(2)0}}{\partial t} + \frac{\partial S^{(1)0}}{\partial t}\frac{\partial S^{(1)0}}{\partial t} - \delta_{mn}\frac{\partial S^{(0)m}}{\partial t}\frac{\partial S^{(0)n}}{\partial t}\Big),$$

$$g_{0n} = \frac{\partial S^{(1)0}}{\partial x^n} + c^{-1}\Big(\frac{\partial S^{(2)0}}{\partial x^n} + \frac{\partial S^{(1)0}}{\partial x^n}\frac{\partial S^{(1)0}}{\partial t} - \delta_{ab}\frac{\partial S^{(0)a}}{\partial x^n}\frac{\partial S^{(0)b}}{\partial t}\Big),$$

$$g_{mn} = \frac{\partial S^{(1)0}}{\partial x^m}\frac{\partial S^{(1)0}}{\partial x^n} - \delta_{ab}\frac{\partial S^{(0)a}}{\partial x^m}\frac{\partial S^{(0)b}}{\partial x^n}. \qquad (42)$$

The substitution of (42) in (7) gives in the first nonvanishing approximation

$$\frac{d^2\xi^m_A}{dt^2} + \Lambda^m_{Aij}\frac{d\xi^i_A}{dt}\frac{d\xi^j_A}{dt} + N^m_{Ai}\frac{d\xi^i_A}{dt} + P^m_A = 0, \qquad (43)$$

where

$$\Lambda^k_{ij} = \tfrac{1}{2}\gamma^{ks}(\partial\gamma_{is}/\partial x^j + \partial\gamma_{js}/\partial x^i - \partial\gamma_{ij}/\partial x^s),$$

$$N_{ks} = 2\delta_{mn}(\partial S^{(0)m}/\partial x^k)(\partial^2 S^{(0)n}/\partial x^s\,\partial t),$$

$$P_k = \delta_{mn}(\partial S^{(0)m}/\partial x^k)(\partial^2 S^{(0)n}/\partial t^2) + \tfrac{1}{2}\partial\varphi/\partial x^k \qquad (44)$$

and

$$\gamma_{ab} = \delta_{ij}(\partial S^{(0)i}/\partial x^a)(\partial S^{(0)j}/\partial x^b). \qquad (45)$$

We shall first prove that the condition (A) is sufficient. The substitution of (42) in (A) gives

$$\gamma^{ks}\partial^2 S^{(0)m}/\partial x^k\,\partial x^s = 0. \qquad (46)$$

From (46) in conjunction with (IV) we obtain

$$S^{(0)m} = c^m_n x^n + V^m t + \varepsilon^m, \qquad (47)$$

where c^m_n, V^m, and ϵ^m are constants and

$$\delta_{ij}c^i_m c^j_n = \delta_{mn}.$$

From (44) and (45) we obtain

$$\gamma_{ab} = \delta_{ab}, \qquad \Lambda^k_{mn} = 0, \qquad N_{mn} = 0, \qquad P_s = \tfrac{1}{2}\partial\varphi/\partial x^s.$$

Thus (43) gives the Newtonian equations of motion.

To prove the necessity of (A) we note that (43) has definite solutions for arbitrary initial conditions regarding ξ^S_A and $d\xi^S_A/dt$; we can therefore consider these quantities as arbitrary. The second derivatives $d^2\xi^S_A/dt^2$ are by definition related through (14). Multiplying (14) by Lagrangian multipliers and adding it to (43), we obtain

$$\frac{d^2\xi^k_A}{dt^2}(\lambda^s_{Ak} - \delta^s_{Ak}) + \Lambda^s_{Aij}\frac{d\xi^i_A}{dt}\frac{d\xi^j_A}{dt} + N^s_{Ai}\frac{d\xi^i_A}{dt} + (F^k_A\lambda^s_{Ak} - P^s_A) = 0. \qquad (48)$$

By equating the coefficients of $d\xi^S_A/dt$ and $d^2\xi^S_A/dt^2$ to zero and taking (IV) into account, we have

$$S^{(0)m} = c^m_k x^k + V^m t + \varepsilon^m, \quad \gamma_{ks} = \delta_{ks}. \qquad (49)$$

It follows from (49) that (A) is fulfilled.

5. PROOF OF THEOREM (b) AND CONCLUDING REMARKS

In Infeld's method the condition (A) is replaced by (V). When (V) is fulfilled, then (A) is also fulfilled. The converse is not true: from (A) it follows, as has been shown, that

$$g^{(0)}_{00} = 1, \qquad g^{(0)}_{0n} = \frac{\partial S^{(1)0}}{\partial x^n}, \qquad g^{(0)}_{mn} = -\delta_{mn} + \frac{\partial S^{(1)0}}{\partial x^m}\frac{\partial S^{(1)0}}{\partial x^n},$$

which denotes that (V) is not fulfilled. Thus (V) not only contains the zero-order harmonic conditions but is stronger than the latter. Therefore it also follows that (V) is not necessary for deriving the Newtonian equations of motion.

The correspondence principle requires that (43) should coincide with (14) for large c. Since (43) is independent of c, the proof of theorem (a) is also a proof that the zero-order harmonic conditions follows from the correspondence principle.

It follows from (15) that in conjunction with the postulates (I)—(IV) the harmonic conditions (A) determine the coordinate system accurately up to Galilean transformations. This accounts for the fact that the coordinate conditions (A) are sufficient for deriving the Newtonian equations of motion.[10]

The same question has also been considered by Meister and Papapetrou,[11] who arrived at the opposite conclusion that the Newtonian equations of motion do not depend on the zero-order harmonic conditions. However, the fulfillment of (V) was assumed in the proof of this conclusion. Yet, as we have seen, the latter not only contains, but is stronger than, the zero-order harmonic conditions. In [11] it was essentially only proved that the Newtonian equations of motion do not depend on harmonic conditions of higher than the zero order.

All the foregoing considerations indicate that the harmonic coordinate system has a very important role in the deduction of the equations of motion of particles from the gravitational field equations.

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