RESTRICTIONS IMPOSED ON THE REGGE TRAJECTORY BY MAXIMAL ANALYTICITY

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The restrictions imposed on the possible trajectories of the Regge poles $\alpha(t)$ by the analytic properties of the function α are investigated. It is being assumed, that the functions α possess "maximal analyticity" [i.e., admit of representations of the form (1) or (2)]. Inequalities satisfied by the functions for t below threshold are established and the corresponding restrictions on the masses of the particles belonging to a given trajectory as well as on the decrease of cross sections as a function of the scattering angle at large energies (in particular, restrictions on the structure of the "diffraction peak") have been established.

1. INTRODUCTION

IN connection with the important role played by the concept of moving Regge poles [1] in the investigation of the asymptotic behavior of cross sections of various processes in the high-energy region [2-5] and in the clarification of the spectrum of particles, the reconstruction of the trajectories $\alpha(t)$ of these poles from experimental data is of special interest.

In order to explain the observed picture of diffraction scattering, Gribov [2] and Chew and Frautschi [3] have proposed the existence of a trajectory possessing the quantum numbers of the vacuum (the Pomeranchuk trajectory $\alpha_p(t)$, with $\alpha_p(0) = 1$. The analysis of the πN and NN scattering data has allowed to determine the behavior of the vacuum trajectory for small negative t, and also to obtain some information about the other trajectories, which give smaller contributions to the cross sections in the high-energy region (cf. in this connection [8, 14]). In principle scattering data allow one to determine trajectories $\alpha(t)$ for $t \le 0$. On the other hand, Goldberger and Blankenbecler [6] and Chew and Frautschi [7] have expressed the hypothesis that the masses of the particles are determined by the pole trajectories in the region $t \ge 0$.

Thus there appears the problem of extrapolating the pole trajectories into the region $t > 0$, and also into the region of negative t of larger absolute values than those for which $\alpha(t)$ is known from the present experimental data. The present paper is devoted to one of the aspects of this problem, namely to an investigation of those restrictions on the possible course of the trajectories imposed by the analytic properties of the functions α. We shall establish inequalities to which the functions $\alpha(t)$ are subjected for values of t lying below threshold (e.g., $t < 4\mu^2$ for the vacuum trajectory; μ is the pion mass), and we shall find the restrictions which these inequalities impose on the masses of the particles which lie on a given trajectory, as well as on the decrease of cross sections with increasing scattering angle at high energies (in particular, the structure of the "diffraction peak").

We shall assume that the functions $\alpha(t)$ admit the representation

$$\alpha(t) = \frac{1}{\pi}\int_1^\infty \frac{\tilde{\alpha}(t')\,dt'}{t'-t}, \qquad \tilde{\alpha}(t') \geqslant 0, \tag{1}$$

or a representation with one subtraction

$$\alpha(t) = \alpha(t_2) + \frac{t-t_2}{\pi}\int_1^\infty \frac{\tilde{\alpha}(t')\,dt'}{(t'-t)(t'-t_2)}, \qquad \tilde{\alpha}(t') \geqslant 0 \tag{2}$$

(the threshold value of energy has been put equal to one). Such representations have been proposed by Gribov and Pomeranchuk [9] and are the most natural from the standpoint of "maximal analyticity": the functions α have branch points (cuts) only for such values of t, where this follows with necessity from the unitarity condition for the partial wave amplitudes. The positive character of the spectral function $\tilde{\alpha}$ is necessary in order that a pole, which has wandered off into the upper half-plane of t near the threshold, should remain there. One subtraction is sufficient since the scattering amplitude, which is proportional to $s^{\alpha(t)}$, has no essential singularity for $|t| \to \infty$ [9].

At present there does not exist a complete proof of the Regge pole hypothesis (cf. [10]), hence one can make no rigorous affirmations about the analytic properties of the trajectories of these poles. In particular, Oehme[11] has discovered in the problem of the relativistic Schrödinger equation with a Coulomb potential the existence of complex branch points of the functions $\alpha(t)$. The non-relativistic analog of the cause of the appearance of complex branch points is the "falling into the center." We hope that in a more realistic theory than the one discussed by Oehme[11], the behavior of the "potential" near $r = 0$ will forbid such branch points.

All our results refer to the case of boson trajectories (to which, in particular the Pomeranchuk trajectory belongs). As Gribov[12] has shown recently, the analytic properties of the fermion trajectories are somewhat more complicated than the properties implied by Eqs. (1) or (2).

2. THE CASE OF A DISPERSION REPRESENTATION WITHOUT SUBTRACTIONS

Let us find the restrictions on the possible form of the trajectory of the pole $\alpha(t)$, assuming that the function α admits a representation of the form (1) and assuming the value of this function and its derivative known in a point t_1 ($t_1 < 1$, otherwise t_1 is arbitrary).

The representation (1) implies that $\mathrm{Im}\,t/\mathrm{Im}\,\alpha(t) \geq 0$. In other words, $\alpha(t)$ is a Nevanlinna function (or R-function). In the Appendix we derive the following inequality for an R-function $f(x)$, in the domain $x < 1$, if $f(x)$ is real in this domain:

$$f(x_1) + f'(x_1)(x - x_1) \leqslant f(x) \leqslant f(x_1) + f'(x_1)(x - x_1)(1 - x_1)/(1 - x) \qquad (3)$$

($x_1 < 1$; the prime denotes a derivative). Using this inequality and observing that, (1) implies that for $t < 1$, $\alpha(t) > 0$, we obtain

$$\max\{0; \alpha(t_1) + \alpha'(t_1)(t - t_1)\} \leqslant \alpha(t) \leqslant \alpha(t_1) + \alpha'(t_1)(t - t_1)(1 - t_1)/(1 - t), \qquad (4)$$

where $\max\{a, b\}$ is the larger of the quantities a or b.

The domain in the (α, t) plane in which the trajectory $\alpha(t)$ can lie according to Eq. (4) is cross-hatched in Fig. 1. This domain is bounded by the t-axis and by the curves

$$B_1: \quad \alpha = \alpha(t_1) + \alpha'(t_1)(t - t_1),$$

$$B_2: \quad \alpha = \alpha(t_1) + \alpha'(t_1)(t - t_1)(1 - t_1)/(1 - t).$$

We note that Eq. (4) also implies the inequality

$$\alpha(t) \geqslant \alpha'(t)(1 - t) \quad (t < 1);$$

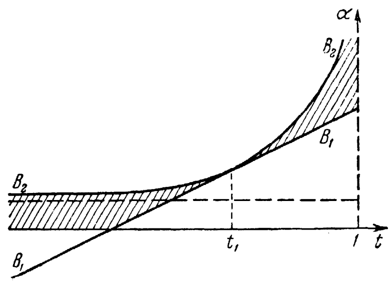

FIG. 1

in particular, for the Pomeranchuk trajectory $\alpha'_P(0) \leq \alpha_P(0) = 1$ (from the analysis of experimental data given in [8] $\alpha'_P(0) \approx 0.1$).

The inequality (4) also yields limitations on the decrease of the scattering a amplitude with the increase of the scattering angle θ at large energies $\sqrt{s}$:

$$\max\{s^{-\alpha(0)};\ s^{\varphi_1(s,\cos\theta)}\} \leqslant A(s, \cos\theta)/A(s, 1) \leqslant s^{\varphi_2(s,\cos\theta)},$$
$$\varphi_1 = -{}^1/_2\alpha'(0) \quad s(1 - \cos\theta),$$
$$\varphi_2 = -\alpha'(0)s(1 - \cos\theta)[2 + s(1 - \cos\theta)]^{-1}. \qquad (5)$$

These limitations are valid in that region of scattering angles in which, on the one hand the scattering amplitude $A(s, t) \sim \Sigma r(t) s^{\alpha(t)}$ is determined by the same Regge pole as for $\theta = 0$ and, on the other hand, the "coupling constants" of the Regge poles $r(-{}^1/_2 s(1 - \cos\theta))$ can be considered as slowly varying functions of the energy (in comparison with the exponential factor s^{α}). Particularly, if at $|t| \to \infty$, $r(t)$ neither vanishes nor becomes infinite and the trajectory of the pole which determines the scattering at $\theta = 0$ does not intersect another trajectory with the same quantum numbers, then the restrictions (5) are valid for all scattering angles.

Besides the restrictions on the width of the "diffraction peak" (5) one can also determine the minimal possible value for the mass $m(l)$ of the particle with spin l belonging to a given trajectory:

$$[m(l)]^2 \geqslant 1 - \frac{\alpha'(t_1)(1 - t_1)^2}{l + \alpha'(t_1)(1 - t_1) - \alpha(t_1)} \quad (t_1 \leqslant 0). \qquad (6)$$

This relation is an immediate consequence of Eq. (4) if one takes into account the fact that the mass of the particle with spin 1 satisfies the equation $\alpha(m^2) = 1$.

3. THE CASE OF A DISPERSION REPRESENTATION WITH ONE SUBTRACTION

Let us turn now to the case in which the function α admits the representation (2). As before, the function α is an R-function, and therefore one can apply to it the inequality (3) and all the inequalities deriving from it. The function

$$[\alpha(t)-\alpha(t_2)]/(t-t_2),$$

for $t_2 < 1$ (and otherwise arbitrary), is also an R-function. Applying Eq. (3) to this function we obtain

$$\max\{0;\ g_1+g_2(t-t_1)\}\leqslant\frac{\alpha(t)-\alpha(t_2)}{t-t_2}$$

$$\leqslant g_1+g_2(t-t_1)\frac{1-t_1}{1-t}, \qquad (7)$$

where

$$g_1=\frac{\alpha(t_2)-\alpha(t_1)}{t_2-t_1}\geqslant 0,$$

$$g_2=\frac{\alpha(t_2)-\alpha(t_1)-\alpha'(t_1)(t_2-t_1)}{(t_2-t_1)^2}\geqslant 0.$$

As was to be expected, in order to impose more severe restrictions on the form of the trajectory $\alpha(t)$ than the ones considered in the preceding section, one may assume one further characteristic of the trajectory (for instance the value of α in the point $t_2 \neq t_1$). If one knows a particle belonging to the trajectory $\alpha(t)$ one may take as t_2 the square of its mass. Then $\alpha(t_2)$ equals the spin of that particle. If the curvature of the trajectory near t_1 is known, it is convenient to go to the limit $t_2 \to t_1$ and to transform (7) to the simpler form

$$a_1(t)\leqslant\alpha(t)\leqslant a_2(t)\qquad (t>t_1),$$
$$a_2(t)\leqslant\alpha(t)\leqslant a_1(t)\qquad (t_1>t>t_3),$$
$$a_2(t)\leqslant\alpha(t)<\alpha(t_1)-\tfrac{1}{2}[\alpha'(t_1)]^2/\alpha''(t_1)\qquad (t<t_3), \quad (8)$$

where

$$t_3=t_1-\alpha'(t_1)/\alpha''(t_1),$$

$$a_1(t)=\alpha(t_1)+\alpha'(t_1)(t-t_1)+\tfrac{1}{2}\alpha''(t_1)(t-t_1)^2,$$

$$a_2(t)=\alpha(t_1)+\alpha'(t_1)(t-t_1)$$
$$+\tfrac{1}{2}\alpha''(t_1)(t-t_1)^2(1-t_1)/(1-t).$$

The region in the (α, t) plane which, according to Eqs. (8) can contain the trajectory $\alpha(t)$ is the one cross-hatched in Fig. 2. This region is bounded by the curves

C_1: $\quad \alpha=\alpha(t_1)+\alpha'(t_1)(t-t_1)+\tfrac{1}{2}\alpha''(t_1)(t-t_1)^2,$

C_2: $\quad \alpha=\alpha(t_1)+\alpha'(t_1)(t-t_1)$
$\quad +\tfrac{1}{2}\alpha''(t_1)(t-t_1)^2(1-t_1)/(1-t),$

C_3: $\quad \alpha=\alpha(t_1)-\tfrac{1}{2}[\alpha'(t_1)]^2/\alpha''(t_1).$

The inequalities restricting the decrease of the scattering amplitude with increasing scattering angle at large energies are the following:

$$s^{\varphi_3(s,\cos\theta)}\leqslant A(s,\cos\theta)/A(s,1)\leqslant s^{\varphi_4(s,\cos\theta)},$$

$$\varphi_3=-\tfrac{1}{2}\alpha'(0)s(1-\cos\theta)$$
$$+\tfrac{1}{4}\alpha''(0)s^2(1-\cos\theta)^2[2+s(1-\cos\theta)]^{-1},$$

$$\varphi_4=\min\{-\tfrac{1}{2}\alpha'(0)s(1-\cos\theta)+\tfrac{1}{8}\alpha''(0)s^2(1-\cos\theta)^2,$$
$$-\tfrac{1}{2}[\alpha'(0)]^2[\alpha''(0)]^{-1}\}. \qquad (9)$$

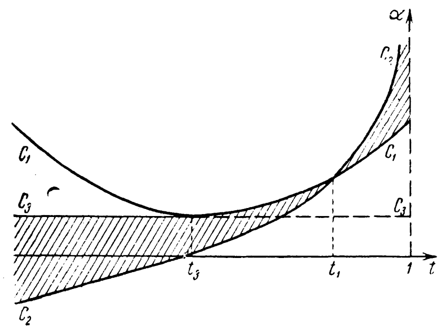

FIG. 2

The conditions of applicability of these formulas are the same as for Eq. (5).

The minimal possible mass of the particle with spin l belonging to the trajectory $\alpha(t)$ is determined from the equation

$$\alpha(t_1)+\alpha'(t_1)(m^2_{min}-t_1)$$
$$+\tfrac{1}{2}\alpha''(t_1)(1-t_1)(m^2_{min}-t_1)^2/(1-m^2_{min})=l$$
$$(t_1\leqslant 0) \qquad (10)$$

or from the equation

$$\alpha(t_1)+g_1(m^2_{min}-t_2)$$
$$+g_2(1-t_1)(m^2_{min}-t_1)(m^2_{min}-t_2)/(1-m^2_{min})=l \quad (t_1\leqslant 0;$$
$$l>\alpha(t_2)). \qquad (10')$$

We observe, that the established restrictions on the behavior of the functions $\alpha(t)$ become weaker as t approaches unity or $-\infty$. In particular we are unable to determine the maximal spin of a stable particle belonging to a given Regge trajectory: for $t \to 1$ the inequalities (4), (7), and (8) restrict the function $\alpha(t)$ only from below. Though the function α is bounded for $t \to -\infty$ (or even vanishes in the no-subtraction case), knowledge of the value of this function and of its first two derivatives in the point t_1, together with the representation (2) [or (1)] do not allow a determination of the quantity $\alpha(-\infty)$ (or the law according to which $\alpha(t)$ goes to zero).

In conclusion I wish to express my gratitude to N. I. Akhiezer for a valuable comment, to I. Ya. Pomeranchuk for a useful discussion and interest in this work, and also to I. S. Shapiro for valuable observations.

APPENDIX

For a proof of the inequality (3) we shall start from the general representation of an R-function which is real for $x < 1$ [Eq. (1) is a special case of this]:

$$f(x) = \mu x + \nu + \int_1^\infty \frac{1+ux}{u-x}\, d\sigma(u), \qquad (\mu \geqslant 0;\ \mathrm{Im}\,\nu = 0), \tag{A.1}$$

where σ is a non-decreasing function of bounded variation (cf. e.g.[13]). According to Eq. (A.1) we have

$$\frac{f(x)-f(x_1)}{x-x_1} = \mu + \int_1^\infty \frac{1+u^2}{(u-x)(u-x_1)}\, d\sigma(u),$$

$$f'(x_1) = \mu + \int_1^\infty \frac{1+u^2}{(u-x_1)^2}\, d\sigma(u). \tag{A.2}$$

A comparison of these expressions immediately yields the restriction on $f(x)$ from below given in Eq. (3). Further, by subtracting the second line of Eq. (A.2) form the first line, we obtain

$$\frac{f(x)-f(x_1)}{x-x_1} - f'(x_1) = (x-x_1)\int_1^\infty \frac{(1+u^2)\, d\sigma(u)}{(u-x_1)^2(u-x)}. \tag{A.3}$$

Observing that the right hand side of (A.3) does not decrease if one replaces $u - x$ in the denominator of the integrand by $1 - x$, and utilizing Eq. (A.2) we obtain the restriction from above on $f(x)$ in Eq. (3).

We note that the inequality (3) can be obtained as a limiting case from a well-known result of N. I. Akhiezer and M. G. Kreĭn, which establishes restrictions on the possible values of an R-function $f(x)$ for complex x, if it is known that $f(x_0) = w_0$ ($\mathrm{Im}\, x_0 > 0$) and that furthermore the function $f(x)$ is real and continuous in a given interval on the real axis (cf. [13], page 162). Using this proof one obtains simultaneously sufficiently rigid restrictions on the behavior of the function $\alpha(t)$ in the complex t-plane. We shall not reproduce these restrictions since we are not aware of physical quantities in which there appear Regge poles with complex t.

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