ON THE THEORY OF INTERNAL CONVERSION OF GAMMA RAYS

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Internal conversion of radiation of the magnetic and electric types in the K-shell is considered for transitions of arbitrary multipolarity. Closed-form expressions for the conversion coefficients and angular distribution of the conversion electrons are obtained with an accuracy to terms of the order of αZ.

THE calculation of the internal-conversion coefficient with allowance for the Coulomb field of the nucleus leads to radial integrals of a complicated form, which can be obtained only by numerical methods. Therefore, along with tabulating the coefficients calculated numerically by using exact Dirac wave functions of the electron in the Coulomb field, interest attaches to various approximate methods, which yield expressions for the conversion coefficients in closed form.

In the present paper we calculate the internal conversion coefficients, using for the electron wave function in the continuous spectrum the Furry-Sommerfeld-Maue function [1,2]

$$\psi = e^{\pi\xi/2}\Gamma(1+i\xi)e^{i\mathbf{pr}}[1-\gamma_4\boldsymbol{\gamma}\nabla/2\varepsilon]F[-i\xi, 1; -i(pr+\mathbf{pr})]u(\mathbf{p}), \quad (1)$$

where $\xi = \alpha Z\epsilon/p$; ϵ are $\mathbf{p}$ are the energy and momentum; $u(\mathbf{p})$ is the Dirac bispinor of the free electron; γ_4 and $\boldsymbol{\gamma}$ are Dirac matrices; $F[a; c; x]$ is the confluent hypergeometric function (it is assumed here and throughout that $\hbar = c = m = 1$).

Calculations with the aid of this function enable us to obtain the internal-conversion coefficient with allowance for terms of order αZ (Z is the nuclear charge and α is the fine-structure constant). The character of the employed approximation becomes clear if we compare the function (1), expanded in powers of the momenta l, with the exact function, also expanded in the momenta. It turns out here that the exact function goes over into (1) if we neglect in each term of the exact series the quantity $\alpha^2 Z^2/l^2$ compared with unity (see [3]). Therefore in those cases when small l give an insignificant contribution, the results obtained by using (1) are close to accurate. In the general case, the Furry-Sommerfeld-Maue function has an accuracy of order αZ. In the internal-conversion process, the value of the momentum l is determined by the difference between the spins of the initial and final state of the nucleus. Therefore the error introduced in the calculations by the approximate wave function of the final state decreases with increasing multipolarity of the transitions.

We note that the wave function of the final state of the electron should have at large distances the form of a sum of a plane and spherical convergent waves, which corresponds to the occurrence of a particle in a continuous spectrum. The function (1) satisfies this condition.

The differential conversion coefficient, which is the ratio of the probability of the conversion transition with emission of an electron in the momentum interval from $\mathbf{p}$ to $\mathbf{p} + d\mathbf{p}$ to the probability of emission of the γ quantum in the same nuclear transition, is determined by the expression

$$d\beta_L^{(\lambda)} = \frac{\alpha\omega}{4}\sum |M_{LM}^{(\lambda)}|^2 \frac{d\mathbf{p}}{(2\pi)^3}\delta(\varepsilon_i + \omega - \varepsilon_f),$$
$$M_{LM}^{(\lambda)} = \int \bar{\psi}_f B_{LM}^{(\lambda)}\psi_i d\mathbf{r}, \quad (2)$$

where ω is the energy released in the nuclear transition, L is the difference between the spins of the initial and final state of the nucleus, ψ_i and ψ_f are the electron wave functions, $B_{LM}^{(\lambda)}$ is the potential of the magnetic ($\lambda = 0$) or electric ($\lambda = 1$) multipole radiation, and the summation is carried out over the polarizations of the electrons in the initial and final states.

For the wave function of the initial state (K-shell) we choose in accordance with the accuracy of the analysis the function

$$\psi_i = \psi_{iI} + \psi_{iII},$$
$$\psi_{iI} = N_i e^{-\eta r}u_0, \qquad \psi_{iII} = -\tfrac{1}{2}N_i\eta\gamma_4\boldsymbol{\gamma}\mathbf{n}e^{-\eta r}u_0, \quad (3)$$

where $N_i = \eta^{3/2}\pi^{-1/2}$, $\eta = aZ$, $\mathbf{n} = \mathbf{r}/r$, and $u_0 = u(\mathbf{p})_{p=0}$ is the bispinor of the electron at rest.

Using an expansion analogous to that obtained by Gordon [7]

$$\Gamma(1-i\xi)\,e^{-i\mathbf{pr}}F[i\xi,1;\,i(pr+\mathbf{pr})]$$
$$=\sum_{lm}4\pi(-i)^l\frac{\Gamma(l+1-i\xi)}{(2l+1)!}$$
$$\times\,(2pr)^l e^{-ipr}F\,[l+1+i\xi,2l+2;\,2ipr]Y_{lm}(\boldsymbol{\nu})Y^*_{lm}(\mathbf{n}),\quad(4)$$

(where $\nu = \mathbf{p}/p$, $Y_{lm}(\nu)$ is a spherical function) and the formula for the gradient of the spherical function, we obtain the following representation of the wave function of the final state

$$\bar{\psi}_f(r)=\bar{\psi}_{f\,\mathrm{I}}+\bar{\psi}_{f\,\mathrm{II}};\quad(5)$$

$$\bar{\psi}_{f\,\mathrm{I}}=\bar{u}(\mathbf{p})\sum_{lm}N^*_l(2pr)^l e^{-ipr}F_1(r)\,\mathbf{Y}_{lm}(\boldsymbol{\nu})\,Y^*_{lm}(\mathbf{n}),$$

$$\bar{\psi}_{f\,\mathrm{II}}=\bar{u}(\mathbf{p})\frac{\gamma_4}{2\varepsilon}\sum_{lm}N^*_l(2pr)^l e^{-ipr}$$
$$\times\Big\{ipF_1(r)\sqrt{\frac{l}{2l+1}}\,\boldsymbol{\gamma}\mathbf{Y}^*_{l,\,l-1,\,m}(\mathbf{n})\,Y_{lm}(\boldsymbol{\nu})$$
$$-ipF_1(r)\sqrt{\frac{l+1}{2l+1}}\,\boldsymbol{\gamma}\mathbf{Y}^*_{l,\,l+1,\,m}(\mathbf{n})\,Y_{lm}(\boldsymbol{\nu})$$
$$+\Big[\Big(\frac{l+i\xi}{r}+ip\Big)F_1(r)+\frac{l+1-i\xi}{r}F_2(r)\Big]$$
$$\times\sqrt{\frac{l}{2l+1}}\,\boldsymbol{\gamma}\mathbf{Y}_{l,\,l-1,\,m}(\boldsymbol{\nu})\,\mathbf{Y}^*_{lm}(\mathbf{n})$$
$$+\Big[\Big(\frac{l+1-i\xi}{r}-ip\Big)F_1(r)$$
$$-\frac{l+1-i\xi}{r}F_2(r)\Big]\sqrt{\frac{l+1}{2l+1}}\,\boldsymbol{\gamma}\mathbf{Y}_{l,\,l+1,\,m}(\boldsymbol{\nu})\,Y^*_{lm}(\mathbf{n})\Big\},\quad(6)$$

where

$$F_1(r)=F(l+1+i\xi;\;2l+2;\;2ipr),$$
$$F_2(r)=F(l+i\xi,\,2l+2;\,2ipr),$$
$$N^*_l=4\pi(-i)^l e^{\pi\xi/2}\frac{\Gamma(l+1-i\xi)}{(2l+1)!}.$$

$\mathbf{Y}_{jlm}(\mathbf{n})$ is a spherical vector (for definitions see [4,5]). Along with the system of spherical vectors $\mathbf{Y}_{jlm}(\mathbf{n})$, $l = j$, $j \pm 1$ we shall use also the system $\mathbf{Y}^{(\mu)}_{jm}(\mathbf{n})$, $\mu = 0, \pm 1$ (see [6]).

The second terms in the wave functions, i.e., $\psi_{i\mathrm{II}}$ and $\psi_{f\mathrm{II}}$, have a relative order αZ. In the calculation of the matrix elements $M^{(\lambda)}_{LM}$ it is therefore necessary to discard their products, which have a relative order $(\alpha Z)^2$.

CONVERSION OF MAGNETIC-TYPE RADIATION

We represent the matrix element of the conversion of a magnetic multipole in the form of a sum of three terms

$$M^{(0)}_{LM}=M_1+M_2+M_3,\qquad M_1=\int\bar{\psi}_{f\,\mathrm{I}}B^{(0)}_{LM}\psi_{i\,\mathrm{I}}d\mathbf{r},$$
$$M_2=\int\bar{\psi}_{f\,\mathrm{I}}B^{(0)}_{LM}\psi_{i\,\mathrm{II}}d\mathbf{r},\qquad M_3=\int\bar{\psi}_{f\,\mathrm{II}}B^{(0)}_{LM}\psi_{i\,\mathrm{I}}d\mathbf{r}$$

and calculate the values of M_1, M_2, and M_3, using the following expression for the potential:

$$B^{(0)}_{LM}(\mathbf{r})=-i\boldsymbol{\gamma}\mathbf{Y}^{(0)}_{LM}(\mathbf{n})\,G_L(\omega r),$$
$$G_L(\omega r)=(2\pi)^{3/2}i^L(\omega r)^{-1/2}H^{(1)}_{L+1/2}(\omega r),\quad(7)$$

where $H^{(1)}_{L+1/2}(\omega r)$ is a Hankel function of the first kind.

The integration over the angles in M_1 is readily carried out by taking into account the fact that

$$\sum_m Y_{lm}(\boldsymbol{\nu})\int Y^*_{lm}(\mathbf{n})\,\mathbf{Y}_{lLM}(\mathbf{n})\,dO_n=\mathbf{Y}_{lLM}(\boldsymbol{\nu})\delta_{lL}.\quad(8)$$

The angle integral in M_2, which is equal to

$$\int Y^*_{lm}(\mathbf{n})\,(\boldsymbol{\gamma}\mathbf{Y}_{LLM}(\mathbf{n}))(\boldsymbol{\gamma}\mathbf{n})dO_{\mathbf{n}},\quad(9)$$

reduces to (8) if we use the relations

$$\gamma_i\gamma_k=\delta_{ik}+i\varepsilon_{ikl}\Sigma_l,\qquad\Sigma=[\boldsymbol{\gamma}\boldsymbol{\gamma}]/2i,\quad(10)$$

$$-i\,[\mathbf{n}\mathbf{Y}_{L,\,L,\,M}(\mathbf{n})]=\sqrt{\frac{L+1}{2L+1}}\,\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})$$
$$+\sqrt{\frac{L}{2L+1}}\,\mathbf{Y}_{L,\,L+1,\,M}(\mathbf{n}),\quad(11)$$

where ϵ_{ikl} is a unit antisymmetrical tensor of the third rank.

In M_3 there arise angle integrals of two types:

$$\sum_m(\boldsymbol{\gamma}\mathbf{Y}_{ll\pm1,\,m}(\boldsymbol{\nu}))\int Y^*_{lm}(\mathbf{n})(\boldsymbol{\gamma}\mathbf{Y}_{LLM}(\mathbf{n}))dO_{\mathbf{n}},\quad(12)$$

$$\sum_m Y_{lm}(\boldsymbol{\nu})\int(\boldsymbol{\gamma}\mathbf{Y}^*_{l,\,l\pm1,\,m}(\mathbf{n}))\,(\boldsymbol{\gamma}\mathbf{Y}_{LLM}(\mathbf{n}))\,dO_{\mathbf{n}}.\quad(13)$$

The first can be calculated in the following manner: we express the spherical vector of the ν direction in terms of a spherical function, using the differential relations

$$\mathbf{Y}_{l,\,l-1,\,m}(\boldsymbol{\nu})=\Big\{\sqrt{\frac{l}{2l+1}}\,\boldsymbol{\nu}+\frac{p}{\sqrt{l(2l+1)}}\nabla_{\boldsymbol{\nu}}\Big\}Y_{lm}(\boldsymbol{\nu}),\quad 14)$$

$$\mathbf{Y}_{l,\,l+1,\,m}(\boldsymbol{\nu})=\Big\{-\sqrt{\frac{l+1}{2l+1}}\,\boldsymbol{\nu}+\frac{p}{\sqrt{l(2l+1)}}\nabla_{\boldsymbol{\nu}}\Big\}Y_{lm}(\boldsymbol{\nu}).\quad(14a)$$

We then take the integral in accordance (8) and following this we differentiate with respect to ν. The latter can be readily carried out if it is noted that

$$\mathrm{div}\,\mathbf{Y}_{LLM}(\boldsymbol{\nu})=0,$$
$$\mathrm{rot}\,\mathbf{Y}_{LLM}(\boldsymbol{\nu})=\frac{i}{p}\{\mathbf{Y}^{(1)}_{LM}(\boldsymbol{\nu})+\sqrt{L(L+1)}\,\mathbf{Y}^{(-1)}_{LM}(\boldsymbol{\nu})\}.\quad(15)$$

The second type of integral, (13), is calculated in analogy with the integral arising in M_2.

As a result we obtain the following expression for the matrix element:

$$M_{LM}^{(0)} = -i\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{LLM}(\boldsymbol{\nu})\,u_0 N_L^* N_i I_1(L, L)$$

$$-\frac{i\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\Sigma}\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})\,\gamma_4 u_0 N_L^* N_i \sqrt{\frac{L+1}{2L+1}}\,\Omega$$

$$-\frac{i\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\Sigma}\mathbf{Y}_{L,\,L+1,\,M}(\boldsymbol{\nu})\,\gamma_4 u_0 N_L^* N_i \sqrt{\frac{L}{2L+1}}\,\Lambda,$$

$$\Omega = -\frac{L+2+i\xi}{L-i\xi} I_1(L, L) + \frac{i(L+1)(L+i\xi)}{p(L-i\xi)} I_2(L, L)$$

$$+\frac{i(L+1)(L+1-i\xi)}{p(L-i\xi)} I_3(L, L),$$

$$\Lambda = -\frac{L-1-i\xi}{L+1+i\xi} I_1(L, L) - \frac{iL(L+1-i\xi)}{p(L+1+i\xi)} I_2(L, L)$$

$$+\frac{iL(L+1-i\xi)}{p(L+1+i\xi)} I_3(L, L), \qquad (16)$$

where

$$I_1(l, n) = \int_0^\infty (2pr)^l e^{-ipr} F[l+1+i\xi,\ 2l+2;\ 2ipr]\, G_n(\omega r)\, e^{-\eta r} r^2 dr,$$

$$I_2(l, n) = \int_0^\infty (2pr)^l e^{-ipr} F[l+1+i\xi,\ 2l+2;\ 2ipr]\, G_n(\omega r)\, e^{-\eta r} r dr,$$

$$I_3(l, n) = \int_0^\infty (2pr)^l e^{-ipr} F[l+i\xi,\ 2l+2;\ 2ipr]\, G_n(\omega r)\, e^{-\eta r} r dr, \qquad (17)$$

I are radial integrals, the evaluation of which will be considered below. In the derivation of (16) we used the relations between the confluent hypergeometric functions

$$F[a+1,\ b+2;\ x] = \frac{b(b+1)(x-b)}{ax^2} F[a, b, x] + \frac{b^2(b+1)}{ax^2} F[a-1, b;\ x],$$

$$F[a-1,\ b-2;\ x] = \frac{(a-1)(b-2+x)}{(b-2)(b-1)} F[a, b;\ x] + \frac{b-a}{b-1} F[a-1, b;\ x]. \qquad (18)$$

After summing in the usual fashion the squares of the moduli of the matrix elements over the polarizations of the electrons, we obtain the differential conversion coefficient

$$d\beta_L^{(0)} = \frac{\alpha\omega p\varepsilon}{4(2\pi)^3}\left\{\left| Q_1 - \frac{L+1}{2L+1} Q_2 - \frac{L}{2L+1} Q_3\right|^2 |\mathbf{Y}_{LLM}(\boldsymbol{\nu})|^2 + \frac{L(L+1)}{(2L+1)^2} |Q_2 - Q_3|^2 |Y_{LM}(\boldsymbol{\nu})|^2\right\} dO_\nu; \qquad (19)$$

$$Q_1 = N_L^* N_i \sqrt{\frac{\varepsilon-1}{\varepsilon}} I_1(L, L), \qquad Q_2 = N_L^* N_i \frac{i\alpha Z}{2}\sqrt{\frac{\varepsilon+1}{\varepsilon}}\,\Omega,$$

$$Q_3 = N_L^* N_i \frac{i\alpha Z}{2}\sqrt{\frac{\varepsilon+1}{\varepsilon}}\,\Lambda. \qquad (20)$$

Integration over the electron emission angles yields the total conversion coefficient of the magmetic multipole on the K-shell

$$\beta_L^{(0)} = \frac{\alpha\omega p\varepsilon}{4(2\pi)^3}\left\{\left| Q_1 - \frac{L+1}{2L+1} Q_2 - \frac{L}{L+1} Q_3\right|^2 + \frac{L(L+1)}{(2L+1)^2} |Q_2 - Q_3|^2\right\}. \qquad (21)$$

In the limiting case of small Z, i.e., in the Born approximation ($\xi \ll 1$), the values of Q_2 and Q_3 can be neglected in comparison with Q_1. In this case

$$(\beta_L^{(0)})_B = \frac{\alpha\omega p\varepsilon}{4(2\pi)^3} |q|^2,$$

$$q = \lim_{\xi\to 0} Q_1 = 4\pi(-i)^L \frac{L!}{(2L+1)!} (\alpha Z)^{3/2}\pi^{-1/2} \sqrt{\frac{\varepsilon-1}{\varepsilon}} \int_0^\infty (2pr)^L e^{-ipr} \times F[L+1, 2L+2;\ 2ipr]\, G_L(\omega r) r^2 dr. \qquad (22)$$

Noting that

$$F[L+1,\ 2L+2;\ 2ipr] = \frac{(2L+1)!}{L!} (2pr)^{-L} e^{ipr} \sqrt{\frac{\pi}{2pr}} J_{L+1/2}(pr), \qquad (23)$$

$$\int_0^\infty J_{L+1/2}(pr)\, H_{L+1/2}^{(1)}(\omega r)\, r dr = \frac{2}{\pi i}\left(\frac{p}{\omega}\right)^{L+1/2} \frac{1}{p^2-\omega^2}, \qquad (24)$$

where $J_{L+1/2}(pr)$ are Bessel functions, we obtain the well known formula [8] for the conversion coefficient in the Born approximation

$$(\beta_L^{(0)})_B = 2\alpha(\alpha Z)^3 \frac{1}{\omega}\left(1+\frac{2}{\omega}\right)^{L+1/2}. \qquad (25)$$

CONVERSION OF ELECTRIC TYPE OF RADIATION

The potential of the electric multipole radiation $B_{LM}^{(1)}(\mathbf{r})$ is given by the expression

$$B_{LM}^{(1)}(\mathbf{r}) = \sqrt{\frac{L}{L+1}}\gamma_4 Y_{LM}(\mathbf{n})\, G_L(\omega r) - i\sqrt{\frac{2L+1}{L+1}}\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})\, G_{L-1}(\omega r). \qquad (26)$$

The right half of (26) consists of two terms, the first corresponding to a scalar potential and the second to a vector potential. Accordingly, we represent the matrix element of the conversion also in the form of two parts:

$$M_{LM}^{(1)} = -\sqrt{\frac{L}{L+1}}\,S - \sqrt{\frac{2L+1}{L+1}}\,V,$$

$$S = \int \bar{\psi}_f \gamma_4 Y_{LM}(\mathbf{n})\, G_L(\omega r)\, \psi_i d\mathbf{r},$$

$$V = i\int \bar{\psi}_f \boldsymbol{\gamma} \mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})\, G_{L-1}(\omega r)\, \psi_i\, d\mathbf{r}.$$

Let us calculate the values of S and V, retaining the principal terms, terms of order αZ, and neglecting terms of order $(\alpha Z)^2$. Proceeding in analogy with the case of the magnetic multipole, we write

$$S = S_1 + S_2 + S_3, \qquad S_1 = \int \bar{\psi}_{fI}\gamma_4 Y_{LM}(\mathbf{n})\, G_L(\omega r)\, \psi_{iI}\, d\mathbf{r},$$

$$S_2 = \int \bar{\psi}_{fI}\gamma_4\, Y_{LM}(\mathbf{n})\, G_L(\omega r)\, \psi_{iII}\, d\mathbf{r},$$

$$S_3 = \int \bar{\psi}_{fII}\gamma_4 Y_{LM}(\mathbf{n})\, G_L(\omega r)\, \psi_{iI} d\mathbf{r}.$$

Integrating S_1 over the angles, we obtain

$$S_1 = \bar{u}(\mathbf{p})\,\gamma_4\, u_0 N_L^* N_i Y_{LM}(\boldsymbol{\nu})\, I_1(L, L). \tag{27}$$

To calculate S_2 we note that

$$\sum_{lm} \varphi(l)\, Y_{lm}(\boldsymbol{\nu}) \int Y_{lm}^*(\mathbf{n})\, Y_{LM}(\mathbf{n})\, \mathbf{n}\, dO_{\mathbf{n}}$$

$$= \varphi(L-1)\sqrt{\frac{L}{2L+1}}\,\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})$$

$$-\varphi(L+1)\sqrt{\frac{L+1}{2L+1}}\,\mathbf{Y}_{L,\,L+1,\,M}(\boldsymbol{\nu}). \tag{28}$$

Therefore

$$S_2 = -\frac{\alpha Z}{2}\bar{u}(\mathbf{p})\Big\{N_{L-1}^*\sqrt{\frac{L}{2L+1}}\, I_1(L-1, L)\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})$$

$$- N_{L+1}^*\sqrt{\frac{L+1}{2L+1}}\, I_1(L+1, L)\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L+1,\,M}(\boldsymbol{\nu})\Big\} N_i u_0. \tag{29}$$

If we recognize that

$$N_{L-1}^* = i\frac{2L(2L+1)}{L-i\xi} N_L^*, \qquad N_{L+1}^* = (-i)\frac{L+1-i\xi}{(2L+1)(2L+3)} N_L^* \tag{30}$$

and use the recurrence relations (18), we get

$$S_2 = -\frac{\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})\, u_0 N_L^* N_i\sqrt{\frac{L}{2L+1}}$$

$$\times\Big[-\frac{L+i\xi}{L-i\xi} I_1(L, L) + \frac{iL(L+i\xi)}{p(L-i\xi)} I_2(L, L)$$

$$+\frac{iL(L+1-i\xi)}{p(L-i\xi)} I_3(L, L)\Big]$$

$$-\frac{\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L+1,\,M}(\boldsymbol{\nu})\, u_0 N_L^* N_i$$

$$\times\sqrt{\frac{L+1}{2L+1}}\Big[\frac{L+1-i\xi}{L+1+i\xi} I_1(L, L)$$

$$+\frac{i(L+1)(L+1-i\xi)}{p(L+1+i\xi)} I_2(L, L)$$

$$-\frac{i(L+1)(L+1-i\xi)}{p(L+1+i\xi)} I_3(L, L)\Big]. \tag{31}$$

The quantity S_3 is calculated in analogy with S_2. Using relations (18) and (30), we obtain the following result

$$S_3 = -\frac{\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})\, u_0 N_L^* N_i\sqrt{\frac{L}{2L+1}}$$

$$\times\Big[\frac{2}{L-i\xi} I_1(L, L) - \frac{i(L+i\xi)}{p(L-i\xi)} I_2(L, L)$$

$$-\frac{i(L+1-i\xi)}{p(L-i\xi)} I_3(L, L)\Big]$$

$$-\frac{\alpha Z}{2}\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L+1,\,M}(\boldsymbol{\nu})\, u_0 N_L^* N_i\sqrt{\frac{L+1}{2L+1}}$$

$$\times\Big[\frac{2}{L+1+i\xi} I_1(L, L) + \frac{i(L+1-i\xi)}{p(L+1+i\xi)} I_2(L, L)$$

$$-\frac{i(L+1-i\xi)}{p(L+1+i\xi)} I_3(L, L)\Big]. \tag{32}$$

To calculate the vector part of the matrix element V, we write it also in the form of three terms

$$V = V_1 + V_2 + V_3,$$

$$V_1 = i\int \bar{\psi}_{fI}\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})\, G_{L-1}(\omega r)\, \psi_{iI}\, d\mathbf{r},$$

$$V_2 = i\int \bar{\psi}_{fI}\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})\, G_{L-1}(\omega r)\, \psi_{iII}\, d\mathbf{r},$$

$$V_3 = i\int \bar{\psi}_{fII}\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n})\, G_{L-1}(\omega r)\, \psi_{iI}\, d\mathbf{r}.$$

The integration over the angles in V_1 leads to the expression

$$V_1 = i\bar{u}(\mathbf{p})\,\boldsymbol{\gamma}\mathbf{Y}_{L,\,L-1,\,M}(\boldsymbol{\nu})\, u_0 N_{L-1}^* N_i I_1(L-1, L-1). \tag{33}$$

The angle integration V_2 is also readily carried out, if we use the representation of the vector $\mathbf{Y}_{L,L-1,M}(\mathbf{n})$ with the aid of the transverse vector $\mathbf{Y}_{LM}^{(1)}(\mathbf{n})$ and the longitudinal vector $\mathbf{Y}_{LM}^{(-1)}(\mathbf{n})$:

$$\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n}) = \sqrt{\frac{L+1}{2L+1}}\,\mathbf{Y}_{LM}^{(1)}(\mathbf{n}) + \sqrt{\frac{L}{2L+1}}\,\mathbf{Y}_{LM}^{(-1)}(\mathbf{n}) \tag{34}$$

and the properties of the $\boldsymbol{\gamma}$ matrices given by formula (10). The result is

$$V_2 = \frac{i\alpha Z}{2}\bar{u}(\mathbf{p})\Big\{\sqrt{\frac{L+1}{2L+1}}\,\boldsymbol{\Sigma}\mathbf{Y}_{LM}^{(0)}(\boldsymbol{\nu})$$

$$+\sqrt{\frac{L}{2L+1}}\, Y_{LM}(\boldsymbol{\nu})\Big\}\gamma_4 u_0 N_{L-1}^* N_i$$

$$\times\Big[-\frac{L-i\xi}{L+i\xi} I_1(L-1, L-1)$$

$$-\frac{iL(L-i\xi)}{p(L+i\xi)} I_2(L-1, L-1)$$

$$+\frac{iL(L-i\xi)}{p(L+i\xi)} I_3(L-1, L-1)\Big]. \tag{35}$$

In the expression for V_3 there are three angle integrals:

$$\sum_{lm}' Y_{lm}(\mathbf{v}) \int (\gamma \mathbf{Y}_{lm}^{(-1)*}(\mathbf{n}))(\gamma \mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n}))\, dO_{\mathbf{n}}, \quad (36)$$

$$\sum_{lm}' (\gamma \mathbf{Y}_{l,\,l\pm1,\,m}(\mathbf{v})) \int Y_{lm}^{*}(\mathbf{n})(\gamma \mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n}))\, dO_{\mathbf{n}}. \quad (37)$$

The first is perfectly analogous to the integral in the expression for V_2. The calculation of the two others is carried out by the same procedure as the angle integral (12). As a result we obtain the expression

$$V_3 = \frac{i\alpha Z}{2}\bar{u}(\mathbf{p})\left\{\sqrt{\frac{L}{2L+1}}Y_{LM}(\mathbf{v}) - \sqrt{\frac{L+1}{2L+1}}\Sigma\mathbf{Y}_{LM}^{(0)}(\mathbf{v})\right\}\gamma_4 u_0 N_{L-1}^{*}N_i$$
$$\times\left[-\frac{2}{L+i\xi}I_1(L-1,L-1) - \frac{i(L-i\xi)}{p(L+i\xi)}I_2(L-1,L-1) + \frac{i(L-i\xi)}{p(L+i\xi)}I_3(L-1,L-1)\right]. \quad (38)$$

Gathering together the obtained values of S and V, we obtain the matrix element for the conversion of the electric multipole

$$M_{LM}^{(1)} = -\bar{u}(\mathbf{p})\left\{\gamma_4 Y_{LM}(\mathbf{v})\sqrt{\frac{L}{L+1}}(R_1+P_2)\sqrt{\frac{\varepsilon}{\varepsilon+1}} + \gamma_4\Sigma\mathbf{Y}_{LM}^{(0)}(\mathbf{v})P_3\sqrt{\frac{\varepsilon}{\varepsilon+1}}\right.$$
$$+\gamma\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{v})\sqrt{\frac{2L+1}{L+1}}\left(\frac{L}{2L+1}R_2+P_1\right)$$
$$\left.\times\sqrt{\frac{\varepsilon}{\varepsilon-1}} + \gamma\mathbf{Y}_{L,\,L+1,\,M}(\mathbf{v})\sqrt{\frac{L}{2L+1}}R_3\sqrt{\frac{\varepsilon}{\varepsilon-1}}\right\}u_0, \quad (39)$$

where the following notation is introduced:

$$R_1 = N_L^{*}N_i\sqrt{\frac{\varepsilon+1}{\varepsilon}}I_1(L,L),$$

$$R_2 = -\frac{\alpha Z}{2}N_L^{*}N_i\sqrt{\frac{\varepsilon-1}{\varepsilon}}\left\{-\frac{L-2+i\xi}{L-i\xi}I_1(L,L) + \frac{i(L-1)(L+i\xi)}{p(L-i\xi)}I_2(L,L) + \frac{i(L-1)(L+1-i\xi)}{p(L-i\xi)}I_3(L,L)\right\},$$

$$R_3 = -\frac{\alpha Z}{2}N_L^{*}N_i\sqrt{\frac{\varepsilon-1}{\varepsilon}}\left\{\frac{L+3-i\xi}{L+1+i\xi}I_1(L,L) + \frac{i(L+2)(L+1-i\xi)}{p(L+1+i\xi)}[I_2(L,L)-I_3(L,L)]\right\}; \quad (40)$$

$$P_1 = iN_{L-1}^{*}N_i\sqrt{\frac{\varepsilon-1}{\varepsilon}}I_1(L-1,L-1),$$

$$P_2 = \frac{i\alpha Z}{2}N_{L-1}^{*}N_i\sqrt{\frac{\varepsilon+1}{\varepsilon}}\left\{-\frac{L+2-i\xi}{L+i\xi}I_1(L-1,L-1) - \frac{i(L+1)(L-i\xi)}{p(L+i\xi)}[I_2(L-1,L-1) - I_3(L-1,L-1)]\right\},$$

$$P_3 = \frac{i\alpha Z}{2}N_{L-1}^{*}N_i\sqrt{\frac{\varepsilon+1}{\varepsilon}}\left\{-\frac{L-2-i\xi}{L+i\xi}I_1(L-1,L-1) - \frac{i(L-1)(L-i\xi)}{p(L+i\xi)}[I_2(L-1,L-1) - I_3(L-1,L-1)]\right\}. \quad (40a)$$

The radial integrals $I(l,n)$ are defined by formulas (17).

Substituting expression (39) in (2) and summing over the electron polarizations, we obtain the differential conversion coefficient

$$d\beta_L^{(1)} = \frac{\alpha\omega\varepsilon p}{4(2\pi)^3}\left\{\left|\frac{L}{2L+1}(R_2+R_3)+P_1-iP_3\right|^2|\mathbf{Y}_{LM}^{(0)}(\mathbf{v})|^2 + \frac{L}{L+1}\left|R_1 - i\frac{L}{2L+1}R_2 + i\frac{L+1}{2L+1}R_3 - iP_1 + P_2\right|^2|Y_{LM}(\mathbf{v})|^2\right\}dO_{\mathbf{v}}. \quad (41)$$

Integrating over the directions of electron emission, we obtain the conversion coefficient of the electron multipole on the K-shell (for two electrons)

$$\beta_L^{(1)} = \frac{\alpha\omega p\varepsilon}{4(2\pi)^3}\left\{\left|\frac{L}{2L+1}(R_2+R_3)-P_1+iP_3\right|^2 + \frac{L}{L+1}\left|R_1 - i\frac{L}{2L+1}R_2 + i\frac{L+1}{2L+1}R_3 + iP_1 - P_2\right|^2\right\}. \quad (42)$$

In the limiting case of small Z and $\xi \ll 1$ (Born approximation), we can neglect the values R_2, R_3 and P_2, P_3 compared with R_1 and P_1. In this case

$$(\beta_L^{(1)})_B = \frac{\alpha\omega p\varepsilon}{4(2\pi)^3}\left\{|\mathcal{P}|^2 + \frac{L}{L+1}|\mathcal{R}-i\mathcal{P}|^2\right\};$$

$$\mathcal{P} = \lim_{\xi\to0}P_1 = i4\pi(-i)^{L-1}\frac{\Gamma(L)}{(2L-1)!}(\alpha Z)^{3/2}\pi^{-1/2}\sqrt{\frac{\varepsilon-1}{\varepsilon}} \times\int_0^\infty(2pr)^{L-1}e^{-ipr}F[L,2L;2ipr]\,G_{L-1}(\omega r)\,r^2\,dr,$$

$$\mathcal{R} = \lim_{\xi\to0}R_1 = 4\pi(-i)^L\frac{\Gamma(L+1)}{(2L+1)!}(\alpha Z)^{3/2}\pi^{-1/2}\sqrt{\frac{\varepsilon+1}{\varepsilon}} \times\int_0^\infty(2pr)^L e^{-ipr} \times F[L+1,2L+2;2ipr]\,G_L(\omega r)\,r^2\,dr. \quad (43)$$

Taking formulas (23) and (24) into consideration, we obtain

$$\mathcal{P} = 4\,(2\pi)^{3/2}\,\frac{(\alpha Z)^{3/2}}{\omega\sqrt{2p\varepsilon}}\left(\sqrt{1+\frac{2}{\omega}}\right)^{L-1/2},$$

$$\mathcal{R} = -\,i4\,(2\pi)^{3/2}\,\frac{(\alpha Z)^{3/2}}{\omega\sqrt{2p\varepsilon}}\sqrt{\frac{\varepsilon+1}{\varepsilon-1}}\left(\sqrt{1+\frac{2}{\omega}}\right)^{L+1/2}.$$

Substituting these expression in (43), we obtain the conversion coefficient of the electric multipole in the Born approximation:

$$(\beta_L^{(1)})_{\mathrm{B}} = 2\alpha\,(\alpha Z)^3\,\frac{1}{\omega}\left(1+\frac{2}{\omega}\right)^{L-1/2}\frac{(L+1)\,\omega^2+4L\,(\omega+1)^2}{(L+1)\,\omega^2}. \tag{44}$$

When Z = 0 this formula agrees with the well known formula obtained in the Born approximation by Dancoff and Morrison [8].

We note that in the case of magnetic multipole radiation, both functions lead to the same result.

CALCULATION OF THE RADIAL INTEGRALS

The general formulas (21) and (42) contain the quantities Q, R, and P, which are determined in accordance with (20), (40), and (40a) through the radial integrals I_1, I_2, and I_3. These radial integrals can be expressed in terms of the hypergeometric functions F [a, b; c; x]. To this end, let us use the well known representation of the Hankel function of half-integer order

$$G_L(\omega r) = (2\pi)^{3/2}\,i^L\,\frac{H^{(1)}_{L+1/2}(\omega r)}{\sqrt{\omega r}} = 8\pi\sum_{k=0}^{L}\frac{(-1)^k\,(L+k)!\,e^{i\omega r}}{k!\,(L-k)!\,(2i\omega r)^{k+1}}. \tag{45}$$

Substituting this expression, say, in I_1(L, L) yields

$$I_1\,(L, L) = 8\pi\sum_{k=0}^{L}\frac{(-1)^k\,(L+k)!\,(2p)^L}{k!\,(L-k)!\,(2i\omega)^{k+1}}\,J_k,$$

$$J_k = \int_0^\infty e^{-[i(p-\omega)+\eta]r}\,r^{L-k+1}\,F\,[L+1+i\xi;\ 2L+2;\ 2ipr]\,dr$$

$$= (L-k+1)!\,[i\,(p-\omega)+\eta]^{k-L-2}\,F\,[L+1+i\xi, L-k+2;\ 2L+2;\ 2ip/(\eta+i\,(p-\omega))]. \tag{46}$$

The last equation follows from formula

$$\int_0^\infty e^{-\lambda r}\,r^n F\,[a, c; \varkappa r]\,dr$$

$$= \Gamma\,(n+1)\,\lambda^{-n-1}F\left[a, n+1; c; \frac{\varkappa}{\lambda}\right],\quad \mathrm{Re}\,\lambda>0,\ n>-1. \tag{47}$$

We analogously calculate the integrals I_2(L, L) and I_3(L, L).

The most distinguishing feature of the hypergeometric functions contained in the final result is that the second and third parameters of these functions are integers. In this case the hypergeometric series can be converted into a polynomial, an important consideration in numerical evaluation. Such a conversion can be readily carried out, for example, with the aid of the well known relation

$$F\,[a, b; c; x] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,x^{b}$$

$$\times F\left[b, b+1-c;\ a+b+1-c;\ \frac{x-1}{x}\right]$$

$$+\,\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-x)^{c-a-b}\,x^{b-c}$$

$$\times F\left[1-b, c-b;\ c+1-a-b;\ \frac{x-1}{x}\right]. \tag{48}$$

In all the expressions encountered in the present work, c and b in formula (48) represent positive integers, with c > b. Consequently, the quantities b + 1 − c and 1 − b are either equal to zero or to a negative integer, and the series for the hypergeometric functions of the right half of (48) end after a finite number of terms. It must be noted that this peculiarity in the series of the present work is due to the character of the approximations in the wave functions. Calculations with exact wave functions do not lead to termination of the series.

If we calculate with the aid of formula (21) the internal-conversion coefficient of magnetic quadrupole radiation for the specific case Z = 40 and $\omega = 1$, we obtain $(\beta_2^{(0)})_{\mathrm{FSM}} = 7.5\times10^{-3}$. In the Born approximation we obtain for this case $(\beta_2^{(0)})_{\mathrm{B}} = 5.6\times10^{-3}$. According to the tables of Sliv and Band [9] we have in this case $\beta_2^{(0)} = 8.7\times10^{-3}$.

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