DYNAMIC ELASTICITY MODULI, ROTATION OF THE PLANE OF POLARIZATION OF ELASTIC WAVES, AND COUPLED LONGITUDINAL-TRANSVERSE WAVES IN MAGNETO-POLARIZED METALS

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The equations of motion determining the features of the absorption and propagation of elastic waves in magneto-polarized metals are derived in the free-electron model for a broad range of frequencies and constant magnetic-field intensities and for arbitrary directions of propagation and magnetic field orientation. Explicit expressions for the components of the dynamic modulus of elasticity involved in the equations of motion are derived for the case of strong magnetic fields. The expressions are given in terms of the atomic constants, the electron mean free path, the ultrasound frequency and magnetic field intensity. Formulas for the specific rotation of the plane of polarization, the axial ratio of the ellipse, and the components of the dynamic elasticity modulus are obtained in a broad range of frequencies and fields in terms of the cyclotron frequency Ω, the frequency and wave vector $\mathbf{k}$ of the ultrasound, the velocity v and the mean free path l of the conduction electrons on the Fermi surface. A monotonic increase of the specific rotation with increasing kl up to saturation and oscillations of the coupling coefficient with a change of kv_0/Ω due to geometric resonance are predicted.

In the theoretical consideration of problems connected with the propagation of ultrasound in magneto-polarized media interest is commonly confined to the absorption features of ultrasound. However, along with the absorption features of ultrasound in magneto-polarized media one can also expect the presence of new as yet experimentally unobserved phenomena, such as the rotation of the plane of polarization of the elastic waves [1-5] or the presence of coupled longitudinal-transverse and transverse-longitudinal waves. [2]

Phenomenologically these phenomena can be described by introducing into the discussion the tensor of the dynamic modulus of elasticity. [6,7] In this paper an attempt is made to construct a microscopic quasiclassical theory of these moduli for the case of metals placed in a constant magnetic field $\mathbf{H}_0$.

1. Let us consider the conditions of propagation of a plane elastic wave $u = u_0 \exp i(\omega t - \mathbf{k}\cdot\mathbf{r})$ in a metal. To explain the main features of the expected effects and their dependence on the magnetic field intensity and on the ultrasound frequency, and to estimate approximately the size of the proposed effects, we confine ourselves for the time being to the case of free electrons.

We describe the motion of the electrons by means of Boltzmann's kinetic equation and accept Pippard's postulate [8] that the electron distribution function relaxes to the equilibrium Fermi distribution function $f_0(\mathscr{E}) = \{\exp[(\mathscr{E} - \mu)/kT] + 1\}^{-1}$ in the coordinate system moving along with the volume element of the medium. It is therefore convenient to carry out the whole calculation in a coordinate system fixed in the medium, i.e., moving and being deformed together with the medium.

The kinetic equation in the relaxation-time approximation [1)] will in this system of coordinates be of the form

$$\partial f/\partial t + \mathbf{v}\partial f/\partial \mathbf{r} + \dot{\mathbf{p}}\partial f/\partial \mathbf{p} = -\nu\,(f - f_0(\mathscr{E})), \quad (1)$$

where $\mathscr{E}$ is the energy of the electron:

$$\mathscr{E} = \mathscr{E}_0(\mathbf{p}) - v_i p_j \varepsilon_{ij} - m v_i \dot{u}_i, \quad (2)$$

$$\dot{\mathbf{p}} = -e\,(\mathbf{E} + c^{-1}[\dot{\mathbf{u}}\mathbf{H}_0]) - (e/c)\,[\mathbf{v}\mathbf{H}_0] - \nabla_{\mathbf{r}}\mathscr{E}, \quad (3)^*$$

$\mathbf{u}$ are the displacements, and ϵ_{ij} is the strain tensor of the medium. $\mathbf{E}$ is the intensity of the alternating electric field appearing upon propagation of ultrasound, ν is the relaxation frequency inversely proportional to the relaxation time τ,

[1)]The possibility of introducing the relaxation time has been discussed, for instance, by Gurevich.[9]

*$[\mathbf{v}\mathbf{H}_0] = \mathbf{v} \times \mathbf{H}_0$.

and $\mathscr{E}_0(\mathbf{p})$ is the electron energy in the undeformed lattice.

The solution of Eq. (1) is sought in the form

$$f = f_0(\mathscr{E}) - \chi \partial f_0 / \partial \mathscr{E}. \quad (4)$$

The dependence of the chemical potential μ on the deformations is obtained from the condition that the metal is electrically neutral at equilibrium. This means that in the coordinate system at rest with respect to the medium the electron densities in the undeformed and quasistatically deformed states must be equal. With account of (2) this yields

$$\mu = \mu_0 + \delta_{ij} N \varepsilon_{ij} / \langle 1 \rangle, \quad (5)$$

where we have introduced the abbreviated notation

$$\langle x \rangle = -\frac{2}{h^3} \int x \frac{\partial f_0}{\partial \mathscr{E}} d\mathbf{p}, \quad (6)$$

and N is the equilibrium electron density in the undeformed state.

Following [9], we go over in Eq. (1) to the variables τ^*, p_H, and $\mathscr{E}$ where

$$\tau^* = \Omega t, \qquad \Omega = eH_0/mc. \quad (7)$$

The linearized kinetic equation is obtained in the form

$$(i\omega - i\mathbf{kv} + \nu)\chi + \Omega \partial\chi / \partial\tau^* = Q; \quad (8)$$

$$Q = -(v_i p_j + \delta_{ij} N / \langle 1 \rangle)\dot{\varepsilon}_{ij} - e v_i E_i^*, \quad (9)$$

$$E_i^* = E_i + c^{-1}[\dot{\mathbf{u}}\mathbf{H}_0]_i + e^{-1}(\nabla\mu)_i + (m/e)\ddot{u}_i. \quad (10)$$

To find the intensity of the alternating electric field, we use the equations of electrodynamics

$$\text{rot } \mathbf{E} = -c^{-1}\partial\mathbf{H}/\partial t, \qquad \text{rot } \mathbf{H} = (4\pi/c)\,\mathbf{j} \quad (11)^*$$

and the condition of neutrality of the medium, which for equilibrium processes takes on the form

$$\langle \chi \rangle = 0. \quad (12)$$

In the deformed system of coordinates the current density $\mathbf{j}$ is equal to the electron current:

$$\mathbf{j} = e\langle \mathbf{v}\chi \rangle. \quad (13)$$

The ions will be considered in the continuous-medium approximation. The equation of motion of the ions is of the form

$$\rho\ddot{u}_i = \frac{\partial}{\partial x_m}\lambda^0_{imjl}\varepsilon_{jl} + \rho^{(i)}\left(E_i + \frac{1}{c}[\dot{\mathbf{u}}\mathbf{H}_0]_i\right) - \frac{m}{e\tau} j_i, \quad (14)$$

where ρ and $\rho^{(i)}$ are respectively the mass and charge density of the ions. λ^0_{imjl} is the unrenormalized tensor of the elastic moduli. The second term in (14) represents the Lorentz force exerted on the ions by the electromagnetic field produced by the elastic wave and by the external magnetic field. The third term represents the Stuart-Tolman reaction force exerted on the lattice. [10]

A consistent solution of Eqs. (8), (11), (12), and (14) yields the solution of the stated problem. Excluding from these equations all quantities except the variables of the lattice vibrations, we obtain an equation of motion of the form

$$\rho\ddot{u}_i = -k_m k_j \lambda_{imjl} u_l, \quad (15)$$

where λ_{imjl} is the tensor of the dynamic elasticity moduli.

After integration by parts with respect to τ^* the right-hand side of (15) takes on the form

$$\begin{aligned}\lambda_{imjl}k_m k_j u_l = {} & (\lambda^0_{imjl} - \delta_{im}\delta_{jl}N^2/\langle 1 \rangle)\,k_m k_j u_l - (e/c)\,N\,[\dot{\mathbf{u}}\mathbf{H}_0]_i \\ & - iN^2\delta_{im}\delta_{jl}k_m k_j \dot{u}_l \langle \tilde{1} \rangle / \langle 1 \rangle \langle \mathbf{k}\tilde{\mathbf{v}} \rangle - \nu N k_i \dot{u}_l \langle \tilde{p}_l \rangle / \langle \mathbf{k}\tilde{\mathbf{v}} \rangle \\ & + e^2 N^2 \sigma_0^{-1} B_{\alpha\beta}\,[\delta_{i\alpha}(1 + \sigma_0/\sigma) - k_i \langle \tilde{v}_\alpha^\perp \rangle / \langle \mathbf{k}\tilde{\mathbf{v}} \rangle]\,[\delta_{\beta l} \\ & + i(\langle \tilde{1} v_\beta^\perp \rangle - \langle \tilde{1} \rangle \langle \mathbf{k}\tilde{\mathbf{v}} v_\beta^\perp \rangle / \langle \mathbf{k}\tilde{\mathbf{v}} \rangle)\,\delta_{jl}k_j / \langle 1 \rangle \\ & + \nu N^{-1}(\langle v_\beta^\perp \tilde{p}_l \rangle - \langle \mathbf{k}\tilde{\mathbf{v}} v_\beta^\perp \rangle \langle \tilde{p}_l \rangle / \langle \mathbf{k}\tilde{\mathbf{v}} \rangle)]\,\dot{u}_l,\end{aligned} \quad (16)$$

where we have introduced the notation

$$B^{-1}_{\alpha\beta} = \delta_{\alpha\beta} - e^2\sigma_0^{-1}(\langle \tilde{v}_\alpha^\perp v_\beta^\perp \rangle - \langle \tilde{v}_\alpha^\perp \rangle \langle \mathbf{k}\tilde{\mathbf{v}} v_\beta^\perp \rangle / \langle \mathbf{k}\tilde{\mathbf{v}} \rangle), \quad (17)$$

$$\sigma_0 = -ic^2k^2/4\pi\omega, \qquad \sigma = Ne^2\tau/m. \quad (18)$$

The operator $\sim$ is defined as follows:

$$\tilde{x} = \Omega^{-1}\int_{-\infty}^{\tau^*} d\tau_1 x(\tau_1)\exp\left[\int_{\tau^*}^{\tau_1}\Omega^{-1}(\nu + i\omega - i\mathbf{kv}(\tau'))\,d\tau'\right]. \quad (19)$$

The superscript $\perp$ indicates that the projection of the corresponding vector must be taken on the plane perpendicular to the wave vector $\mathbf{k}$ of the elastic wave, and the components perpendicular to the wave vector are summed over α and β.

We note that account of the Stuart-Tolman effect in (14) leads to the appearance of a term σ_0/σ in the factor $1 + \sigma_0/\sigma$ in Eq. (16) and in the following equations.

Determining the dependence of ν on τ^* from the equation of motion of an electron in a magnetic field, introducing spherical coordinates, aligning the z axis with $\mathbf{H}_0$, and making use of the relation

$$e^{iz\sin\psi} = \sum_{n=-\infty}^{\infty} J_n(z)\,e^{in\psi}, \quad (20)$$

where $J_n(z)$ is the n-th order Bessel function, we obtain

$$\langle \tilde{v}_j \rangle = \frac{3\sigma}{2\Omega\tau v_0 e^2}\sum_{k,m,n} e^{i3\pi(k-n)/2}\int_0^\pi \frac{\sin\vartheta\,d\vartheta}{-\gamma + iy_3 + i(n-m)}$$

$$\times J_k(y_1)\,J_{k+m-n}(y_2)$$

*rot = curl.

$$\times \{\cos \vartheta J_n(y_1) J_m(y_2) \delta_{j3} + i \sin \vartheta [J_n'(y_1) J_m(y_2) \delta_{j2} - J_n(y_1) J_m'(y_2) \delta_{j1}]\}, \quad (21)$$

$$\sigma_{ij} = -e^2 \langle v_i \widetilde{v_j} \rangle = \frac{3\sigma}{2\Omega\tau} \sum_{k,m,n} e^{i3\pi(k-n)/2} \int_0^\pi \frac{\sin\vartheta\, d\vartheta}{-\gamma + iy_3 + i(n-m)}$$
$$\times \{-\cos\vartheta J_n(y_1) J_m(y_2) \delta_{i3} + i \sin\vartheta [J_n(y_1) J_m'(y_2) \delta_{i1} - J_n'(y_1) J_m(y_2) \delta_{i2}]\} \{\cos\vartheta J_k(y_1) J_{k+m-n}(y_2) \delta_{i3}$$
$$+ i \sin\vartheta [J_k(y_1) J_{k+m-n}'(y_2) \delta_{i1} - J_k'(y_1) J_{k+m-n}(y_2) \delta_{i2}]\}, \quad (22)$$

$$\langle v_i \widetilde{1} \rangle = -i\sigma_{ij} k_j / e^2 (\nu + i\omega),$$
$$\langle \widetilde{1} \rangle = (\langle 1 \rangle + i\mathbf{k} \langle \widetilde{\mathbf{v}} \rangle)/(\nu + i\omega), \quad (23)$$

where

$$y_1 = X_1 \sin\vartheta, \quad y_2 = X_2 \sin\vartheta, \quad y_3 = X_3 \cos\vartheta,$$
$$X_i = k_i v_0/\Omega, \quad \gamma = (\nu + i\omega)/\Omega, \quad (24)$$

$J_n'(y) = \partial J_n/\partial y$; v_0 is the electron velocity on the Fermi surface.

The equation of motion (15), together with relations (16) and (21)—(23), determines completely the features of the propagation and absorption of ultrasound resulting from the interaction of the elastic (ultrasound) waves with the conduction electrons in magneto-polarized media within a rather broad range of frequencies and magnetic fields for the case when the conduction electrons can be considered free.

In view of the fact that in writing down the equations of electrodynamics we have neglected the displacement currents, the frequencies of the ultrasound must be lower than the plasma oscillation frequencies of the electrons. The magnetic fields must also not exceed values for which it becomes already essential to take into account quantization in the magnetic field.

2. The number of independent components of the quasistatic tensor of the elasticity moduli is determined by the symmetry of the medium. The presence of magnetic polarization should lead to a decrease in the symmetry of the system and to an increase in the number of independent components of the tensor of the elasticity moduli. The presence of spatial dispersion (i.e., the dependence of the dynamic tensor of the elasticity moduli on the orientation and magnitude of the wave vector) should also lead to a decrease in the symmetry of the system. However, the situation is simplified considerably if we consider the range of frequencies and constant magnetic field intensities in which the anisotropy due to the magnetic field considerably exceeds the anisotropy due to the spatial dispersion.

It is to be expected that the magnitudes of the above-mentioned anisotropies are determined respectively by Ω/v_0 (or $\Omega\tau$) and k (or kl) where l is the mean free path of the electron on the Fermi surface. Thus, in the region of "strong" magnetic fields satisfying the condition

$$X_i \ll 1, \quad (25)$$

the number of independent components of the dynamic tensor of the elasticity moduli should be determined solely by the symmetry of the medium and of the magnetic field. Condition (25) means that the wavelength of the ultrasound should exceed considerably the radius of the cyclotron orbit of the electron.

For frequencies when

$$\sigma_0 \ll \sigma, \text{ i.e. } k^2 \ll k_0^2 = 4i\pi\omega\sigma/c^2, \quad (26)$$

Eq. (16) can be considerably simplified and reduced to the relation

$$\lambda_{imjl} k_m k_j u_l = (\lambda'_{imjl} + i\lambda''_{imjl}) k_m k_j u_l; \quad (27)$$

$$\lambda'_{imjl} = \lambda^0_{imjl} + \tfrac{1}{3} p_0 v_0 N \delta_{im} \delta_{jl}, \quad (28)$$

$$\lambda''_{imjl} k_m k_j u_l = -N\omega \left(\tfrac{1}{3} p_0 l \delta_{im} \delta_{jl} k_m k_j + \nu m \delta_{il} - N e^2 \sigma_{il}^{-1} + (e/c)\, \varepsilon_{ilm} H_{0m}\right) u_l, \quad (29)$$

where ϵ_{ilm} is the absolutely antisymmetrical unit tensor of third-rank. λ'_{imjl} represents the renormalized components of the elasticity moduli. Their number is determined by the symmetry of the medium.

Calculating the tensor of the dynamic electrical conductivity σ_{il} assuming (25) and $\omega\tau \ll 1$ for different orientation of the wave vector k, and substituting it in (29), we obtain a system of equations for the determination of the components of the tensor λ''_{imjl} from which we find

$$\lambda''_{1111} = \lambda''_{2222} = \frac{1 + (\Omega\tau)^2}{1 + 4(\Omega\tau)^2} \lambda''_{1111}(0), \quad \lambda''_{3333} = \lambda''_{3333}(0),$$
$$\lambda''_{1122} = \frac{1 - 2(\Omega\tau)^2}{1 + 4(\Omega\tau)^2} \lambda''_{1122}(0), \quad \lambda''_{1133} = \lambda''_{1133}(0),$$
$$\lambda''_{1212} = \frac{1}{1 + 4(\Omega\tau)^2} \lambda''_{1212}(0), \quad \lambda''_{1313} = \frac{1}{1 + (\Omega\tau)^2} \lambda''_{1313}(0), \quad (30)$$

where

$$\lambda''_{2211} = \lambda''_{1122}, \quad \lambda''_{3311} = \lambda''_{2233} = \lambda''_{3322} = \lambda''_{1133},$$
$$\lambda''_{1221} = \lambda''_{2112} = \lambda''_{2121} = \lambda''_{1212},$$
$$\lambda''_{1331} = \lambda''_{3113} = \lambda''_{3131} = \lambda''_{2323} = \lambda''_{2332} = \lambda''_{3223} = \lambda''_{3232} = \lambda''_{1313}, \quad (31)$$

and also

$$\lambda''_{1332} = -\frac{\Omega\tau}{1+(\Omega\tau)^2}\lambda''_{1313}(0), \quad \lambda''_{1112} = -\frac{2\Omega\tau}{1+4(\Omega\tau)^2}\lambda''_{1313}(0), \tag{32}$$

where

$$\lambda''_{1323} = \lambda''_{3132} = \lambda''_{3123} = -\lambda''_{3213} = -\lambda''_{3231} = -\lambda''_{2313} = -\lambda''_{2331} = \lambda''_{1332},$$

$$\lambda''_{1121} = \lambda''_{1222} = \lambda''_{2122} = -\lambda''_{1211} = -\lambda''_{2111} = -\lambda''_{2212} = -\lambda''_{2221} = \lambda''_{1112}. \tag{33}$$

Here $\lambda''_{ijkl}(0)$ is the value of λ''_{ijkl} in the absence of the constant magnetic field ($H_0 = 0$)

$$\lambda''_{1111}(0) = \lambda''_{3333}(0) = \tfrac{4}{3}\lambda''_{1313}(0),$$

$$\lambda''_{1122}(0) = \lambda''_{1133}(0) = -\tfrac{2}{3}\lambda''_{1313}(0),$$

$$\lambda''_{1313}(0) = \lambda''_{1212}(0) = \lambda''_{2323}(0) = \omega\eta(0), \tag{34}$$

where

$$\eta(0) = \tfrac{1}{5}Nmv_0l \tag{35}$$

is the coefficient of the shear viscosity of the electron gas in the absence of the constant magnetic field.

If we express the components of the moduli λ''_{1212}, λ''_{1313}, and λ''_{2323} in terms of the coefficients of the shear viscosity (for $H_0 \neq 0$), then we obtain for the latter expressions analogous to those obtained by Steinberg [11] for the case $kl \ll 1$. The expression for the component λ''_{1332} coincides with that previously obtained by one of the authors. [2]

From (30) and (32) it follows that the tensor λ''_{imjl} has two types of components.

1) Components symmetric with respect to a transposition of the first and second pair of indices $\lambda''_{imjl} = \lambda''_{jlim}$ [Eqs. (30) and (31)]. These components are responsible for the absorption of ultrasound. The presence of a magnetic field leads to a change in the magnitudes of these components and to an increase in the number of independent components. In the free-electron approximation the number of this type of independent components is determined by the symmetry imposed by the magnetic field (transverse-isotropic symmetry).

2) Components antisymmetric with respect to the transposition of the first and second pair of indices $\lambda''_{imjl} = -\lambda''_{jlim}$ [Eqs. (32) and (33)]. These components are an odd function of the constant magnetic field and vanish in a magnetic-polarized medium ($H_0 = 0$) or in the quasistatic mode ($\omega = 0$).

It is easily seen that the presence of the tensor components of the type λ_{1332} is connected with the phenomenon of the rotation of the plane of polarization of elastic waves, while the presence of the tensor components of the type λ_{1112} is connected with the fact that in magneto-polarized media there can appear coupled longitudinal-transverse and coupled transverse-longitudinal elastic waves. [2]

We have thus obtained on the basis of microscopic theory formulas that allow for the case of metals to estimate numerically the magnitude of the different components of the dynamic tensor of the elasticity moduli which has previously been introduced [6] phenomenologically for magneto-elastic media, and to explain the character of their dependence on the magnetic-field intensity and on the ultrasound frequency; it also allows to foresee the nature of their temperature dependence (if the nature of the temperature dependence of the mean free path is known from experimental or some theoretical considerations).

3. Let us now consider in more detail the phenomenon of the rotation of the plane of polarization of elastic waves in a broad range of frequencies, which should be observed when an elastic wave propagates along the direction of the constant magnetic field (the z axis). In addition we align the field along the principal crystallographic direction of the crystals belonging to the cubic, hexagonal, tetragonal, or orthorhombic system. In this case the equations of motion of an elastic medium will take on the form

$$\rho\ddot{u}_l = -(\lambda_{i33l} + \lambda_{i3l3})k_3^2u_l/2. \tag{36}$$

An analysis of Eq. (16) with account of Eqs. (21)—(23) shows that in the case considered the components of the dynamic tensor of the elasticity moduli λ_{imjl} of the type λ_{1333}, λ_{3331}, λ_{3313}, λ_{3332}, λ_{3323}, and λ_{2333} vanish. However, components of the type

$$\lambda_{1332} = \lambda_{1323} = -\lambda_{2331} = -\lambda_{2313} \tag{37}$$

do not vanish. The interaction of the elastic oscillations with the conduction electrons leads also to the appearance of imaginary terms of the components of the moduli of the type $\lambda_{1331} = \lambda_{1313}$, $\lambda_{2332} = \lambda_{2323}$ and λ_{3333}. These imaginary terms determine the absorption and we shall hereafter neglect them.

Solution of Eqs. (36) with account of (37) shows that in the case considered a longitudinal and two circularly polarized transverse waves can propagate independently in the metal. This means that in the case of normal incidence of a plane-polarized wave on a plane parallel layer of metal (the direction of incidence is chosen along the z axis) one must expect in the latter a rotation of the plane of polarization of the elastic wave and the appearance of its ellipticity. It can be shown [2] that the constant of rotation of the plane of polarization κ (i.e., the angle through which the plane of polarization is rotated when the wave propagates through a distance of one wavelength referred to unit magnetic field intensity) is determined by the relation

$$\varkappa = k_3^2 (\mathrm{Im}\lambda_{2313}) / 2\omega H_0 (\rho\lambda_{1331})^{1/2}, \qquad (38)$$

while the axial ratio of the ellipse after the passage of a distance z turns out to be

$$\frac{b}{a} = \pm \,\mathrm{th}\left[\frac{k_3^2 z}{2\omega(\rho\lambda_{1331})^{1/2}} \,\mathrm{Re}\,\lambda_{2313}\right]. \qquad (39)^*$$

Taking into account Eqs. (16)—(18) and (22), one can obtain for the component λ_{2313} the following expressions:

$$\lambda_{2313} = i\omega NeH_0\alpha/ck_3^2, \qquad (40)$$

$$\alpha = 1 - \frac{(1+\sigma_0/\sigma)(\sigma+\sigma_0)\sigma_{21}}{\Omega\tau\,[(\sigma_{11}+\sigma_0)(\sigma_{22}+\sigma_0)-\sigma_{12}\sigma_{21}]}. \qquad (41)$$

Substituting (40) in (38), we obtain

$$\varkappa = (Ne/2\rho c s_t)\,\mathrm{Re}\,\alpha, \qquad (42)$$

where $s_t = (\lambda_{1331}/\rho)^{1/2}$ is the velocity of transverse sound.

For the components of the tensor σ_{ij} for $\mathbf{k} \parallel \mathbf{H}_0$ we obtain in accordance with (22)

$$\sigma_{11} = \tfrac{3}{8}\sigma\,[(g_1^+ + g_1^-) + i\,(g_2^+ + g_2^-)],$$
$$\sigma_{21} = -\sigma_{12} = i\tfrac{3}{8}\sigma\,[(g_1^+ - g_1^-) + i\,(g_2^+ - g_2^-)], \qquad (43)$$

where [3]

$$g_1^\pm = \frac{1}{(k_3 l)^3}\Big\{c^\pm\,[\mathrm{arctg}\,a^\pm + \mathrm{arctg}\,b^\pm] + (\omega\pm\Omega)\,\tau\ln\frac{1+(a^\pm)^2}{1+(b^\pm)^2} - 2k_3 l\Big\},$$

$$g_2^\pm = -\frac{1}{2(k_3 l)^3}\Big\{c^\pm\ln\frac{1+(a^\pm)^2}{1+(b^\pm)^2} - 4(\omega\pm\Omega)\,\tau\,[\mathrm{arctg}\,a^\pm + \mathrm{arctg}\,b^\pm] + 4k_3 l\,(\omega\pm\Omega)\,\tau\Big\}, \qquad (44)\dagger$$

$$a^\pm = k_3 l + (\omega\pm\Omega)\,\tau, \qquad b^\pm = k_3 l - (\omega\pm\Omega)\,\tau,$$
$$c^\pm = 1 + (k_3 l)^2 - [(\omega\pm\Omega)\,\tau]^2.$$

Formula (41) will be much simplified if we consider the still rather broad range of frequencies satisfying condition (26). If in addition we also assume that $\omega \ll \Omega$, then (41) takes on the form

$$\alpha = 1 + \frac{4}{3}\frac{1}{\Omega\tau}\frac{g_2^+}{(g_1^+)^2 + (g_2^+)^2}. \qquad (45)$$

Thus, with these approximations α becomes pure real and the components of the type λ_{2313} consequently become pure imaginary. In accordance with (39) this means that in this case one should not expect the appearance of appreciable ellipticity (circular magnetic dichroism).

In the region of low frequencies for which $k_3 l \ll 1$ or in the region of high fields for which $X_3 \ll 1$ we obtain

$$\alpha = \tfrac{1}{5}(k_3 l)^2/[1 + (\Omega\tau)^2]. \qquad (46)$$

This expression coincides with that previously obtained in [2].

In the region of comparatively high frequencies, such that $k_3 l \gg 1$ with $k_3 l \gg \Omega\tau$, satisfying however conditions (26) and $\omega \ll \Omega$, we have

$$\alpha = 1 - \frac{16}{3\pi^2}\Big(1 + \frac{16-\pi^2}{2\pi}\frac{1}{k_3 l}\Big) \approx 0.46 - 0.53\frac{1}{k_3 l}. \qquad (47)$$

In the region of weak fields $\Omega\tau \ll 1$ and in a broad range of frequencies satisfying only conditions (26) and $\omega \ll \Omega$

$$\alpha = 1 + \frac{4}{3}(k_3 l)^3 \frac{\mathrm{arctg}\,(k_3 l) - k_3 l}{\{[1+(k_3 l)^2]\,\mathrm{arctg}\,(k_3 l) - k_3 l\}^2}. \qquad (48)$$

Let us now consider the region of high frequencies such that $k \gg k_0$. In this case in accordance with (41) and (43)

$$\alpha = 1 - (3i/8\Omega\tau)\,[(g_1^+ - g_1^-) + i\,(g_2^+ - g_2^-)]. \qquad (49)$$

In the region of small fields such that $|(\omega\pm\Omega)\,\tau| \ll 1$ we obtain

$$\alpha = \Big(1 + \tfrac{3}{4}\alpha_1\Big) + i\tfrac{3}{2}\,\omega\tau\alpha_2, \qquad (50)$$

$$\alpha_1 = 4\,(k_3 l)^{-2}((k_3 l)^{-1}\mathrm{arctg}\,k_3 l - 1),$$
$$\alpha_2 = \tfrac{1}{2}\alpha_1 + 2/[1 + (k_3 l)^2]. \qquad (51)$$

If in addition $k_3 l \gg 1$, then

$$\alpha = 1 - \frac{3}{(k_3 l)^2} + i\frac{3\pi}{2}\frac{\omega}{k_3 v_0}\frac{1}{(k_3 l)^2}. \qquad (52)$$

Equation (52) is also valid under less stringent conditions, namely when $|(\omega\pm\Omega)\,\tau| \ll k_3 l$ and $k_3 l \gg 1$.

The real part of (52) determines the rotation of the plane of polarization, and tends to unity as $k_3 l \to \infty$. In this case, considered previously in [1], the rotation of the plane of polarization is completely determined by the Lorentz force exerted on the ions by the constant magnetic field. The imaginary part of (52) determines the ellipticity. Allowing for the fact that ω/k_3 differs little from s_t—the propagation velocity of transverse sound—

*th = tanh.

†arctg = $\tan^{-1}$.

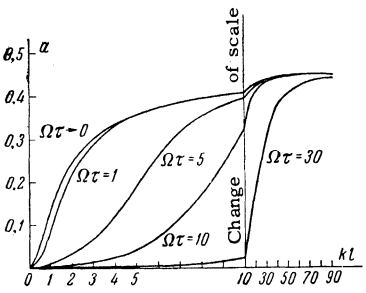

FIG. 1

we see that the imaginary part of α is proportional to the small quantity s_t/v_0.

Figure 1 shows plots of the quantity α, which determines the rotation of the plane of polarization, against the magnitude of the wave vector (and consequently also the ultrasound frequency), calculated for various values of the constant magnetic field. These plots are calculated by means of Eqs. (45)—(48) for ranges of fields and frequencies satisfying conditions (26) and $\omega \ll \Omega$. We have plotted the dimensionless quantities $k_3 l$ and $\Omega\tau$, so that the temperature dependence of the expected effects can be estimated.

Analyzing Fig. 1, one can conclude that it is convenient to measure the angle of rotation of the plane of polarization in the range of frequencies for which $k_3 l > 10$, when the specific rotation depends weakly on the intensity of the constant magnetic field and the angle of rotation of the plane of polarization can be increased by increasing the magnetic field intensity.

If we take $N \sim 10^{22}$ cm^{-3}, $\rho \sim 10$ g/cm^3, $s_t \sim 10^5$ cm/sec, then we obtain for κ_{max} the estimate

$$\varkappa_{max} = Ne/2\rho c s_t \sim 10^{-4}\ \text{rad}\cdot\text{cm}^{-1}\ \text{Oe}^{-1}. \tag{53}$$

This means that an appreciable effect can be expected for fields $H_0 \sim 10^4$ Oe.

An estimate of the angle of rotation $\varphi = \kappa H_0$ of the plane of polarization per unit length for the case $\Omega\tau \gg 1$ and $kv_0/\Omega \sim 1$, given in the papers of Kjeldaas [3] and Kotkin, [4] can be obtained, for instance, from Eqs. (42) and (46).

Let us now estimate the expected ellipticity for frequency ranges satisfying the condition $k \gg k_0$. Assuming $\operatorname{Im}\alpha \ll 1$, we obtain in accordance with (38)—(40), (42), and (52)

$$b/a = \tfrac{3}{2}\pi\varkappa_{max}(s_t/v_0) z H_0 (k_3 l)^{-2}. \tag{54}$$

Taking (53) into account and setting $s_t/v_0 \sim 1/300$ and $H_0 \sim 10^4$ Oe, we find that at a distance z of 1 cm one can expect the ratio b/a to be less than one per cent.

4. Let us now consider the case when a plane elastic wave propagates perpendicular to the direction of a constant magnetic field oriented along the principal crystallographic direction of cubic, hexagonal, tetragonal, and orthorhombic crystals, when coupled longitudinal-transverse and transverse-longitudinal waves should appear.

Let a wave propagate along the x axis. In this case the equations of motion take on the form

$$\rho\ddot{u}_i = -k_1^2(\lambda_{i11l} + \lambda_{i1l1})u_l/2. \tag{55}$$

An analysis of Eq. (16) with account of (21)—(23) shows that in the case under consideration the components of the type λ_{1113}, λ_{1131}, λ_{2113}, λ_{2131}, λ_{3111}, λ_{3112}, and λ_{3121} of the dynamic tensor of the elasticity modulus λ_{imjl} vanish. However the components

$$\lambda_{1112} = \lambda_{1121} = -\lambda_{2111} \tag{56}$$

do not vanish.

The solution of Eqs. (55) with account of (56) shows that in the case considered there can propagate in the metal a transverse wave whose plane of polarization coincides with the xz plane and two coupled longitudinal-transverse and transverse-longitudinal waves the plane of polarization of whose transverse components coincides with the xy plane.

The coupling coefficient for the coupled longitudinal-transverse or transverse-longitudinal waves is determined by the component λ_{1112}. Taking into account (16), (21), and (22), we obtain the following expression for λ_{1112}:

$$\lambda_{1112} = -iNmv_0\beta\omega/k_1, \tag{57}$$

$$\beta = X_1^{-1}\left\{1 - \frac{(\sigma_0+\sigma)\sigma_{21}}{\Omega\tau[(\sigma_0+\sigma_{22})\sigma_{11}+\sigma_{21}^2]}\right\}. \tag{58}$$

The components of the tensor σ_{ij} for the relative orientation of $\mathbf{k}$ and $\mathbf{H}_0$ under consideration are according to (22) equal to [12]

$$\begin{aligned}
\sigma_{11} &= \frac{3\sigma}{(k_1 l)^2}(1+i\omega\tau)\left[1 - \sum_{n=-\infty}^{\infty}\frac{(1+i\omega\tau)g_n(X_1)}{1+i(\omega-n\Omega)\tau}\right],\\
\sigma_{22} &= 3\sigma\sum_{n=-\infty}^{\infty}\frac{s_n(X_1)}{1+i(\omega-n\Omega)\tau},\\
\sigma_{21} &= -\sigma_{12} = -\frac{3}{2}\frac{\sigma}{k_1 l}\sum_{n=-\infty}^{\infty}\frac{(1+i\omega\tau)g_n'(X_1)}{1+i(\omega-n\Omega)\tau};\\
g_n(X_1) &= \int_0^{\pi/2}\sin\vartheta J_n^2(X_1\sin\vartheta)\,d\vartheta,\\
s_n(X_1) &= \int_0^{\pi/2}\left[\frac{\partial J_n(X_1\sin\vartheta)}{\partial X_1}\right]^2\sin\vartheta\,d\vartheta,
\end{aligned} \tag{59}$$

$$g_n'(X_1) = \partial g_n(X_1)/\partial X_1. \tag{60}$$

Let us now consider some special cases. Let $k_1 \ll k_0$ and $X_1 \ll 1$ (the case of strong fields or low frequencies). Then we obtain for λ_{1112} relation (32). In terms of the variables X_1 it will have the form (57) where

$$\beta = \frac{2}{5}\frac{(\Omega\tau)^2}{1+4(\Omega\tau)^2}X_1. \tag{61}$$

We now consider the case when $k \ll k_0$ and the condition for geometric resonance is fulfilled, i.e., $X_1 \sim 1$. For $X_1 \leq 1$ the inequality $\Omega/\omega \geq v_0/s$ is fulfilled where $v_0/s \gg 1$ and $s = \omega/k_1$ is the ultrasound velocity. Therefore for $\Omega\tau \gg 1$ when the electron has enough time to complete many turns in the cyclotron orbit without undergoing collisions, the main role in the sums occurring in (59) will be played only by the term with $n = 0$. In this case we obtain for β the expression

$$\beta = \frac{1}{X_1}\left[1 + \frac{1}{3}\frac{0.5g_0'X_1}{s_0(1-g_0)+(0.5g_0')^2}\right]. \tag{62}$$

Thus the component λ_{1112} turns out in the cases considered to be a pure imaginary quantity.

Let us estimate the ratio of the amplitude of the longitudinal component $(u_1^0)_{t,l}$ to the amplitude of the transverse component $(u_2^0)_{t,l}$ of the transverse-longitudinal wave or of the amplitude of the transverse component $(u_2^0)_{t,l}$ to the longitudinal component $(u_1^0)_{l,t}$ of the longitudinal-transverse wave. Assuming $|\lambda_{1112}| \ll |\lambda_{1111} - \lambda_{2112}|$, we obtain

$$(u_1^0/u_2^0)_{t,l} = (u_2^0/u_1^0)_{l,t} = \lambda''_{1112}(\lambda_{1111} - \lambda_{2112})^{-1}, \tag{63}$$

where λ''_{1112} is the imaginary term of the component of the modulus λ_{1112}.

For an estimate of the maximum possible value of this ratio we substitute for λ''_{1112} the value of the coefficient of β in (57), and in place of $(\lambda_{1111} - \lambda_{2112})$ we substitute $p_0v_0N/3$ which determines in the expression (28) for the elasticity moduli the term due to the elasticity of the conduction electrons which is equal in order of magnitude to the magnitude of the components of the resulting modulus of elasticity tensor. Then

$$(u_2^0/u_1^0)_{l,t} \approx 3s_{l,t}/v_0, \qquad (u_1^0/u_2^0)_{t,l} \approx 3s_{t,l}/v_0,$$

where $s_{l,t}$ and $s_{t,l}$ are the velocities of the longitudinal-transverse and transverse-longitudinal

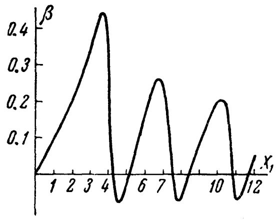

FIG. 2

waves which, neglecting the terms quadratic in λ_{1112}, are $s_{l,t} \approx s_l = s_l = (\lambda_{1111}/\rho)^{1/2}$ and $s_{t,l} \approx s_t = (\lambda_{2112}/\rho)^{1/2}$. Figure 2 shows the dependence of β, to which λ_{1112} is proportional, on X_1 plotted according to formulas (61) and (62) under the condition $\Omega\tau \gg 1$ and $k^2 \ll k_0^2$. The dependence of the functions g_0, s_0, and g_0' on X_1 is taken from the paper of Cohen et al.[12] The oscillations of β and consequently also of λ_{1112} are related to the geometric resonance.

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