RADIATION EMITTED BY ATOMS MOVING IN THE FIELD OF A STANDING WAVE

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The radiation from atoms moving in the field of a standing wave is considered. The expressions derived are applied to the calculation of the power radiated by spontaneous and stimulated emission from a gaseous quantum generator.

1. INTRODUCTION

In a number of publications [1-5] the probability of stimulated emission from a stationary atom has been evaluated, where the atom is immersed in a monochromatic field of frequency ω which is close to the frequency ω_{mn} of the transition $m \to n$. The formula derived remains in force also in those cases when the atom moves (gas), but the external field consists of a progressive plane wave. In fact, in the reference frame moving with center of mass of the atom, the frequency will be $\omega - \mathbf{k}\cdot\mathbf{v}$, [$\mathbf{v}$ is the velocity of the atom and $\mathbf{k}$ is the wave vector] (Doppler effect) and it is therefore sufficient to replace ω by $\omega - \mathbf{k}\cdot\mathbf{v}$ and average over the velocity. If the atom also moves in a field that consists of a superposition of progressive waves of the same frequency but various directions of propagation, then $\omega - \mathbf{k}\cdot\mathbf{v}$ will be different for the various waves. Thus the interaction of a moving atom with such a field is equivalent to the interaction of a stationary atom with a non-monochromatic field and the formulae of [1-5] are as a rule inapplicable. Under such conditions a number of singularities also arise in the spontaneously emitted radiation.

It is namely such a case which occurs in quantum generators—the field set up in them is a superposition of two progressive waves propagated in opposite directions. Because of this, we consider below the radiation emitted by atoms in the field:

$$E = E_1 \cos(\omega t - \mathbf{k}_1\cdot\mathbf{R} + \delta_1) + E_2 \cos(\omega t - \mathbf{k}_2\cdot\mathbf{R} + \delta_2);$$

$$|\mathbf{k}_1| = |\mathbf{k}_2|. \qquad (1.1)$$

The interaction of an atom with such a field is described by the system of equations

$$i(\dot{a}_m + \gamma_m a_m) = V a_n, \quad i(\dot{a}_n + \gamma_n a_n) = V^* a_m. \qquad (1.2)$$

Here a_m, a_n and $1/2\gamma_m$, $1/2\gamma_n$ are the probability amplitudes and lifetimes of the states of the atom, and,

$$V = p_{mn} E \hbar^{-1} e^{i\omega_{mn}t} = e^{i(\omega_{mn}-\omega)t}\,[G_1 e^{i(\mathbf{k}_1\mathbf{R}-\delta_1)} + G_2 e^{i(\mathbf{k}_2\mathbf{R}-\delta_2)}];$$

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{v}(t - t_0), \; G_{1,2} = p_{mn} E_{1,2}/2\hbar, \qquad (1.3)$$

where p_{mn} is the dipole matrix element and $\mathbf{R}_0$ is the coordinate of the atom at the initial moment of time t_0.

Having determined a_m and a_n from (1.2), we can evaluate the current $\langle\Psi^*|\,j\,|\Psi\rangle$ induced by the field, and the power $P(t - t_0, \mathbf{R}, \mathbf{R}_0)$ of the stimulated emission at the point $\mathbf{R}$ at the moment of time t, produced by an atom excited at the point $\mathbf{R}_0$:

$$P(t - t_0, \mathbf{R}, \mathbf{R}_0) = \langle\Psi^*|\,\mathbf{E}\mathbf{j}\,|\Psi\rangle = \hbar\omega_{mn}\,\mathrm{Re}\,\{2ip_{mn}E\hbar^{-1}a_m^* a_n e^{i\omega_{mn}t}\}. \qquad (1.4)$$

With the aid of the function $P(t - t_0, \mathbf{R}, \mathbf{R}_0)$ it is possible to evaluate the various characteristics of forced emission of individual atoms, and media.

In what follows we will be interested in the following quantities:

$$A(\mathbf{R}_0, \mathbf{v}) = \int_{t_0}^{\infty} P[t - t_0, \mathbf{R}(t), \mathbf{R}_0]\,dt,$$

$$A(\mathbf{R}, \mathbf{v}) = \int_{-\infty}^{t} P[t - t_0, \mathbf{R}, \mathbf{R}_0(t_0)]\,dt_0. \qquad (1.5)$$

The first of these is the total energy radiated by some particular atom after its excitation at time t_0 at the point $\mathbf{R}_0$; the second is the average energy of stimulated emission at the point $\mathbf{R}$, per excited atom (with velocity $\mathbf{v}$). Averaging $A(\mathbf{R}_0, \mathbf{v})$ over $\mathbf{R}_0$ and $\mathbf{v}$ or $A(\mathbf{R}, \mathbf{v})$ over $\mathbf{R}$ and $\mathbf{v}$, and multiplying by the number of excitations Q per cm^3 per second, we obtain the average power of stimulated emission per unit volume.

$$\langle P\rangle = \hbar\omega_{mn} Q\overline{W} = Q\ \langle A\ (\mathbf{R}_0,\ \mathbf{v})\rangle = Q\ \langle A\ (\mathbf{R},\ \mathbf{v})\rangle. \quad (1.6)$$

Knowing the fundamental system of solutions for equations (1.2), it is possible to calculate the probability of spontaneous emission. We designate these solutions by a_{m1}, a_{n1} and a_{m2}, a_{n2}. We introduce into (1.2) a small perturbation V_μ: the perturbation corresponds to a propagating monochromatic wave with wave vector $k_\mu = (\omega_\mu/c)/n_\mu$. By including this perturbation to first order, it is not difficult to obtain:

$$\Psi = (a_n + c_m^+ + c_m^-)\,\psi_m e^{-iE_m t/\hbar} + (a_n + c_n^+ + c_n^-)\,\psi_n e^{-iE_n t/\hbar}; \quad (1.7)$$

$$c_m^+ = I_{m2}a_{m1} - I_{m1}a_{m2}, \quad c_m^- = -I_{n2}a_{m1} + I_{n1}a_{m2},$$
$$c_n^+ = I_{m2}a_{n1} - I_{m1}a_{n2}, \quad c_n^- = -I_{n2}a_{n1} + I_{n1}a_{n2}; \quad (1.8)$$

$$I_{ml} = \frac{i}{W_0}\int\limits_{t_0}^{t} V_\mu^*(t')\,a_m(t')\,a_{ml}(t')\exp\{(\gamma_m + \gamma_n)(t' - t_0)\}\,dt,$$
$$I_{nl} = \frac{i}{W_0}\int\limits_{t_0}^{t} V_\mu(t')\,a_n(t')\,a_{nl}(t')\exp\{(\gamma_m + \gamma_n)(t' - t_0)\}\,dt'. \quad (1.9)$$

Here W_0 is the Wronskian of the fundamental system of solutions at the initial instant of time, t_0.

The physical significance of the functions $c_m^\pm$ and $c_n^\pm$ can be understood from the scheme of radiative transitions in Fig. 1. Under the influence of a strong field the atom executes transitions from state m to state n and vice versa (vertical arrows in Fig. 1). In addition, an atom may make a transition from state m to state n, and radiate a photon ω_μ. The contribution of such transitions to the probability amplitude of states m and n is given by the functions c_m^+ and c_n^+. Transitions are also possible which correspond to the absorption of photons of frequency, ω_μ. Their contribution is given by the functions c_m^- and c_n^-. From Fig. 1 the role of the strong field in radiative transitions of frequency ω_μ is evident. If the strong field is absent, then for the initial conditions $a_m(t_0) = 1$ and $a_n(t_0) = 0$ only photons of frequency ω_μ can be emitted, whereas for $a_m(t_0) = 0$ and $a_n(t_0) = 1$ they can only be absorbed. The presence of a strong field leads to "mixing" of the states m and n, as a result of which both emission and absorption take place for any kind of initial conditions.

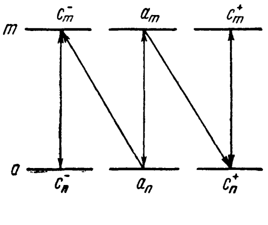

FIG. 1. Diagram of transitions between the states m and n of an atom under the influence of strong (vertical arrows) and weak fields.

We note that the values of the sums $c_m^+ + c_m^-$ and $c_n^+ + c_n^-$ follow directly from a calculation of the probability amplitudes. The fact that they separate out in this way leads, as is evident from (1.8) and (1.9), to the indication that $c_{m,n}^+$ is expressible in terms of V_μ^* and $c_{m,n}^-$, in terms of V_μ.

The probabilities of absorption and emission of a photon in the frequency interval $\Delta\omega_\mu$ and over angles ΔO are given by (1.10) and (1.11) respectively:

$$w_a(\mathbf{k}_\mu)\,\Delta\omega_\mu\Delta O = 2\gamma_m\int\limits_{t_0}^{\infty}|c_m^-|^2\,dt + 2\gamma_n\int\limits_{t_0}^{\infty}|c_n^-|^2\,dt, \quad (1.10)$$

$$w_l(\mathbf{k}_\mu)\,\Delta\omega_\mu\Delta O = 2\gamma_m\int\limits_{t_0}^{\infty}|c_m^+|^2\,dt + 2\gamma_n\int\limits_{t_0}^{\infty}|c_n^+|^2\,dt. \quad (1.11)$$

The true absorption (or emission) actually observed experimentally is determined by the difference of these quantities. The probability of spontaneous emission can be calculated from (1.11) by substituting into the expression, for c_m^+ and c_n^+ the value of V_μ which corresponds to the interaction of an atom with null oscillations of the field:

$$|V_\mu|^2 = 8\pi\frac{\hbar\omega}{2}\frac{p_{mn}^2}{\hbar^2}F(\mathbf{k}_\mu)\frac{\Delta\mathbf{k}_\mu}{(2\pi)^3} = \frac{A_{mn}}{8\pi^2}F(\mathbf{k}_\mu)\,\Delta\omega_\mu\Delta O, \quad (1.12)$$

where A_{mn} is the Einstein coefficient for spontaneous emission from an isolated atom for the transition m → n. The function $F(\mathbf{k}_\mu)$ depends on the number of oscillators of the field, Δn, per unit volume in the interval $\Delta \mathbf{k}_\mu$, in the following way: $\Delta n = (2\pi)^{-3}F(\mathbf{k}_\mu)\Delta\mathbf{k}_\mu$; for free space $F(\mathbf{k}_\mu) = 1$.

Equations (1.10)–(1.12) are valid for sufficiently small values of the perturbation V_μ such that the integral transition probability at frequencies ω_μ is small compared to $\gamma_m + \gamma_n$. In particular, the equation for spontaneous emission is valid for the condition, $A_{mn} \equiv 2\gamma_{mn} \ll 2(\gamma_m + \gamma_n)$.

2. WEAK FIELD

In the general case the system (1.2) with the perturbation (1.3) is not integrable. We therefore consider first the case of a weak field, when it is possible to limit oneself to first order perturbation theory. This enables one to elucidate some of the characteristic properties of radiation from moving atoms. Integrating system (1.2) in this approximation for the initial conditions $a_m = 1$, $a_n = 0$ when $t = t_0$, we obtain

$$P(\Theta,\ \mathbf{R},\ \mathbf{R}_0) = 2\hbar\omega_{mn}e^{-2\gamma_m\Theta}\,\mathrm{Re}\left\{G_1^2\frac{1 - e^{-S_1\Theta}}{S_1} + G_2^2\frac{1 - e^{-S_2\Theta}}{S_2}\right.$$
$$\left. + G_1G_2e^{iL}\frac{1 - e^{-S_1\Theta}}{S_1} + G_1G_2e^{-iL}\frac{1 - e^{-S_2\Theta}}{S_2}\right\}; \quad (2.1)$$

$$S_{1,2} = i\Omega_{1,2} + \gamma_m - \gamma_n, \quad \Omega_{1,2} = \omega - \omega_{mn} - \mathbf{k}_{1,2}\mathbf{v},$$
$$L = \delta_1 - \delta_2 - (\mathbf{k}_1 - \mathbf{k}_2)\,\mathbf{R},\ \Theta = t - t_0. \tag{2.2}$$

The time dependence of P is determined by two factors—relaxation processes (damping with constants γ_m and γ_n) and by the reciprocating motion of an atomic oscillator in a field. The latter factor is associated with the phase difference between the field at the point **R** and the current $\langle\Psi\,|\,\mathbf{j}\,|\,\Psi\rangle$ induced in the atom, this phase difference being determined by all interaction processes occurring between the atom and the field. In the general case when $\mathbf{k}_1 \neq \mathbf{k}_2$, the parameter $P(\Theta, \mathbf{R}, \mathbf{R}_0)$ depends not only on Ω_1, Ω_2, γ_m, γ_n, and $\Theta = t - t_0$, but also on **R** (or $\mathbf{R}_0$) and on the relative orientation of the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and **v**. In order to explain specifically these interference effects for $\mathbf{k}_1 \neq \mathbf{k}_2$, it is useful to compare the cases $\mathbf{k}_1 = \mathbf{k}_2$ (progressive wave) and $\mathbf{k}_1 = -\mathbf{k}_2$ (standing wave) with each other.

For $\mathbf{k}_1 = \mathbf{k}_2$ we have

$$P(\Theta, \mathbf{R}, \mathbf{R}_0) = 2\hbar\omega_{mn}\,[G_1^2 + G_2^2 + 2G_1G_2\cos(\delta_1 - \delta_2)]\,e^{-2\gamma_m\Theta}\,\mathrm{Re}\,\frac{1 - e^{-S_1\Theta}}{S_1}; \tag{2.3}$$

$$A(\mathbf{R}_0, \mathbf{v}) = \hbar\omega_{mn}\frac{\gamma_m + \gamma_n}{\gamma_m}\,\frac{G_1^2 + G_2^2 + 2G_1G_2\cos(\delta_1 - \delta_2)}{\Omega_1^2 + (\gamma_m + \gamma_n)^2}. \tag{2.4}$$

If $\Omega_1 = 0$, then $P > 0$ for all Θ; if on the other hand $\Omega_1 \neq 0$ then oscillations that depend on Ω_1, γ_m, and γ_n occur, and P can take on both positive and negative values. At the same time $A(\mathbf{R}_0, \mathbf{v})$ is always positive, as it should be for the chosen initial conditions. The magnitude of P depends only on Θ and is independent of **R** and $\mathbf{R}_0$. Similarly, $A(\mathbf{R}_0, \mathbf{v})$ is independent of $\mathbf{R}_0$. In other words, all points in space are equivalent when $\mathbf{k}_1 = \mathbf{k}_2$.

We now pass to the case $\mathbf{k}_1 = -\mathbf{k}_2$, assuming for simplicity $\omega = \omega_{mn}$ and writing out only the expression for $A(\mathbf{R}_0, \mathbf{v})$:

$$A(\mathbf{R}_0, \mathbf{v}) = \hbar\omega_{mn}\frac{(\gamma_m + \gamma_n)/\gamma_m}{(kv)^2 + (\gamma_m + \gamma_n)^2}\Big\{G_1^2 + G_2^2 + 2G_1G_2\frac{\gamma_m}{\gamma_m + \gamma_n} \times \frac{[\gamma_m(\gamma_m + \gamma_n) - (kv)^2]\cos\eta - kv(\gamma_m + \gamma_n)\sin\eta}{(kv)^2 + \gamma_m^2}\Big\};$$
$$\eta = 2\mathbf{k}_1\mathbf{R}_0 + \delta_2 - \delta_1,\ k = |\mathbf{k}_1|. \tag{2.5}$$

Here v is the projection of **v** on the direction $\mathbf{k}_1$. It is easily shown that in this case also $A(\mathbf{R}_0, \mathbf{v}) > 0$. However in contrast to (2.4), the interference term in (2.5) depends on the point of excitation, namely because it contains terms proportional to $\cos\eta$ and $\sin\eta$. In this case the first term is an even function of v and the second changes sign when the sign of v is changed. For atoms excited at nodes and antinodes of a standing wave the second term is zero: if $\mathbf{R}_0$ is a point $\lambda/8$ away from a node or antinode, then the term has its maximum value. This result arises from the fact that a standing wave has symmetry about nodes or antinodes but is unsymmetrical about all other points. The term in (2.5) which is antisymmetric with respect to v, can play a very important role. For example, when $(kv)^2 = \gamma_m(\gamma_m + \gamma_n)$ we have

$$A(\mathbf{R}_0, \mathbf{v}) = \frac{\hbar\omega_{mn}}{\gamma_m(\gamma_n + 2\gamma_m)}\Big\{G_1^2 + G_2^2 \pm 2G_1G_2\sqrt{\frac{\gamma_m}{\gamma_m + \gamma_n}}\sin\eta\Big\}, \tag{2.6}$$

where the two signs correspond to the two directions of motion of the atom. Thus when $\sqrt{\gamma_m(\gamma_m + \gamma_n)} \sim 1$ a change in sign of v, and also a change of the coordinates of the point of excitation, changes $A(\mathbf{R}_0, \mathbf{v})$ by a large factor. For $|kv| \gg \gamma_m + \gamma_n$, and neglecting small terms, we obtain from (2.5)

$$A(\mathbf{R}_0, \mathbf{v}) = \hbar\omega_{mn}\frac{\gamma_m + \gamma_n}{\gamma_m(kv)^2}\Big\{G_1^2 + G_2^2 - 2G_1G_2\frac{\gamma_m}{\gamma_m + \gamma_n}\cos\eta\Big\}. \tag{2.7}$$

The condition $|\mathbf{k}\cdot\mathbf{v}| \gg \gamma_m + \gamma_n$ implies that the atom covers many wavelengths during the lifetime of the excited state. Nevertheless $A(\mathbf{R}_0, \mathbf{v})$ depends significantly on changes of the initial coordinates of an atom by amounts of the order of $\lambda/8$, although it does not depend on the direction of motion of the atom. All these effects are closely connected with the properties of the reciprocating motion of the atomic oscillator as it moves in the standing wave field.

As a result of averaging (2.5) over v (here and in what follows the velocity distribution of the atoms will be assumed to be isotropic) the antisymmetric term in v disappears. If, in addition, the average velocity $\bar{v}$ is large enough so that $k\bar{v} \gg \gamma_m + \gamma_n$, then the interference term in (2.5) is of the order of $(\gamma_m + \gamma_n)/k\bar{v}$ and can be neglected.

We now evaluate the function $A(\mathbf{R}, \mathbf{v})$ which determines the power radiated to a fixed point of the volume, **R** [see (1.8)]. Assuming for simplicity that $G_1 = G_2 \equiv G$, we have for $\mathbf{k}_1 = -\mathbf{k}_2$:

$$A(\mathbf{R},\mathbf{v}) = \hbar\omega_{mn}\frac{\gamma_m+\gamma_n}{\gamma_m}$$
$$\times G^2\left\{\left[1+\cos\xi+\frac{kv}{\gamma_m+\gamma_n}\sin\xi\right]\right.$$
$$\times\left[\frac{1}{\Omega_1^2+(\gamma_m+\gamma_n)^2}+\frac{1}{\Omega_2^2+(\gamma_m+\gamma_n)^2}\right]$$
$$\left.+\frac{kv}{\gamma_m+\gamma_n}\sin\xi\frac{(\omega-\omega_{mn})^2}{[\Omega_1^2+(\gamma_m+\gamma_n)^2][\Omega_2^2+(\gamma_m+\gamma_n)^2]}\right\};$$
$$\xi = 2\mathbf{k}_1\mathbf{R}+\delta_2-\delta_1. \tag{2.8}$$

In accordance with (1.1), the square of the amplitude of the field at the point $\mathbf{R}$ is equal to $E^2 = 2(\hbar/p_{mn})^2 G^2[1+\cos\xi]$. From (2.8) it is evident that $A(\mathbf{R},\mathbf{v})$ contains terms with the factor $[kv/(\gamma_m+\gamma_n)]\sin\xi$, which violate the direct proportionality between $A(\mathbf{R},\mathbf{v})$ and E^2. Their role is most important when $|kv| \gg \gamma_m+\gamma_n$. In this case the sign of $A(\mathbf{R},\mathbf{v})$ can be either positive or negative depending on $\mathbf{R}$ and the sign of $\mathbf{v}$. This means that some regions of space emit whereas others absorb; for a fixed region in space emission changes to absorption when the direction of motion of the atom is changed.

On averaging over the velocities, terms linear in v drop out. Therefore $\langle A(\mathbf{R},\mathbf{v})\rangle \sim E^2$ and is positive. Thus, although radiation from individual atoms at the point $\mathbf{R}$ is controlled by their interaction with the field over a region of dimensions of the order of $|\mathbf{v}|/(\gamma_m+\gamma_n)$, the total radiation from a fixed element of volume of gas is proportional to the square of the field at the point $\mathbf{R}$.

We introduce now an expression for the quantities averaged over the velocity $\mathbf{v}$ and the spatial coordinates. From (1.6) and (2.1) it is clear that on averaging over $\mathbf{R}$, the interference terms drop out and

$$\overline{W} = \frac{\gamma_m+\gamma_n}{\gamma_m}\frac{1}{\sqrt{\pi}}$$
$$\times\int_{-\infty}^{\infty} e^{-v^2/\bar{v}^2}\left[\frac{G_1^2}{\Omega_1^2+(\gamma_m+\gamma_n)^2}+\frac{G_2^2}{\Omega_2^2+(\gamma_m+\gamma_n)^2}\right]\frac{dv}{\bar{v}}, \tag{2.9}$$

i.e., the average probability of stimulated emission in the standing wave is determined by the sum of the probabilities for each of the progressive waves.

3. STRONG FIELD

In considering the saturation effect, it is necessary to find an exact solution of system (1.2). As has already been noted, (1.2) cannot be integrated for arbitrary values of the parameters. However for $\mathbf{k}_1 = -\mathbf{k}_2$ and for some special values of the parameters, the solution has a comparatively simple form. Namely, if

$$G_1 = G_2 \equiv G;\quad \omega = \omega_{mn};\quad \gamma_m = \gamma_n \equiv \gamma, \tag{3.1}$$

then the solution of the system (1.2) is

$$a_m = Aa_{m1}+Ba_{m2},\quad a_{m1} = e^{-\gamma t}\cos f(t),\quad a_{m2} = e^{-\gamma t}\sin f(t),$$
$$a_n = Aa_{n1}+Ba_{n2},\quad a_{n1} = -ie^{i(\delta_1+\delta_2)}a_{m2},\quad a_{n2} = -ie^{i(\delta_1+\delta_2)}a_{m1};$$
$$f(t) = \int_{t_0}^{t}|V|\,dt = \frac{2G}{\mathbf{kv}}[\sin(\mathbf{kR}-\delta)-\sin(\mathbf{kR}_0-\delta)],$$
$$\delta = (\delta_1-\delta_2)/2. \tag{3.2}$$

The first two of the conditions (3.1) are close to those which occur in actual operations: the reflection coefficients of the interferometer mirrors are high, and the amplitudes of the two waves set up in the generator system are roughly equal; the case $\omega = \omega_{mn}$ is also the most interesting. As is known, this is not achieved with regard to the condition $\gamma_m = \gamma_n$ in the optical region of the spectrum. Nevertheless we consider this case since it is the only one which permits analytic investigation.

The constants A and B in (3.2) are governed by the initial conditions. Below, the case, $a_m(t_0) = 1$ and $a_n(t) = 0$ is considered; then

$$a_m = e^{-\gamma(t-t_0)}\cos f(t);\quad a_n = -ie^{-\gamma(t-t_0)}\sin f(t)\,e^{i(\delta_1+\delta_2)}. \tag{3.3}$$

Substituting (3.3) in (1.5) one can show that,

$$A(\mathbf{R}_0,\mathbf{v}) = {}^1/_2\,\hbar\omega_{mn}\left\{1-2\gamma\int_{t_0}^{\infty}e^{-2\gamma(t-t_0)}\cos\,2f(t)\,dt\right\}. \tag{3.4}$$

Further computation cannot be carried out in general form. However, for the optical and near infrared region of the spectrum one can use the fact that the Doppler width is generally much larger than the broadening produced by damping, i.e., $k\bar{v} \gg 2\gamma$. The simplifications connected with this condition are apparent from the following considerations: $\cos 2f(t)$ is a periodic function of t with a period $2\pi/(\mathbf{k}\cdot\mathbf{v})$; during the oscillation period the factor $e^{-2\gamma t}$ changes to $1-e^{-4\pi\gamma/\mathbf{k}\cdot\mathbf{v}}$; when $k\bar{v} \gg 2\gamma$ this change is negligibly small for the majority of the atoms. Consequently the chief contribution in (3.4) is made by the value of the function $\cos 2f(t)$ averaged over one period, and is equal to

$$\cos\left(\frac{4G}{\mathbf{kv}}\sin\frac{\eta}{2}\right)J_0\left(\frac{4G}{\mathbf{kv}}\right),$$

where J_0 is a Bessel function of the first kind and order zero. Using this result, we find [1)]

[1)]When both states are excited it is necessary to replace in the formula for the radiated power the probability of excitation of the state m by the difference in probabilities of excitation of the states m and n.

$$A(\mathbf{R}_0, \mathbf{v}) = \frac{\hbar\omega_{mn}}{2}\left\{1 - \cos\left[\frac{4G}{k\mathbf{v}}\sin(\mathbf{k}\mathbf{R}_0 - \delta)\right] J_0\left(\frac{4G}{k\mathbf{v}}\right)\right\}; \quad (3.5)$$

$$\overline{W} = \frac{1}{2}\{1 - I_0\}, \qquad I_0 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} J_0^2\left(\frac{4G}{k\mathbf{v}}\right) e^{-v^2/\bar{v}^2}\frac{dv}{\bar{v}}$$
$$(\mathbf{v} \gg 2\gamma;\ G > \gamma). \quad (3.6)$$

From (3.5) it is evident that for large enough values of $4G \sim \mathbf{k}\cdot\mathbf{v}$, there is a specific dependence of $A(\mathbf{R}_0, \mathbf{v})$ on $\mathbf{R}_0$ which, as can be shown, is preserved even after averaging over $\mathbf{v}$. This difference from the weak field case [compare the discussion of (2.5)] is associated with the properties of the reciprocating motion of an atomic oscillator under saturation conditions.

As an estimate shows, the error in (3.6) does not exceed $2\sqrt{\pi}\,(2\gamma/k\bar{v})\ln 2$ and in many cases it is unimportant. An exception occurs for small values of $G < \gamma$, when the second term I_0 in the curly brackets in (3.6) approaches the value unity, and the value of $\overline{W}$ itself turns out to be small. Thus (3.5) and (3.6) do not enable one to procede to the limiting case $G \to 0$.

In the limiting case $G \gg k\bar{v}$, (3.6) takes the form

$$\overline{W} = \frac{1}{2}\left\{1 - \frac{1}{\pi^{3/2}}\frac{k\bar{v}}{4G}\right\} \qquad (G > k\bar{v} \gg 2\gamma). \quad (3.7)$$

It is worth noting that in (3.7) the term $4G/k\bar{v}$ appears (the ratio of the "saturation width" to the Doppler width) and not the square of this parameter, as might have been expected on the basis of the corresponding equation for a traveling wave.

For intermediate values of the parameter $4G/k\bar{v}$, a general analysis of (3.6) is complicated, and a numerical integration has been performed, the results of which are depicted in Fig. 2, curve 1. For comparison, curves corresponding to a traveling wave are also given in this figure. Curves 2 and 3 correspond to the amplitude of a traveling wave being equal to the sum of G_1 and G_2, (curve 2) and $(G_1^2 + G_2^2)^{1/2}$ (curve 3). Curve 4 corresponds to stationary atoms in a standing-wave field, where the distribution of frequencies of the atoms is assumed to be Gaussian with natural width $k\bar{v}$. The general character of all the graphs is roughly the same. As can be seen, curves 1 and 2 practically coincide; this can be considered as a demonstration that two waves with amplitudes G and wave vectors $\mathbf{k}$ and $-\mathbf{k}$ are equivalent (in the sense of stimulated emission) to a single progressive wave of amplitude 2G and arbitrary direction of propagation. In other words saturation conditions, the waves G_1 and G_2 have the property of "amplitude addition," whereas in the weak field case it is the squares of the amplitudes which are combined [see (2.9)].

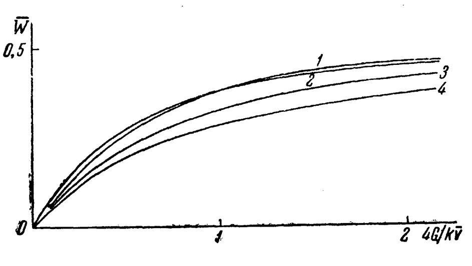

FIG. 2. Dependence of the probability of induced emission, $\overline{W}$, on $4G/k\bar{v}$.

4. POWER OF GENERATED RADIATION

The amplitude of the field established in a quantum generator can be evaluated from the energy balance. If the reflection coefficients of the mirrors are large enough, then

$$\left(\frac{4G}{k\bar{v}}\right)^2 = \frac{4}{\sqrt{\pi}}\frac{2\gamma}{k\bar{v}}\frac{Q}{Q_0}\overline{W}(G); \qquad Q_0 = \frac{\hbar\gamma k\bar{v}}{\sqrt{\pi}\,p_{mn}^2 R}, \quad (4.1)$$

where Q_0 is the threshold value of the excitation probability Q at which generation ensues, $\mathbf{R}$ is the resolving power of the interferometer or, in another terminology, the Q of the resonator.

From Fig. 2 and (4.1) it follows that near the excitation threshold, i.e., for small $4G/k\bar{v}$,

$$\overline{W} \cong 0.60\frac{4G}{k\bar{v}};$$
$$\left(\frac{4G}{k\bar{v}}\right)^2 = 1.83\left[\frac{2\gamma}{k\bar{v}}\frac{Q}{Q_0}\right]^2 \qquad \left(2 < \frac{Q}{Q_0} < 0.15\frac{k\bar{v}}{2\gamma}\right). \quad (4.2)$$

Thus for small $4G/k\bar{v}$ the generated flux is proportional to the excitation probability and for a given excess over the excitation threshold (i.e., for Q/Q_0 = constant) the degree of saturation decreases as $(2\gamma/k\bar{v})^2$. The use of (4.2) is limited to a region where the graph in Fig. 2 is a straight line (upper limit) and where the condition (3.6) is valid, namely $G > \gamma$. Since typical values for $k\bar{v}/2\gamma$ are $= 10^2$, Eq. (4.2) covers all excitation-probability cases of practical interest, except regions right near the threshold where $Q/Q_0 < 2$.

For considerable saturation ($4G > k\bar{v}$), the value of $\overline{W}$ is close to its limiting value of 1/2, and Eq. (4.2) reduces to

$$\left(\frac{4G}{k\bar{v}}\right)^2 = \frac{2}{\sqrt{\pi}}\frac{2\gamma}{k\bar{v}}\frac{Q}{Q_0}, \qquad \frac{Q}{Q_0} \gtrsim \frac{k\bar{v}}{2\gamma}. \quad (4.3)$$

Thus in order to achieve a significant degree of saturation, the limiting threshold must be exceeded by a factor of roughly $k\bar{v}/2\gamma$. We note that such a result occurs in the case of crystals with non-uniform broadening and arises directly from averaging the emission probability over the transition frequencies.

5. SPONTANEOUS EMISSION

We now pass to a consideration of the spontaneous emission from an atom moving in a standing-wave field. It will be convenient for what follows to expand (3.3) in a Fourier series and represent the wave function of an atom perturbed by a strong field in the form

$$\Psi = \psi_m \sum_s A_s \exp\{-i[E_m/\hbar - s\mathbf{kv} - i\gamma]t\} + \psi_n \sum_{s'} B_{s'} \exp\{-i[E_n/\hbar - s'\mathbf{kv} - i\gamma]t\}, \quad (5.1)$$

where A_s and $B_{s'}$ are the coefficients of the Fourier expansion. According to (5.1) an atom moving in a standing-wave field is described by wave functions of the same type as a system with quasi-stationary states, m, s; n, s′. The energies of these states $E_m - s\hbar\mathbf{k}\cdot\mathbf{v}$ and $E_n - s'\hbar\mathbf{k}\cdot\mathbf{v}$ form two equidistant systems of sublevels with a splitting $\mathbf{k}\cdot\mathbf{v}\hbar$ equal to the Doppler shift (see Fig. 3). The amplitudes A_s and B_s depend on the field and on the initial conditions. It can be shown that these coefficients are significantly different from zero for $|s|, |s'| \lesssim 2G/\mathbf{k}\cdot\mathbf{v}$, while the maximum values of $|A_s|$ and $|B_s|$ are reached near the boundaries of this region.

We note that the damping constants of all s- and s′- sublevels are identical and equal to γ. This follows from the result that $\gamma_m = \gamma_n$ and $\omega = \omega_{mn}$. In the other cases it is to be expected that the damping constants will depend on s, s′, $\omega - \omega_{mn}$, and the field strength.

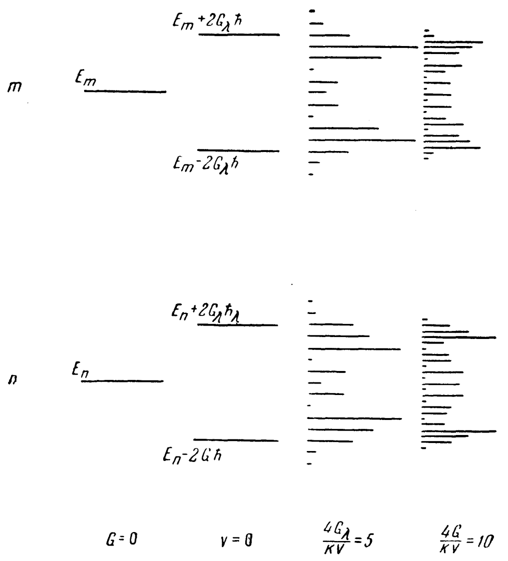

FIG. 3. System of sublevels s and s′. The length of the lines are proportional to $|A_s|^2$ and $|B_{s'}|^2$.

Substituting (1.8) in (1.11) and using (1.9) and (3.2) we obtain, under the initial conditions $a_m(t_0) = 1$ and $a_n(t_0) = 0$, the following expression for the spectral and angular probability density for spontaneous emission $w(\mathbf{k}_\mu)$:

$$w(\mathbf{k}_\mu) = \frac{\gamma_{mn}\gamma}{8\pi^2} F(\mathbf{k}_\mu) \int_{t_0}^{\infty} e^{-2\gamma(t-t_0)}\, dt \left\{ \left| \int_{t_0}^{t} [1 + \cos 2f]\, e^{i\Omega t'}\, dt' \right|^2 + \left| \int_{t_0}^{t} \sin 2f e^{i\Omega t'}\, dt' \right|^2 \right\},$$

$$\Omega = \omega_\mu - \omega_{mn} - \mathbf{k}_\mu \mathbf{v}. \quad (5.2)$$

Equation (5.2) relates to atoms having a fixed speed and must be averaged over **v**.

In calculating the inner integrals of (5.2) it is convenient to expand $\cos 2f(t')$ and $\sin 2f(t')$ in series analogous to (5.1). Squaring the result of the integration leads to a series of squares of the moduli of the individual terms and a double series of cross products. If $k\bar{v} \gg 2\gamma$ then, as can be shown, the cross terms can be neglected since their contribution is $2\gamma/k\bar{v}$ smaller than the contribution of the squared moduli. Then, (5.2) takes the form

$$w(\mathbf{k}_\mu) = \frac{\gamma_{mn}}{8\pi^2} F(\mathbf{k}_\mu) \left\{ \frac{1 + J_0^2\left(\frac{4G}{\mathbf{kv}}\right) + 2J_0\left(\frac{4G}{\mathbf{kv}}\right) \cos\left(\frac{4G}{\mathbf{kv}} \sin\frac{\eta}{2}\right)}{(\omega_\mu - \omega_{mn} - \mathbf{k}_\mu \mathbf{v})^2 + (2\gamma)^2} + \sum_{\substack{-\infty \\ j \neq 0}}^{\infty} \frac{J_j^2(4G/\mathbf{kv})}{(\omega_\mu - \omega_{mn} - \mathbf{k}_\mu\mathbf{v} + j\mathbf{kv})^2 + (2\gamma)^2} \right\}, \quad (5.3)$$

where J_j is a Bessel function of the first kind.

From (5.3) it is apparent that the line spontaneously emitted from an atom moving in a standing wave field consists of components of dispersive form and width 2γ. The individual components are spaced by equal distances $\mathbf{k}\cdot\mathbf{v}$ from each other and are almost completely separate from each other when the condition $\mathbf{k}\cdot\mathbf{v} \gg 2\gamma$ is fulfilled. It is for this very reason that the cross-product "interference" terms rejected above are of no importance. The intensities of the components depend on the field strength. In particular, when $G \to 0$, the series in (5.3) reduces to zero, $J_0 \to 1$, and (5.3) leads to the usual expression for spontaneous emission.

Equation (5.3) contains a cosine term which depends on the coordinate $\mathbf{R}_0$ of the excited atom: $\eta/2 = \mathbf{k}\cdot\mathbf{R}_0 - \delta$. If the field is not very strong, $4G/\mathbf{k}\cdot\mathbf{v} \ll 1$, then the change in $\mathbf{R}_0$ depends only slightly on $w(\mathbf{k}_\mu)$. If however $4G/\mathbf{k}\cdot\mathbf{v} \sim 1$, then a change in the coordinate $\mathbf{R}_0$ by a fraction of a wavelength, i.e., by a small fraction of its mean free path, changes the amplitude of the first term from $[1 + J_0(4G/\mathbf{k}\cdot\mathbf{v})]^2$ to $[1 - J_0(4G/\mathbf{k}\cdot\mathbf{v})]^2$, i.e., by a large factor. This effect arises only in saturation ($4G \sim \mathbf{k}\cdot\mathbf{v}$) and is produced by interference effects (see Sec. 2).

We now average (5.3) over the velocities and the coordinates. After averaging over η, the numerator of the first term in (5.3) turns out to be equal to $1 + 3J_0^2$. Averaging over $\mathbf{v}$ is more difficult. We observe, first of all, that $w(\mathbf{k}_\mu)$ depends generally speaking, on the mutual orientation of $\mathbf{k}_\mu$ and $\mathbf{k}$, i.e., on the angle between the axis of the generator and the direction along which the spontaneous emission is observed. We consider first the case when $\mathbf{k}_\mu$ is perpendicular to $\mathbf{k}$. Noting that $k\bar{v} \gg 2\gamma$ and $|\mathbf{k}_\mu| \cong |\mathbf{k}| \equiv k$, we obtain from (5.3)

$$w(\mathbf{k}_\mu) = \frac{\gamma_{mn} F(\mathbf{k}_\mu)}{16\pi^{5/2}\gamma k\bar{v}} \Big\{ [1 + 3I_0] \exp\{-(\omega_\mu - \omega_{mn})^2/(k\bar{v})^2\} + \sum_{\substack{-\infty \\ j\neq 0}}^{\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} J_j^2\left(\frac{4G}{kv}\right) \exp\{-[k^2v^2 + (\omega_\mu - \omega_{mn} + jkv)^2]/(k\bar{v})^2\} \frac{dv}{\bar{v}} \Big\}. \quad (5.4)$$

The first term in (5.4) evidently defines a Gaussian line profile with a Doppler width $k\bar{v}$ and a maximum at the frequency ω_{mn}. This part of the line has the same appearance as a spontaneous emission line from an isolated atom. The difference is only in intensity, which in the given case depends on the amplitude of the external field (I_0 has been calculated in Sec. 3).

The terms of the series in (5.3) give line components displaced with respect to $\omega_\mu = \omega_{mn}$. Actually the integral in (5.4) will have its largest value when the maxima of J_j^2 and the exponentials coincide. It is known that J_j is a maximum when its argument is roughly equal to its order, i.e., $4G \cong |jkv|$. Consequently the coincidence referred to will occur when $|\omega_\mu - \omega_{mn}| \cong 4G$. Since this condition is independent of j, all terms of the series will have their maximum values over one and the same frequency range, $\omega_\mu \cong \omega_{mn} \pm 4G$.

It is not difficult to show that the ratio of the integrated intensities of the displaced and undisplaced parts of the line is $(1 - I_0)/(1 + 3I_0)$. For a weak field $I_0 \to 1$, i.e., all the energy is concentrated in the undisplaced component of the line. In the limiting case $4G \gg k\bar{v}$ the integral $I_0 \to 0$, and the intensity ratio is close to unity. Thus, with increasing G, the integral line intensity decreases, the line broadens, and for large enough field, $4G \sim kv$, satellites show up near the frequencies $\omega_\mu = \omega_{mn} \pm 4G$, each with intensity roughly half the intensity of the undisplaced part of the line.

The results obtained can be interpreted in terms of a simple qualitative explanation if the formation of a spontaneous emission line is interpreted as a consequence of transitions between the systems of sublevels in Fig. 3. To each term in (5.3) it is possible to ascribe a transition between the sublevels $E_m - s\hbar\mathbf{k}\cdot\mathbf{v}$ and $E_n - (s - j)\hbar\mathbf{k}\cdot\mathbf{v}$. The central and two displaced (by $\pm 4G$) parts of the line correspond to transitions between groups of the most densely populated sublevels. We emphasise that this description is only a qualitative interpretation which is applicable under the conditions $kv \gg 2\gamma$, since the sublevels s and s′ are generally speaking not independent. This is indeed the cause, in the general case, of the interference terms in the expression for $w(\mathbf{k}_\mu)$ [see the derivation of (5.3)].

We now consider the case when $\mathbf{k}_\mu$ and $\mathbf{k}$ are parallel. Then

$$w(\mathbf{k}_\mu) = \frac{\gamma_{mn}}{8\pi^2} F(\mathbf{k}_\mu) \Big\{ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1 + 3J_0^2(4G/kv)}{(\omega_\mu - \omega_{mn} - kv)^2 + (2\gamma)^2} e^{-v^2/\bar{v}^2} \frac{dv}{\bar{v}} + \sum_{\substack{-\infty \\ j\neq 0}}^{\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{J_j^2(4G/kv)}{(\omega_\mu - \omega_{mn} + (j-1)kv)^2 + (2\gamma)^2} e^{-v^2/\bar{v}^2} \frac{dv}{\bar{v}} \Big\}. \quad (5.5)$$

Equation (5.5) describes the intensity distribution in a line with a total width of the order of $k\bar{v}$. The most notable feature of this distribution is the existence of a narrow peak with a radiative width 2γ and a maximum at ω_{mn}. This peak corresponds to the term $j = 1$.

$$\frac{\gamma_{mn}}{8\pi^2} F(\mathbf{k}_\mu) \frac{I_1}{(\omega_\mu - \omega_{mn})^2 + (2\gamma)^2};$$

$$I_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} J_j^2\left(\frac{4G}{kv}\right) e^{-v^2/\bar{v}^2} \frac{dv}{\bar{v}}. \quad (5.6)$$

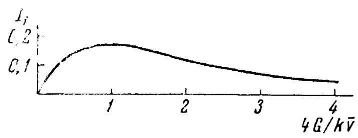

FIG. 4. Dependence of I_1 on $4G/k\bar{v}$.

The production of this narrow line is connected with the mutual compensation of the Doppler shifts $\mathbf{k}_\mu\cdot\mathbf{v}$ and $\mathbf{k}\cdot\mathbf{v}$ when $s - s' = 1$.

The integral I_1, which determines the intensity of the narrow line, tends to zero as $G \to 0$, since the appearance of this line is a strong-field effect. In the limiting cases we have

$$I_1 = \frac{8}{3\pi^{3/2}} \frac{4G}{k\bar{v}} \qquad \left(\frac{4G}{k\bar{v}} \ll 1\right);$$

$$I_1 = \frac{1}{\pi^{3/2}} \frac{k\bar{v}}{4G} \qquad \left(\frac{4G}{k\bar{v}} \gg 1\right). \quad (5.7)$$

Figure 4 shows the general course of I_1 as a function of $4G/k\bar{v}$. The maximum value of I_1 occurs in the region of $4G/k\bar{v} \sim 1$ and is equal to 0.17, which is 7% of the integrated intensity of the whole line. The ratio of the corresponding "peak" intensities will be $0.07\ k\bar{v}/2\gamma$, i.e., it can be very large for $k\bar{v} \gg 2\gamma$. Using (5.7) and (4.2) it is not difficult to show that the "peak" intensity of the narrow line is comparable to the intensity of the broad part when Q exceeds the threshold value by a considerable margin, i.e., the effect can be observed under physically realizable conditions.

We have considered above two limiting cases, spontaneous emission along and perpendicular to the direction of generation. With regard to the radiation in other directions we note the following features. The axial symmetry of the generation and the equivalence of the directions $\mathbf{k}_\mu$ and $-\mathbf{k}_\mu$ result from (5.3). Within the angular range of order $2\gamma/k\bar{v}$ between $\mathbf{k}_\mu$ and $\mathbf{k}$ the narrow line does not change much; on increasing this angle above $2\gamma/k\bar{v}$, the narrow line splits into a doublet, the maxima shift to the positions $\omega_{mn} \pm 4G$, and the line width rapidly increases.

We note that all the analysis is based on the assumption that $F(\mathbf{k}_\mu) = 1$. This assumption has also been intentionally made for directions perpendicular to $\mathbf{k}$. For directions close to the generator axis, however, the function $F(\mathbf{k}_\mu) \neq 1$ and depends on the generator parameters (length, diameter, reflection coefficients of the mirrors, etc.). In each actual case the form of this function must be found by solving the problem of the natural modes in the resonator.

In conclusion we present the equation for the integrated probability of spontaneous emission

$$W_\mu = (\gamma_{mn}/2\gamma)(1 + I_0) = (\gamma_{mn}/\gamma)(1 - \overline{W}).$$

Here, $\overline{W}$ is determined by (5.6).

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