INTERACTION OF FAST PARTICLES WITH LOW FREQUENCY PLASMA OSCILLATIONS

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The effect of low-frequency oscillations of fast particles passing through the plasma is considered. The calculations are performed for arbitrary particle velocities, and in particular ultrarelativistic velocities, and also for arbitrary constant magnetic fields.

1. As is well known, a fast charged particle passing through a plasma loses energy to the excitation of electromagnetic waves in a plasma. In particular, not only high-frequency electron oscillations are excited, but also low frequency oscillations, in which both electrons and ions participate. The loss of energy of the charged particle to the excitation of low-frequency oscillations in an equilibrium plasma is, generally speaking, negligibly small 1) compared with the loss to radiation of high frequency oscillations 2).

In the present paper we consider the passage of a fast charged particle through a plasma in which low-frequency oscillations are excited. In the presence of waves, induced Cerenkov cyclotron emission and absorption of waves by the particle arise in the plasma [3-5]. This interaction increases with the number of waves. Therefore the low-frequency oscillations of the plasma can under certain conditions noticeably affect the interaction between the plasma and the particle. The present paper is devoted to an examination of this effect.

Low-frequency oscillations can be excited in the plasma by a beam of particles [1], by external currents [2,6], simply by an external alternating electromagnetic field, or finally by mechanical or other means. It is assumed that the number of waves in the plasma does not change with time (the spectral density of the oscillations is specified). This assumption can correspond, for example, to two possibilities: a) the intensity of the waves in the plasma is maintained at a fixed level by external factors (sources); b) the relaxation of the waves in the plasma is small (region of transparency for the waves). In the first case the analysis is suitable for all time intervals, whereas in the second it is suitable only for times that are shorter than the relaxation times of the waves.

2. We consider a case of low-frequency oscillations in the presence of a constant external magnetic field, with the following inequalities, which correspond to the region of weak damping of the waves, satisfied:

$$\omega \ll \omega_{Hi}; \quad kv_{Ti} \ll \omega \ll kv_{Te}; \quad kv_{Te} \ll \omega_{He};$$
$$T_e \gg T_i; \quad \omega \ll \omega_{0i}, \tag{1}$$

where ω and k are the frequency and wave vector of the oscillations, ω_{Hi} and ω_{He} are the cyclotron frequencies, while v_{Ti} and v_{Te} are the average thermal velocities of the ions and electrons.

The dielectric tensor $\epsilon_{ij}(\omega, k)$ of the electron-ion plasma has the following form if (1) is satisfied ($\hbar = c = 1$)

$$\varepsilon_{11} = \varepsilon_{22} = 1 + v_A^{-2}, \quad \varepsilon_{33} = -\omega_{0i}^2\omega^{-2}(1 - \omega^2/k^2v_s^2\cos^2\theta),$$
$$\varepsilon_{13} = \varepsilon_{12} = 0, \quad \varepsilon_{23} = -i\omega_{0i}\,\mathrm{tg}\,\theta/\omega v_A, \tag{2*}$$

where θ is the angle between the direction of propagation of the wave and the magnetic field, $\omega_{0i}^2 = 4\pi N_i e_i^2/m_i$ the plasma frequency of the ions, $v_A = H/\sqrt{4\pi N_i m_i}$ the Alfven velocity, and $v_s = \sqrt{T_e/m_i}$ the velocity of sound in a non-isothermal plasma.

The dispersion equations

$$k^2\omega^{-2}\cos^2\theta = \varepsilon_{11}; \quad k^2/\omega^2 = \varepsilon_{22} + \varepsilon_{23}^2/\varepsilon_{33} \tag{3}$$

yield, first, the Alfven wave

$$\omega^2 = k^2\cos^2\theta\, v_A^2/(1 + v_A^2) = k^2v_0^2\cos^2\theta;$$
$$v_0^2 = v_A^2/(1 + v_A^2) \tag{4}$$

1)The intensity of the Cerenkov radiation in a spectral interval $\Delta\omega$ is proportional to $\omega\Delta\omega$.

2)Interest attaches to the generation of low frequency oscillations in a plasma. It is obvious that there are more effective methods for this purpose, for example interaction with beams of charged particles,[1] excitation with currents,[2] etc.

*tg = tan.

and, second, the magnetic-sound oscillations

$$\omega_{\pm}^2 = k^2 v_{\pm}^2;\ v_{\pm}^2 = \frac{v_A^2 + v_s^2(1+v_A^2\cos^2\theta) \pm \{[v_s^2(1+v_A^2\cos^2\theta) - v_A^2]^2 + 4v_A^2 v_s^2 \sin^2\theta\}^{1/2}}{2(1+v_A^2)}. \quad (5)$$

Formulas (4) and (5), which describe (in the absence of collisions) the relativistic magnetohydrodynamic waves, are analogous to those obtained by Khalatnikov [7] within the framework of ordinary hydrodynamics. In the nonrelativistic limit, corresponding to weak magnetic fields $v_A^2 \ll 1$, Eq. (5) leads to the result of Stepanov [8]. If $v_A^2 \lesssim v_s^2$, the nonisothermal behavior is essential if the damping is to be small. When $v_s^2 \ll v_A^2$, the nonisothermal behavior is of no importance; in particular, when $v_s = 0$ formula (5) describes waves also if the inequality satisfied is $\omega \ll kv_{Te}$, which is the inverse of the inequality used to derive (2).

3. Let us consider in the general case a relativistic charged particle moving along a helical line in a plasma. We describe its state by a set of quantum numbers q, viz., p_z—the projection of the particle momentum on the field, $\mu = \pm 1$—the projection of the spin on the field, $x_0 = p_y/eH$—the projection of the center of the Larmor "circle" on the x axis, and the energy

$$\varepsilon_q = \sqrt{p_z^2 + p_\perp^2 + m^2}, \qquad p_\perp^2 = eH(2n - \mu + 1),$$

$$n = 0, 1, 2, \ldots$$

Let $N_{\omega k}$ be the number of waves (quanta) of frequency ω and momentum $\mathbf{k}$, and let $w_q^{q'}(\omega, \mathbf{k})$ be the probability of spontaneous emission of waves per unit time with transition from a state characterized by a set q into a state with set q′.

The decrease in the particle energy is due to the induced emission of the waves, the probability of which is proportional to $w_q^{q'}(\omega, \mathbf{k})(N_{\omega,\mathbf{k}} + 1)$, while the increase in energy is due to induced absorption, proportional to $N_{\omega,\mathbf{k}}$. We denote by $d\epsilon_q/dt$ the change in particle energy per unit time. We have[3)]

$$\frac{d\varepsilon_q}{dt} = -\int_0^\infty \omega d\omega \int d\mathbf{k} \sum_{\nu,\mu'} \{w_{p_z,n,\mu}^{p_z-k_z,n+\nu,\mu'}(\omega,\mathbf{k})(N_{\omega,\mathbf{k}}+1) - w_{p_z+k_z,n-\nu,\mu'}^{p_z,n,\mu}(\omega,\mathbf{k}) N_{\omega,\mathbf{k}}\}. \quad (6)$$

In the term of the right half of (6) connected with the spontaneous emission we can neglect, with good accuracy, the quantum corrections. Taking into account the smallness of the momentum transfer as compared with the particle momentum, we can confine ourselves in the expression for the effect of the waves on the particle (proportional to N) to the first term of the expansion in the momentum transfer:

$$\delta p_z = k_z; \qquad \delta p_\perp^2 = 2eH\nu = 2\omega_H \varepsilon_q \nu; \qquad \omega_H = eH/\varepsilon_q. \quad (7)$$

As a result we obtain

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{ind}} = \int_0^\infty \omega d\omega \int d\mathbf{k} \sum_{\nu=1}^\infty \left(k_z \frac{\partial}{\partial p_z} + \frac{\omega_H \nu}{v_\perp}\frac{\partial}{\partial p_\perp}\right) \times N_{\omega,\mathbf{k}} w_{\nu,p_z,p_\perp}(\omega,\mathbf{k}), \quad (8)$$

where $w_{\nu,p_z,p_\perp}$ is the classical probability. In (7) and (8) allowance is made for the fact that the change in the spin direction is a quantum effect of second order ($\sim\hbar^2$).

The classical probability $w_{\nu,p_z,p_\perp}$ for a particle moving in a plasma along a helical line was calculated by Ginzburg and Éĭdman [10]. The Hamiltonian formalism in [10], however, is too cumbersome for the derivation of general expressions in media with spatial dispersion. A simpler way is to solve directly Maxwell's equations with the right half describing the motion of the charge along the helical line, with subsequent calculation of the work performed by the forces (for the simplest isotropic case see [9]). We obtain

$$w_{\nu,p_z,p_\perp}(\omega,\mathbf{k}) = \tfrac{1}{2}e^2(2\pi)^{-3}\Gamma_i^{\nu,p_z,p_\perp}(\Gamma_j^{\nu,p_z,p_\perp})^* \varepsilon_q^{-2} \times D''_{ij}(\omega, k_\perp, k_z)\,\delta(\omega - \omega_H\nu - k_z v_z); \quad (9)$$

$$\Gamma_1^{\nu,p_z,p_\perp} = 2p_\perp \nu J_\nu(z)/z; \quad \Gamma_2^{\nu,p_z,p_\perp} = -2ip_\perp J'_\nu(z); \quad (10)$$

$$\Gamma_3^{\nu,p_z,p_\perp} = -2p_z \operatorname{sign} k_x J_\nu(z); \quad v_\perp = p_\perp/\varepsilon_q; \quad v_z = p_z/\varepsilon_q; \quad (11)$$

$$\varepsilon_q = \sqrt{m^2 + p_\perp^2 + p_z^2}; \quad z = k_\perp v_\perp/\omega_H; \quad J'_\nu = dJ_\nu(z)/dz, \quad (12)$$

J_n is the Bessel function. In (9) we used the fact that by virtue of the symmetry of the problem the probability depends only on $k_\perp$ and k_z, so that we can put without loss of generality $k_y = 0$ and $k_x = k_\perp \operatorname{sign} k_x$.

The vertex $D''_{ij}(\omega, \mathbf{k})$ is the antihermitian part of the retarded Green's function of the electromagnetic field $D_{ij}(\omega, \mathbf{k})$, satisfying the equation

$$(k^2\delta_{ij} - k_i k_j - \omega^2 \varepsilon_{ij}(\omega, \mathbf{k})) D_{jl}(\omega, \mathbf{k}) = 4\pi\delta_{il}. \quad (13)$$

Using expressions (2) for the dielectric tensor ϵ_{ij} and putting, by virtue of the remark made,

3)The term with absorption is written out here from considerations of detailed balancing.

$k_y = 0$ and $k_x = k_\perp \operatorname{sign} k_x$, we obtain $D_{12} = D_{13} = D_{23} = D_{33} = 0$,

$$D_{11} = \frac{4\pi}{k_z^2 - \omega^2\varepsilon_{11}}, \tag{14}$$

$$D_{12} = \frac{4\pi(1-\omega^2/v_s^2k_z^2)}{k_\perp^2[1-\omega^2k_z^{-2}(v_A^{-2}+v_s^{-2})]+(k_z^2-\omega^2\varepsilon_{11})(1-\omega^2/k_z^2v_s^2)}. \tag{15}$$

Expressions (14) and (15) were derived under the assumption that ω and $k_\perp \ll \omega_{0i}$. To obtain the imaginary parts of D''_{ij} which enter into (7), it is necessary to replace the denominators by delta-functions. For example,

$$D''_{11} = 4\pi^2\delta(k_z^2 - \omega^2\varepsilon_{11}). \tag{16}$$

The change in the particle momentum parallel and perpendicular to the field is obtained from (6) by replacing ω in the integrand in accordance with the following scheme:

$$\begin{pmatrix}\varepsilon_p\\ p_z\\ p_\perp\end{pmatrix} \to \begin{pmatrix}\omega\\ k_z\\ -\omega_H\nu/v_\perp\end{pmatrix}.$$

4. The probabilities written out above were used to calculate the spontaneous emission of low-frequency waves by a charged particle moving along a helical line, and the induced emission and radiation of waves. The results were obtained assuming the velocities $v_\perp$ and v_z to be small compared with the velocity of light.

We do not write out here the results for the induced emission, since the contribution of the low-frequency region to the total deceleration of the particle is small. We note merely that when $v_z \to v_0 = v_A/\sqrt{1+v_A^2}$ the excited frequencies can increase appreciably and go out of the low-frequency region. It is then necessary to use in lieu of (2) a more accurate expression for the dielectric tensor. For the induced processes the situation is different, since the essential region of the frequencies is determined by the non-vanishing of the radiation density (wave density). We consider below only a case in which the waves are present in the low-frequency region.

For the action of the waves on the charged particles we obtain, carrying out the differentiation contained in (8) and using (9)—(12) (a similar result is obtained by the previously employed expansion[5])

$$\frac{d\varepsilon_q}{dt} = -\sum_{\nu=-\infty}^{\infty}\int_0^\infty \omega d\omega \frac{e^2}{4\pi^2 v_z}\int_0^\infty dk_\perp^2 \frac{1}{2\varepsilon_q^2}\Big\{(\Gamma_{i0}\delta\Gamma_{j0}^* + \delta\Gamma_{i0}\Gamma_{j0}^*)D''_{ij0}\overline{N}_{\omega,k_{z0},k_\perp} + \frac{k_{z0}}{p_z}D''_{ij0}\Gamma_{i0}\Gamma_{j0}^*\overline{N}_{\omega,k_{z0},k_\perp} + \frac{k_{z0}^2-\omega^2}{2p_z}\frac{d}{dk_{z0}}(D''_{ij0}\Gamma_{i0}\Gamma_{j0}^*\overline{N}_{\omega,k_{z0},k_\perp})\Big\}, \tag{17}$$

where $\overline{N}$ is the number of quanta $N_{\omega,k}$ averaged over φ:

$$\overline{N}_{\omega,k_z,k_\perp} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi N_{\omega,k_z,k_\perp}, \tag{18}$$

and

$$k_{z0} = \frac{\omega+\nu\omega_H}{v_z}, \quad D_{ij0} = D_{ij}(\omega, k_{z0}, k_\perp), \tag{19}$$

$$\Gamma_{i0} = \Gamma_i(\omega, k_{z0}, k_\perp), \quad \delta\Gamma_{i0} = \delta\Gamma_i(\omega, k_{z0}, k_\perp), \tag{20}$$

$$\delta\Gamma_1 = \omega_H\nu^2 J'_\nu(z)/v_\perp, \quad \delta\Gamma_2 = -i\left(\frac{\omega_H}{v_\perp}\right)(\nu^2 - z^2)\nu J_\nu(z)/z. \tag{21}$$

5. Let us consider first the question of the effect of Alfven waves on a charged particle. The Green's function of the electromagnetic field for Alfven waves is given by (14) and (16). Substituting (16) in (17)—(21), we obtain after simple transformations[4)]

$$\left(\frac{d\varepsilon_q}{dt}\right)_A = \frac{e^2\omega_H^3 v_0}{\varepsilon_q}\sum_{\nu=1}^{\nu_{max}}\nu^3\int_0^{z_{max}} J_\nu^2(z)\,dz\Big\{\frac{1}{1-v_z/v_0}\Big[\frac{2\overline{N}_1(1/v_0^2-1)}{z|1-v_z/v_0|^2} + \frac{\omega_H\nu(1/v_0^2-1)}{z|1-v_z/v_0|^3}\frac{d\overline{N}_1}{d\omega} - \frac{1}{v_\perp^2}\frac{d\overline{N}_1}{dz}\Big] + \frac{1}{1+v_z/v_0}\Big[\frac{2\overline{N}_2(1/v_0^2-1)}{z|1+v_z/v_0|^2} + \frac{\omega_H\nu}{z|1+v_z/v_0|^3}\left(\frac{1}{v_0^2}-1\right)\frac{d\overline{N}_2}{d\omega} - \frac{1}{v_\perp^2}\frac{d\overline{N}_2}{dz}\Big]\Big\}. \tag{22}$$

We introduce here the notation

$$\overline{N}_{1,2}(\omega, z) = \overline{N}_{\omega,k_z,k_\perp} \text{ for } \omega = \omega_{1,2}, \quad k_z = \varepsilon_{1,2}\omega_{1,2}/v_0, \quad k_\perp = \omega_H z/v_\perp, \tag{23}$$

$$\varepsilon_{1,2} = \begin{cases}1 & \text{for } 1\\ -1 & \text{for } 2,\end{cases} \quad \omega_1 = \frac{\omega_H\nu}{|1-v_z/v_0|}, \quad \omega_2 = \frac{\omega_H\nu}{|1+v_z/v_0|}. \tag{24}$$

In (22) the integration limits z_{max} are determined by the non-vanishing of the radiation density. The investigated frequencies correspond to (24) and are obtained from the energy conservation law $k_z = (\omega + \nu_H)/v_z$ and from the dispersion equation of the Alfven waves $k_z^2 = \omega^2/v_0^2$. These frequencies increase with increasing ν. However, one need not worry about whether these frequencies enter in the low-frequency region, since $\overline{N}$ vanishes if this does not take place. We therefore need not put z_{max} and ν_{max} in (22). In exactly the same manner, formulas (22) vanish as $v_z \to v_0$, because

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4)In the derivation of (22) we used integration by parts, assuming that the results of the substitution vanish when $z = z_{max}$.

ω_1 enters the high-frequency region in which, by definition, $\overline{N} = 0$.

Let us emphasize the general nature of this result. In the derivation of (22) we impose no limitations whatever on the values of $v_\perp$ and v_z, which can, in particular, be as close to the velocity of light as desired (we note that the velocity of the Alfven waves v_0 approaches the velocity of light if $v_A \to \infty$). In order for the Alfven waves to act on a charged particle, it is necessary that the frequencies (24) enter the low-frequency region $\omega \ll \omega_{Hi}$. To be specific, let us consider a hydrogen plasma. It is then necessary that a) $\omega_H \nu \ll \omega_{Hi}$, if v_z and v_0 are of the same order of magnitude (that is, this condition should be satisfied at least for the first harmonic $a = 1$), that is, the mass of the particle should appreciably exceed the ion mass, or, b) $v_z \gg v_0$.

In the region of strong magnetic fields or, what is the same, low plasma density, when $v_A \gg 1$, we can realize only the first condition, that is, the waves interact only with the heavy ions. The force of interaction increases with increasing ion charge.

Let us consider some particular cases.

A. Let $\overline{N}$ be independent of ω, $k_\perp$, or k_z in the low-frequency region

$$\overline{N}_{1,2} = \overline{N}_A. \tag{25}$$

Then

$$\left(\frac{d\varepsilon_q}{dt}\right)_A = \frac{2e^2\omega_H^3}{\varepsilon_q} v_0 \overline{N}_A \sum_{\nu=1}^{\nu_{max}} \nu^3 \left\{\frac{2(1+3v_z^2/v_0^2)}{(1-v_z^2/v_0^2)^3}\left(\frac{1}{v_0^2}-1\right)\int_0^\infty \frac{J_\nu^2(z)}{z}dz\right\}. \tag{26}$$

When $v_z > v_0$ the particle is slowed down by the waves, whereas when $v_z < v_0$ it is accelerated[5]. Consequently, by virtue of the statements made above, only heavy ions can be accelerated. It follows from (26) that

$$\left(\frac{d\varepsilon_q}{dt}\right)_A = \begin{cases} \dfrac{-e^2\omega_H^3}{\varepsilon_q v_z^2}\dfrac{v_A^3}{v_z^2}\overline{N}_A \nu_{max}(\nu_{max}+1)(2\nu_{max}+1) \text{ for } v_z \gg v_0, & (27)\\ \dfrac{1}{3}\dfrac{e^2\omega_H^3 v_0}{\varepsilon_q v_A^2}\overline{N}_A \nu_{max}(\nu_{max}+1)(2\nu_{max}+1) \text{ for } v_z \ll v_0. & (28)\end{cases}$$

In (28) ν_{max} for a hydrogen plasma is of the order of $\omega_{Hi}/\omega_H \sim A/Z$, where A is the atomic number, that is, the accelerating force is of the order of

$$(d\varepsilon_q/dt)_A \sim 2Z^{-1}q^2\omega_H^3 A^3 \overline{N}_A v_0/3\varepsilon_q v_A^2, \tag{29}$$

where $e = Zq$ and q is the elementary charge. Assuming $Z \sim A$ and recognizing that $\omega_H \sim 1/A$ for the energies of nonrelativistic or not ultra-relativistic energies, we have $(d\epsilon_q/dt)_A \sim A$. This conclusion, which shows that the accelerating force increases with increasing atomic number, is important for questions involving the origin of cosmic rays.

B. Let us consider now a case when the number of waves decreases rapidly with increasing frequency, namely

$$\frac{1}{N_1}\frac{\omega_H \nu}{|1-v_z/v_0|}\left|\frac{d\overline{N_1}}{d\omega}\right| \gg 1. \tag{30}$$

We then leave in (22) only a term containing $d\overline{N}_1/d\omega$. Putting

$$\frac{d\overline{N_1}}{d\omega} = -\overline{N}', \tag{31}$$

we obtain

$$\left(\frac{d\varepsilon_q}{dt}\right)_A = \begin{cases} \dfrac{e^2\omega_H^3 v_A^3}{\varepsilon_q v_z^4}\nu_{max}^2(\nu_{max}+1)^2\left(\dfrac{\overline{N}'\omega_H v_A}{v_z}\right) \text{ for } v_z \gg v_0, & (32)\\ -\dfrac{1}{4}\dfrac{e^2\omega_H^3 v_0}{\varepsilon_q v_A^2}\nu_{max}^2(\nu_{max}+1)^2(\overline{N}'\omega_H) \text{ for } v_z \ll v_0. & (33)\end{cases}$$

With increasing frequency the number of waves should decrease. It may be of interest to determine when the acceleration condition $v_z < v_0$, corresponding to (26), changes into $v_z > v_0$ which corresponds, for example, to (31) in the class of the power-law wave distribution $\overline{N} \sim 1/\omega^q$. We can readily obtain the answer from (22), namely $q > 2$.

6. Let us consider further the effect of magnetic-sound waves on a charged particle. From (17)—(21) and (15) we obtain

$$\begin{aligned}\left(\frac{d\varepsilon_q}{dt}\right)_m = &-\sum_{\nu=-\infty}^{\infty}\frac{2e^2}{v_z\varepsilon_q}\int_0^{\omega_{max}}\omega d\omega\left|\frac{\alpha_s}{\alpha_{As}}\right|\Bigg\{\Bigg[\omega_H(\nu^2-z^2)\frac{\nu J_\nu'(z)J_\nu(z)}{z}\\ &+\frac{v_\perp^2}{v_z}(J_\nu'(z))^2\left[\frac{k_{z0}^2-\omega^2}{k_{z0}}\left(\frac{1}{\alpha_s}-\frac{1}{\alpha_{As}}\right)+k_{z0}\right]\\ &+\frac{v_\perp^2}{v_z}\frac{k_{z0}^2-\omega^2}{k_{z0}}J_\nu'(z)z\\ &\times J_\nu''(z)\left(\frac{1}{\alpha_0}+\frac{1}{\alpha_s}-\frac{1}{\alpha_{As}}\right)\Bigg]N_{\omega,k_{z0},k_{\perp 0}}\\ &+\frac{v_\perp^2}{v_z}\frac{k_{z0}^2-\omega^2}{2}[J_\nu'(z)]^2\Bigg[\frac{d}{dk_{z0}}N_{\omega,k_{z0},k_{\perp 0}}\\ &+\frac{k_{\perp 0}}{k_{z0}}\left(\frac{1}{\alpha_0}+\frac{1}{\alpha_s}-\frac{1}{\alpha_{As}}\right)\frac{d}{dk_{\perp 0}}N_{\omega,k_{z0},k_{\perp 0}}\Bigg\},\end{aligned} \tag{34}$$

$$\alpha_0 = 1-\omega^2/k_{z0}^2v_0^2; \quad \alpha_s = 1-\omega^2/k_{z0}^2v_s^2; \quad \alpha_{As} = 1-\omega^2/k_{z0}^2v_{As}^2. \tag{35}$$

Here $z = k_{\perp 0}v_\perp/\omega_H$, k_{z0} is given by (19), and the value of $k_{\perp 0}$ is

[5]We note that there is no Cerenkov effect here ($\nu \neq 0$).

$$k_{\perp 0}^2 = -k_{z0}^2 \alpha_0 \alpha_s / \alpha_{As}; \quad k_{z0} = (\omega + \nu\omega_H)/v_z, \tag{36}$$

with

$$v_{As}^{-2} \equiv v_A^{-2} + v_s^{-2}. \tag{37}$$

The limits of integration over the frequencies in (34) are determined by the radiation condition $k_{\perp 0}^2 > 0$.

The result in (34) holds true for all $v_\perp$ and v_z. For (34) we can make the same remarks as in the preceding section (in particular, the resonant denominators cannot be large).

For a cold plasma in the limit as $v_s \to 0$ we have

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{m}} = -\sum_{\nu=-\infty}^{\infty} \frac{2e^2}{v_z \varepsilon_q} \int\limits_{k_{\perp 0}^2>0}^{\omega_{max}} \omega d\omega \left\{\left[\omega_H (\nu^2 - z^2) \frac{\nu J_\nu'(z) J_\nu(z)}{z} + \frac{v_\perp^2 k_{z0}}{v_z} J_\nu'^2(z) - \frac{v_\perp^2}{v_z} k_{z0} \frac{1-\omega^2/k_{z0}^2}{1-\omega^2/k_{z0}^2 v_0^2} J_\nu'^2(z) + \frac{v_\perp^2}{v_z} k_{z0} \frac{1-\omega^2/k_{z0}^2}{1-\omega^2/k_{z0}^2 v_0^2} \times (\nu^2 - z^2) \frac{J_\nu(z) J_\nu'(z)}{z}\right] \overline{N}_{\omega, k_{z0} k_{\perp 0}} + \frac{v_\perp^2}{2v_z} \frac{(k_{z0}^2 - \omega^2)}{k_{z0}} [J_\nu'(z)]^2 \times \left[k_{z0} \frac{d}{dk_{z0}} \overline{N}_{\omega, k_{z0}, k_{\perp 0}} + k_{\perp 0} \alpha_0^{-1} \frac{d}{dk_{\perp 0}} \overline{N}_{\omega, k_{z0}, k_{\perp 0}}\right]\right\}. \tag{38}$$

Here $k_{\perp 0}^2 = k_{z0}^2(\omega^2/v_0^2 k_{z0}^2 - 1)$; $z = k_{\perp 0} v_\perp / \omega_H$.
The limits of integrations are:

from ω_+ to ω_- when $v_z < v_0$ and $\nu < 0$;

the integral vanishes for $\nu > 0$;

from ω_+ to ω_{max} for $v_z > v_0$, $\nu < 0$ and from ω_- to ω_{max} $[\omega_\pm = -\nu\omega_H/(1 \pm v_z/v_0)]$ when $\nu > 0$.

We note that there is no singularity in (38) when $\omega^2 = k_{z0}^2 v_0^2$.

Let us consider some particular cases.

A. An isotropic radiation density distribution which does not depend on $k_\perp$, k_z, and ω vanishes when $\omega > \omega_{max}$, provided $v_z < z_0$. Let us assume that ν_{max} bands (ω_+, ω_-) fit in the range up to ω_{max}, that is,

$$\frac{\omega_H \nu_{max}}{1 - v_z/v_0} < \omega_{max}; \qquad \frac{\omega_H(\nu_{max}+1)}{1 + v_z/v_0} > \omega_{max}.$$

We then obtain

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{m}} = \sum_{\nu=1}^{\nu_{max}} \frac{4e^2 \overline{N}_{\text{m}}}{\varepsilon_q v_0} \frac{\omega_H^3 \nu^3}{(1 - v_z^2/v_0^2)^2} \int\limits_0^{\pi/2} \left(1 - \frac{v_z}{v_0} \cos 2\theta\right) \sin 2\theta \, d\theta \times \left\{\frac{\nu v_0 (1 - v_z^2/v_0^2 - v_\perp^2 \sin^2 2\theta / v_0^2)}{v_\perp \sin 2\theta \sqrt{1 - v_z^2/v_0^2}} J_\nu\left(\nu \frac{v_\perp}{v_0} \frac{\sin 2\theta}{\sqrt{1 - v_z^2/v_0^2}}\right) \times J_\nu'\left(\nu \frac{v_\perp}{v_0} \frac{\sin 2\theta}{\sqrt{1 - v_z^2/v_0^2}}\right) \left[1 + \frac{v_\perp^2 (1 - v_0^2)}{v_z v_0} \frac{(v_z/v_0 - \cos 2\theta)^3}{(1 - v_z^2/v_0^2)^2 \sin^2 2\theta} - \frac{v_\perp^2 v_0}{v_z} \frac{(v_z/v_0 - \cos 2\theta)}{1 - v_z^2/v_0^2}\right] - \frac{(1 - v_0^2) v_\perp^2 (v_z/v_0 - \cos 2\theta)(1 - v_z \cos 2\theta / v_0)^2}{v_z v_0 (1 - v_z^2/v_0^2)^2 \sin^2 2\theta} \times \left[J_\nu'\left(\nu \frac{v_\perp}{v_0} \frac{\sin 2\theta}{\sqrt{1 - v_z^2/v_0^2}}\right)\right]^2\right\}. \tag{39}$$

The result of integration of (39) over the angles is cumbersome and contains indefinite integrals of Bessel functions of index $2(\nu + \alpha)$, where $\alpha = 0$, 1, 2, 3, and 4[6]. To examine particular cases it is simpler to use (39) directly. We thus find that for $v_\perp \ll v_0$

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{m}} = \frac{e^2 \overline{N}_{\text{m}} \omega_H^3}{\varepsilon_q v_0} \left(1 - \frac{v_z^2}{v_0^2}\right)^{-2}; \tag{40}$$

and for $v_\perp \gg v_0$

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{m}} = \frac{4e^2 \omega_H^3 \overline{N}_{\text{m}}}{\varepsilon_q v_0} \frac{v_\perp^2}{v_A^2} \nu_{max}^2 (\nu_{max} + 1)^2 \left(1 - \frac{v_z^2}{v_0^2}\right)^{-4}. \tag{41}$$

Now assumptions were made in (40) and (41) regarding the smallness of $v_\perp$ and v_z with respect to the velocity of light. For $v_\perp > v_0$, of course, $v_0 \ll 1$. Under the conditions corresponding to (29), the accelerating force is proportional to A when $Z \sim A$. We note that the result obtained is valid when ν_{max} is not smaller than unity.

If the plasma density tends to zero, $\nu_{max} \sim v_A^{-1}$, then the low-frequency region narrows down also at a definite density $\nu_{max} < 1$, that is, even the first band will not fit the low-frequency region.

B. If $v_z > v_0$, then we obtain from (38) for radiation which is isotropic in the frequency range $0 < \omega < \omega_{max}$

$$\left(\frac{d\varepsilon_q}{dt}\right)_{\text{m}} = -\sum_{\nu=1}^{\nu_{max}} \frac{2e^2 \omega_H^3 \nu^3}{v_0 \varepsilon_q} \frac{\overline{N}}{(v_z^2/v_0^2 - 1)^2} \int\limits_0^{\theta_{max}^+} \operatorname{sh}\theta \, d\theta \times \left(\frac{v_z}{v_0} \operatorname{ch}\theta - 1\right) \left\{\frac{-\nu v_0 (v_z^2/v_0^2 - 1 - v_\perp^2 \operatorname{sh}^2\theta / v_0^2)}{v_\perp \operatorname{sh}\theta \sqrt{v_z^2/v_0^2 - 1}} \times J_\nu\left(\nu \frac{v_\perp \operatorname{sh}\theta}{v_0 \sqrt{v_z^2/v_0^2 - 1}}\right) J_\nu'\left(\nu \frac{v_\perp}{v_0} \frac{\operatorname{sh}\theta}{\sqrt{v_z^2/v_0^2 - 1}}\right) \times \left[1 + \frac{v_\perp^2}{v_z v_0} \frac{(v_z/v_0 - \operatorname{ch}\theta)}{\operatorname{sh}^2\theta (v_z^2/v_0^2 - 1)^2} \left(v_0^2 \left(\frac{v_z^2}{v_0^2} - 1\right) \times \operatorname{sh}^2\theta - (1 - v_0^2)\left(\frac{v_z}{v_0} - \operatorname{ch}\theta\right)^2\right)\right]\right.$$

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[6]In analogy with the Schott formula, where $\alpha = 0$.

$$+\frac{v_\perp^2}{v_0 v_z}\frac{(1-v_0^2)(\text{ch}\,\theta - v_z/v_0)(1-\text{ch}\,\theta v_z/v_0)^2}{\text{sh}^2\,\theta\,(v_z^2/v_0^2-1)^2}$$

$$\times\left[J'_\nu\left(\nu\frac{v_\perp}{v_0}\frac{\text{sh}\,\theta}{\sqrt{v_z^2/v_0^2-1}}\right)\right]^2+\ldots\Bigg\}, \quad (42)^*$$

where

$$\theta^{\pm}_{max}=\ln\left[\frac{\omega_{max}}{\omega_H\nu}\left(\frac{v_z}{v_0}-\frac{v_0}{v_z}\right)\pm\frac{v_0}{v_z}\right.$$

$$\left.+\left\{\left[\frac{\omega_{max}}{\omega_H\nu}\left(\frac{v_z}{v_0}-\frac{v_0}{v_z}\right)\pm\frac{v_0}{v_z}\right]^2-1\right\}^{1/2}\right]. \quad (43)$$

It is necessary to add also to the expression in the curly bracket in (42) the terms obtained from those written down by replacing $\theta^+_{max}\to\theta^-_{max}$, $v_z\to -v_z$.

Thus, when $v_\perp \ll v_0$ and $v_\perp \ll v_z\omega_H/\omega_{max}\sqrt{v_z^2/v_0^2-1}$, $\omega_{max}/\omega_H \gg 1$, we obtain a decelerating force

$$\left(\frac{d\varepsilon_q}{dt}\right)_m \approx -\frac{2e^2\omega_H^2\omega_{max}}{\varepsilon_q v_z(v_z^2/v_0^2-1)}\overline{N}_m. \quad (44)$$

An account of the thermal motion, as can be seen from (34) causes in first approximation the need for adding to the obtained results expressions that differ from those obtained by a factor $v_s^2/v_A^2 = 4\pi n_i T_e/H^2$. The correction is of the order of the ratio of the average energy density of thermal motion of the electrons to the energy density of the magnetic field.

7. Let us illustrate the significance of the obtained results by means of some estimates. It was noted above that the preferred acceleration of the ions with large atomic numbers [7] by both Alfven waves [see (28)] and by magnetic-sound waves [see (41)] can be of interest in the elucidation of the origin of cosmic rays, for which, as is well known [11, 12], the abundance of the heavy elements is much higher than in nature.

It is more convenient to deal in estimates with the average radiation density ρ, rather than with the number of waves $\overline{N}$.

The values of ρ_A under cosmic conditions are quite varied. One can make a crude estimate of ρ_A, the average energy of the wave per unit volume, by assuming this quantity to be of the order [8] of the average energy $nm_i v^2_{hydr}$ of hydrodynamic motion. In order for acceleration to take place, it is necessary that the accelerating force exceed the decelerating force due to the ionization losses [9]. When $v_z \sim z_A \ll 1$, we obtain $v^2_{hydr} \gtrsim \omega_{0i}e^2\nu_{max}/m_e v_A^2$. For $v_A \sim v_{hydr} \sim 10^{-4}$ we have $n < 10^{10}\nu_{max}^{-2}$ cm^{-3}. If the conditions $v_A \gg 1$, $\epsilon_q^2 \ll m^2 v_A^2$, $\epsilon_q^2 \gg m^2$, and $\epsilon_q^2 \ll m^2 v_\perp^2$ are satisfied, the accelerating force may be strongly dependent on the energy:

$$\left(\frac{d\varepsilon_q}{dt}\right)_m \approx \frac{4\pi^2}{3}\frac{q^2Z^2}{m_i\nu^0_{max}}\varepsilon_q^3\rho_A\frac{1}{\omega_{Hi}m^3}. \quad (45)$$

We took into account here the fact that $\nu_{max} \sim \nu^0_{max}/\epsilon_q$, where ν^0_{max} does not depend on the particle energy.

For a magnetic-sound wave we put

$$\rho_m = \frac{\overline{N}_m\omega_H^4\nu_{max}^4}{16\pi^2|1-v_z/v_0|^4 v_0^3}. \quad (46)$$

In place of (41) we get when $\nu_{max} \gg 1$

$$\left(\frac{d\varepsilon_q}{dt}\right)_m = \frac{64\pi^2q^2Z^2}{m_i\omega_{Hi}}\frac{v_\perp^2}{(1+v_z/v_0)^4}\rho_m. \quad (47)$$

When $v_\perp \sim v_A$ we obtain the same orders of estimated magnitudes as above. We note that for small v_z the accelerating force is proportional to the particle energy $mv_\perp^2$. In the ultrarelativistic limit under the conditions corresponding to (45), we obtain from (40) the energy-independent force

$$\left(\frac{d\varepsilon_q}{dt}\right)_m \approx \frac{Z^2q^2\pi^2}{m_i\omega_{Hi}}\frac{\rho_m}{(\nu^0_{max})^4}. \quad (48)$$

The estimated acceleration time for $v_A \sim 10^{-4}$, $n \sim 1$ cm^{-3} is $\sim 10^3 - 10^4$ years. This estimate corresponds to the minimum time (maximum radiation density).

8. In conclusion let us note the following two circumstances.

A. The acceleration condition $v_z < v_0$ [10] shows that in regions with large magnetic field, where $v_0 \to 1$, the relativistic particles are formed in larger numbers. The latter requirement means that

$$v_A^2 = H^2/4\pi n_i m_i c^2 \gg 1. \quad (49)$$

B. The condition $v_z < v_0$ does not impose a

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*ch = cosh; sh = sinh.

[7]The deduction that the heavy ions have preferred acceleration is the consequence of the fact that more overtones of the gyrofrequency are contained in the low-frequency region for the heavy ions.

[8]The assumption made is optimal and ρ_A is frequently much less.

[9]The heavy-ion radiation losses can be neglected.

[10]If this condition is not satisfied, then when the number of fast particles is sufficiently large the action of the waves on them leads to such a distribution of the waves over the frequencies and the angles, which corresponds to the tendency of the acceleration to occur (see Sec. 5).

limitation on the particle energy, since $v_\perp$ can be arbitrary. If the region in which the magnetic field is present is bounded then, generally speaking, particles with large values of v_z drift out of this region. Consequently a situation is possible when the particles which are retained by the given region are accelerated.

Finally, it must be noted that with increasing ρ the acceleration at the low-frequency waves goes over into ordinary Fermi acceleration and acceleration by means of high-frequency oscillations [4,5] may turn out to be more effective.

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