EFFECT OF COLLISIONS ON THE DISTURBANCES AROUND A BODY MOVING IN A PLASMA

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Formulas for the Fourier components n_q of the disturbances of the electron density around a body moving in a plasma in a magnetic field are deduced in various limiting cases by taking into account collisions of the ions with one another and with other particles. The calculation is performed on the basis of kinetic equations with exact collision integrals. The relation between the expressions for n_q and the plasma dielectric constant is found by taking into account spatial dispersion.

1. FORMULATION OF THE PROBLEM

THE present work is devoted to an account of the influence of collisions of ions with ions and with other particles on the perturbations of the electron and ion density around a body moving in a plasma. In earlier papers [1,2] the authors calculated the Fourier components of the ion-density or electron-density perturbations n_q for the case when

$$qR_0 \ll 1 \qquad (1)$$

(R_0 is the characteristic dimension of the body). The plasma was assumed rarefied, that is, the mean free path of the ions was assumed to be sufficiently large,

$$l \gg R_0. \qquad (2)$$

By virtue of this we neglected in [1] the effect of collisions on n_q. In this case the Fourier component of the perturbation of the ion distribution function satisfies in the absence of a magnetic field the equation

$$i\mathbf{q}(\mathbf{u}-\mathbf{V}_0) f_\mathbf{q} + (e/T) f_0(u)\, i\mathbf{qu}\varphi_\mathbf{q} = I(\mathbf{u}), \qquad (3)$$

where $f_0(u) = n_0(M/2\pi T)^{3/2}\exp(-Mu^2/2T)$, M is the ion mass, $\mathbf{V}_0$ the velocity of the body, n_0 the unperturbed ion density, $\mathbf{u}$ the ion velocity, T the plasma temperature measured in energy units, and $I(\mathbf{u})$ the so-called "integral for the collisions between the ions and the body," which takes into account the scattering of the ions by the surface of the body and by the electric field surrounding it.

Since the velocity of the body is small compared with the average thermal velocity of motion of the electrons, a Boltzmann electron distribution can be assumed. In addition, at low values of q it follows from the Poisson equation that the ion density is equal to the electron density, accurate to terms $\sim(Dq)^2$ (D is the Debye radius in the plasma). Therefore,

$$n_\mathbf{q} = \int f_\mathbf{q} d^3u = n_0 e\varphi_\mathbf{q}/T. \qquad (4)$$

In a magnetic field it is necessary to add to the left half of (3) a term with Lorentz force, so that the equation assumes the form

$$\frac{\partial f_\mathbf{q}}{\partial \mathbf{u}}\frac{e}{Mc}[\mathbf{uH}] + i\mathbf{q}(\mathbf{u}-\mathbf{V}_0) f_\mathbf{q} + \frac{e}{T} f_0(\mathbf{u})\, i\mathbf{qu}\varphi_\mathbf{q} = I(\mathbf{u}). \qquad (5)^*$$

Equation (4) remains the same.

As was already mentioned, no account was taken of the ion collisions in (4) and (5). To be able to do this, however, condition (2) is not sufficient. Certain conditions must be additionally imposed on q. These conditions are different with and without the magnetic field. Namely, the collisions without the magnetic field can be neglected if

$$\mathbf{qV}_0 \gg \nu \text{ or } q\sqrt{T/M} \gg \nu \qquad (6a)$$

(ν is the effective number of ion collisions).

In a magnetic field the condition for neglecting the collisions has the form

$$\mathbf{qV}_0 \gg \nu \text{ or } q_\parallel \sqrt{T/M} \gg \nu, \qquad (6b)$$

where $q_\parallel$ is the projection of the vector $\mathbf{q}$ on the direction of the magnetic field. Condition (6a) is actually satisfied under the conditions of the ionosphere for the values of $\mathbf{q}$ of interest. At the same time, in a magnetic field the limiting case that is inverse to (6b) can also be satisfied, when

$$\mathbf{qV}_0 \sim q_\parallel \sqrt{T/M} \lesssim \nu. \qquad (7)$$

In this case the collisions are very important. We shall henceforth consider precisely this case.

*$[\mathbf{uH}] = \mathbf{u} \times \mathbf{H}$; $\mathbf{qu} = \mathbf{q}\cdot\mathbf{u}$.

In order to take the collisions into account, it is necessary to add the collision integral to the right half of (5). This is usually done by introducing an effective number of collisions. In particular, formulas with collision integral were obtained in the form

$$Y = -\nu\left(f_{\mathbf{q}} - \frac{f_0}{n_0}\int f_{\mathbf{q}} d^3u\right). \tag{8}$$

The formulas obtained in this manner, however, can only be interpolative. Furthermore, the quantity ν contained in (8) is determined only as far as order of magnitude is concerned, and the final result depends strongly on the exact value of ν.

In the present investigation we carry out the calculations with exact collision integrals. It turns out here that in the limiting cases

$$|\mathbf{q}_\perp|\sqrt{T/M} \gg \Omega, \tag{9}$$

$$|\mathbf{q}_\perp|\sqrt{T/M} \ll \Omega \tag{10}$$

($\mathbf{q}_\perp$ is the projection of the vector $\mathbf{q}$ on the plane perpendicular to the magnetic field and $\Omega = eH/Mc$ is the Larmor frequency of the ions), the calculations can be carried through to conclusion.

The collision integrals can be linearized in the present problem, for by virtue of condition (2) the collisions begin to influence $n_{\mathbf{q}}$ only at large distances from the body, where the perturbations of the distribution function are small. If the plasma has a degree of ionization on the order of unity, the principal role is assumed by ion-ion and ion-electron collisions. The linearized integral for collisions of singly-charged ions with one another has the form [3]

$$Y^{(ii)} = -\frac{1}{M}\frac{\partial j_k^{(ii)}}{\partial u_k},$$

$$j_k^{(ii)} = \frac{\pi e^4 L}{M}\int\Big\{f_{\mathbf{q}}(\mathbf{u})\frac{\partial f_0(\mathbf{u}')}{\partial u_l'} + f_0(\mathbf{u})\frac{\partial f_{\mathbf{q}}(\mathbf{u}')}{\partial u_l'} - f_{\mathbf{q}}(\mathbf{u}')\frac{\partial f_0(\mathbf{u})}{\partial u_l}$$

$$- f_0(\mathbf{u}')\frac{\partial f_{\mathbf{q}}(\mathbf{u})}{\partial u_l}\Big\}\frac{w^2\delta_{lk} - w_l w_k}{w^3}d^3u'$$

$$= \frac{\pi e^4 L}{M}\int\Big\{\Big[\frac{\partial f_{\mathbf{q}}(\mathbf{u}')}{\partial u_l'} + \frac{Mu_l'}{T}f_{\mathbf{q}}(\mathbf{u}')\Big]f_0(\mathbf{u})$$

$$-\Big[\frac{\partial f_{\mathbf{q}}(\mathbf{u})}{\partial u_l} + \frac{Mu_l}{T}f_{\mathbf{q}}(\mathbf{u})\Big]f_0(\mathbf{u}')\Big\}\frac{w^2\delta_{lk} - w_l w_k}{w^3}d^3u';$$

$$\mathbf{w} = \mathbf{u} - \mathbf{u}'; \qquad L = \ln\frac{De^2}{T}. \tag{11}$$

The integral for collisions between ions and electrons has a similar form

$$Y^{(ie)} = -\frac{1}{M}\frac{\partial j_k^{(ie)}}{\partial u_k},$$

$$j_k^{(ie)} = -\frac{\pi e^4 L}{M}\Big[\frac{\partial f_{\mathbf{q}}(\mathbf{u})}{\partial u_l} + \frac{Mu_l}{T}f_{\mathbf{q}}(\mathbf{u})\Big]\int f_{0e}(\mathbf{u}')\frac{w^2\delta_{lk} - w_l w_k}{w^3}d^3u'. \tag{12}$$

Here $f_{0e}(\mathbf{u}') = n_0(m/2\pi T)^{3/2}\exp(-mu'^2/2T)$, m is the electron mass (it is easy to show that when collisions are taken into account a Boltzmann electron distribution can also be assumed with the required degree of accuracy).

If the plasma is weakly ionized, then the collisions between the ions and neutral molecules are the most important. The corresponding collision integral can be written in the form [4]:

$$Y^{(im)} = \int W(\mathbf{u}, \mathbf{u}_1; \mathbf{u}', \mathbf{u}_1')\,[f_{\mathbf{q}}(\mathbf{u}')f_{m0}(\mathbf{u}_1')$$

$$- f_{\mathbf{q}}(\mathbf{u})f_{m0}(\mathbf{u}_1)]\,d^3u'd^3u_1d^3u_1'. \tag{13}$$

Here W is the probability of collision between an ion and a molecule with corresponding change in velocity

$$f_{m0}(\mathbf{u}_1) = n_{0m}(M_1/2\pi T)^{3/2}\exp(-M_1u_1^2/2T),$$

M_1 is the mass of the molecules and n_{0m} their unperturbed density. We shall see later on that the collision integrals of the form (1)—(2) and (13) lead to essentially different results.

2. CONNECTION BETWEEN DENSITY PERTURBATIONS AND THE DIELECTRIC CONSTANT OF THE PLASMA

Before we proceed to our main problem, namely taking collisions into account, we derive general formulas which are of independent interest and which enable us to simplify the calculations appreciably. Namely, we show that the term linear in the potential $\varphi_{\mathbf{q}}$ in the equations leads to multiplication of $n_{\mathbf{q}}$ by a factor which is simply related to the dielectric constant of the plasma with allowance for spatial dispersion. The proof will be presented in a form which is also suitable when account is taken of the magnetic field and of the collisions between particles.

As was already mentioned, at small values of q the quantity $f_{\mathbf{q}}$ satisfies an equation that can be written symbolically in the form

$$\hat{L}f_{\mathbf{q}} = -\mathbf{q}\mathbf{u}f_0 e\varphi_{\mathbf{q}}/T + I(\mathbf{u})/i, \tag{14}$$

where $\hat{L}$ is a linear operator acting on the functions of the velocity $\mathbf{u}$. An equation of the form (14) is valid also in the case of a magnetic field and when collisions are taken into account.

It follows from (12) that

$$f_{\mathbf{q}} = -(e/T)\varphi_{\mathbf{q}}\hat{L}^{-1}\mathbf{q}\mathbf{u}f_0 - i\hat{L}^{-1}I(\mathbf{u}),$$

where $\hat{L}^{-1}$ is an operator inverse to $\hat{L}$. Integrating with respect to d^3u and taking (4) into account, we obtain

$$n_{\mathbf{q}} = \frac{1}{i}\int \hat{L}^{-1}I(\mathbf{u})\, d^3u \Big/ \left[1 + \int \hat{L}^{-1}\mathbf{qu}\frac{f_0}{n_0}\, d^3u\right]$$
$$= \int f_1 d^3u \Big/ \left[1 + \int f_2 d^3u\right], \qquad (15)$$

where the functions f_1 and f_2 satisfy the equations

$$\hat{L}f_1 = -iI(\mathbf{u}), \quad \hat{L}f_2 = \mathbf{qu}f_0/n_0.$$

The function $f_{\mathbf{q}}$ is obviously expressed in terms of f_1 and f_2 by the formula

$$f_{\mathbf{q}} = f_1 - (e/T)\, n_0\varphi_{\mathbf{q}} f_2 = f_1 - n_{\mathbf{q}} f_2. \qquad (16)$$

In order to relate the function f_2 with the dielectric constant, we apply to the homogeneous plasma an external longitudinal electric field

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{qr}} = -i\mathbf{q}\varphi.$$

Under the influence of this field the electron density changes by an amount

$$\delta n_e = n_0 e\varphi/T,$$

and the change in the ion density is determined by the equation

$$\delta n_i = \int f d^3u, \quad \hat{L}f = -\mathbf{qu}f_0 e\varphi/T$$

or, taking (15) into account,

$$\delta n_i = -n_0 \frac{e\varphi}{T}\int f_2 d^3u. \qquad (17)$$

As is well known, the divergence of the dielectric polarization of the medium is equal to the negative density of the charges produced under the influence of the field

$$\operatorname{div}\mathbf{P} = -e(\delta n_i - \delta n_e) = \frac{n_0 e\varphi}{T}\left\{\int f_2 d^3u + 1\right\}.$$

On the other hand

$$P_l = (\varepsilon_{lk} - \delta_{lk})E_k/4\pi = -i(\varepsilon_{lk}q_k - q_l)\varphi/4\pi,$$

where $\epsilon_{lk}(\mathbf{q})$ is the dielectric tensor of the medium. Thus,

$$\int f_2 d^3u + 1 = -i(D^2\varepsilon_{lk}q_lq_k - D^2q^2) \approx iD^2\varepsilon_{lk}q_lq_k \qquad (18)$$

and finally

$$n_{\mathbf{q}} = \int f_1(u)\, d^3u/\varepsilon_{ik}q_iq_kD^2. \qquad (19)$$

We now recall that we are carrying out the calculations in a system of coordinates fixed in the body. On going over to a stationary coordinate system, the field which is static in the moving coordinate system and has a vector $\mathbf{q}$ acquires a frequency $\omega = \mathbf{qV}_0$. Therefore

$$\varepsilon_{ik} = \varepsilon_{ik}(\mathbf{qV}_0, \mathbf{q}), \qquad (20)$$

where $\epsilon_{ik}(\omega, \mathbf{q})$ is the dielectric tensor of the stationary plasma. As is well known, the connection between the frequency and the wave vector of longitudinal waves propagating in a plasma is given by the equation

$$\varepsilon_{ik}[\omega(\mathbf{q}), \mathbf{q}]\, q_iq_k = 0. \qquad (21)$$

If the wave-dispersion law defined by (21) is $\omega = \omega(\mathbf{q})$, then for those values of $\mathbf{q}$ satisfying the relation

$$\mathbf{qV}_0 = \omega(\mathbf{q}), \qquad (22)$$

the expression for $n_{\mathbf{q}}$ will have a pole. Since we are interested only in real $\mathbf{q}$, these poles are essential only if they lie at values of $\mathbf{q}$ that are real or complex with small imaginary part. For this it is necessary first that $\omega(\mathbf{q})$ be almost real, that is, that the waves be weakly damped. Second, it is necessary that the phase velocity of the waves in the region of weak damping be smaller than the velocity of the body.

Indeed, it follows from (22) that

$$\mathbf{qV}_0 = qV_0\cos\vartheta = \omega(\mathbf{q}), \quad V_0 > \omega(\mathbf{q})/q. \qquad (23)$$

If (22) has an almost real root, this will lead to a strong scattering of the radiowaves with $\mathbf{q}$ close to this root. (We recall that in the scattering $\mathbf{q}$ is the change of the wave vector [1].) The spatial distribution $n(\mathbf{r})$ then acquires relatively weakly damped oscillating terms. The condition (22), as should be the case, coincides with the condition for Cerenkov radiation of the corresponding waves by the body.

3. SIMPLIFIED EQUATIONS WITH COLLISION INTEGRAL OF THE GENERAL TYPE

We have already mentioned that the collisions are most essential for values of $\mathbf{q}$ satisfying the inequality (7). In this case, if we take into account also the inequality which is satisfied in the ionosphere

$$\Omega \gg \nu,$$

the equations can be greatly simplified. We use henceforth the results of the preceding section and seek $n_{\mathbf{q}}$ in the form (14), where f_1 and f_2 satisfy, if we spell out the meaning of the operator $\hat{L}$ for our case, the equations

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_0)f_1 + \frac{e}{Mc}[\mathbf{uH}]\frac{\partial f_1}{\partial \mathbf{u}} - Y[f_1] = I(\mathbf{u}), \qquad (24)$$

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_0)f_2 + \frac{e}{Mc}[\mathbf{uH}]\frac{\partial f_2}{\partial \mathbf{u}} - Y[f_2] = i\mathbf{qu}\frac{f_0}{n_0}. \qquad (25)$$

Introducing the vectors $\mathbf{q}_\perp$ and $\mathbf{u}_\perp$, which are the projections of $\mathbf{q}$ and $\mathbf{u}$ on the plane perpendicular to $\mathbf{H}$, and the angle β between $\mathbf{q}_\perp$ and $\mathbf{u}_\perp$, we write (24) in cylindrical coordinates:

$$\frac{\partial f_1}{\partial \beta} - i\frac{q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0}{\Omega} f_1 - i\frac{q_\perp u_\perp \cos\beta}{\Omega} f_1 + \frac{1}{\Omega} Y[f_1] = -\frac{I}{\Omega}. \quad (26)$$

We now change to a new unknown function:

$$f_1(\mathbf{u}) = g(\mathbf{u}) \exp(iq_\perp u_\perp \sin\beta/\Omega) = g(u)\, e^{i\gamma\sin\beta},$$
$$\gamma = q_\perp u_\perp/\Omega. \quad (27)$$

$g(\mathbf{u})$ satisfies the equation:

$$\partial g/\partial\beta - i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0)/\Omega + e^{-i\gamma\sin\beta}\Omega^{-1} Y[\exp(i\gamma' \sin\beta')\, g(\mathbf{u}')] = -e^{-i\gamma\sin\beta} I(\mathbf{u})/\Omega,$$
$$\gamma' = q_\perp u'_\perp/\Omega. \quad (28)$$

In order to arrive at the sought simplification, let us expand $g(u)$ in a Fourier series in β:

$$g(\mathbf{u}) = g_0(u_\parallel, u_\perp) + g_1(u_\parallel, u_\perp)\, e^{i\beta} + g_{-1}(u_\parallel, u_\perp)\, e^{-i\beta} + \ldots + g_m(u_\parallel, u_\perp)\, e^{im\beta} + \ldots \quad (29)$$

We now estimate the order of the individual terms in (29) subject to condition (7). All but the zeroth term of the series can be found by comparing the largest first terms in the left and right halves. Thus,

$$g_m \sim I/\Omega.$$

For the zeroth term the derivative $\partial g_0/\partial\beta$ vanishes, and we should estimate g_0 by comparing the second or third term with the right half. Then

$$g_0 \sim \frac{\Omega}{\mathbf{q}\mathbf{V}_0}\cdot\frac{I}{\Omega} \sim \frac{\Omega}{\nu}\cdot\frac{I}{\Omega}.$$

Thus, subject to the condition (7)

$$g_0 \sim \Omega g_m/\nu \gg g_m.$$

As a result we can confine ourselves in g only to the term g_0, taking the zeroth Fourier component of (28), that is, averaging it over the angle β.

Ultimately $g = g_0(u_\parallel, u_\perp)$ satisfies the equation[1)]

$$i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0)\, g(u_\parallel, u_\perp) - \frac{1}{2\pi}\int_0^{2\pi} d\beta e^{-i\gamma\sin\beta} Y[e^{i\gamma'\sin\beta'} g(u'_\parallel, u'_\perp)] = \frac{1}{2\pi}\int_0^{2\pi} d\beta e^{-i\gamma\sin\beta} I(\mathbf{u})\, d\beta. \quad (30)$$

If, as we shall assume henceforth, $I(\mathbf{u})$ does not depend on the angle β, then

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta e^{-i\gamma\sin\beta} I(\mathbf{u})\, d\beta = \frac{1}{2\pi}\int_0^{2\pi} e^{i\gamma\sin\beta} d\beta I(\mathbf{u}) = J_0(\gamma)\, I(\mathbf{u}) \quad (31)$$

(J_0 is the zero-order Bessel function).

We can subject (25) to an analogous simplification. For this purpose we must put

$$f_2 = f_0/n_0 + h(u_\parallel, u_\perp)\, e^{i\gamma\sin\beta}. \quad (32)$$

The final equation for h will differ from (30) for g only in the substitution

$$I(\mathbf{u}) \to \mathbf{q}\mathbf{V}_0 f_0(\mathbf{u})/n_0. \quad (33)$$

We note that in cases (9) and (10) the answer for $1 + \int f_2 d^3u$ can be written down immediately without calculation. Indeed, if

$$q_\perp \sqrt{T/M} \ll \Omega,$$

then the right half of (25), and hence also the function f_2, is small so that the integral of f_2 can be neglected. Therefore

$$1 + \int f_2 d^3u \approx 1, \qquad q_\perp\sqrt{T/M} \ll \Omega. \quad (34)$$

In the second limiting case, when

$$q_\perp\sqrt{T/M} \gg \Omega,$$

the right half of the equation for h turns out to be small because of the fast oscillations of the function $\exp(i\gamma\sin\beta)$, so that we can put $f_2 = f_0/n_0$ and

$$1 + \int f_2 d^3u \approx 1 + \int \frac{f_0}{n_0} d^3u = 2, \qquad q_\perp\sqrt{T/M} \gg \Omega. \quad (35)$$

To illustrate the method we shall apply Eq. (30) to the collision integral (8), for which we have already obtained a solution[2]. Substituting (8) in (30) and calculating the integrals with respect to $d\beta$, we obtain

$$[i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu]\, g - \nu\frac{f_0}{n_0} J_0(\gamma)\int J_0(\gamma)\, g d^3u = J_0(\gamma)\, I(\mathbf{u}). \quad (36)$$

From this we get

$$\int f_1 d^3u = \int J_0(\gamma)\, g d^3u = \int\frac{J_0^2(\gamma)}{i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu} du^3 \times\left[1 - \frac{\nu}{n_0}\int\frac{f_0 J_0^2(\gamma)}{i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu} d^3u\right]^{-1}. \quad (37)$$

If we now put, as was done in [1,2],

$$I(\mathbf{u}) = -\sigma_0 V_0 f_0, \quad (38)$$

where σ_0 is the transverse cross section of the body, then, writing down an analogous formula for the function f_2 and calculating the integrals with

1) The idea of our approximation is analogous to the approximation developed by Budker and Belyaev.[5] We, however, do not assume that f and φ vary slowly in the direction perpendicular to **H**. Our formulas therefore coincide with those that follow from the approximation[5] only in the particular case $q_\perp\sqrt{T/M} \ll \Omega$.

respect to d^3u, we obtain [2)]

$$n_q = -\sigma_0 V_0 n_0 \sqrt{M/2T} F(a_2) \exp(-q_\perp^2 T/M\Omega^2) I_0(q_\perp^2 T/M\Omega^2) \times q_\parallel^{-1}\left[2 + \frac{i\mathbf{q}\mathbf{V}_0 - 2\nu}{i\mathbf{q}\mathbf{V}_0 - \nu} i a_2 F(a_2) \exp\left(-\frac{q_\perp^2 T}{M\Omega^2}\right) I_0\left(\frac{q_\perp^2 T}{M\Omega^2}\right)\right]^{-1};$$

$$F(a_2) = \left(\sqrt{\pi} + 2i\int_0^{a_2} e^{x^2} dx\right) e^{-a_2^2}, \qquad a_2 = \frac{\mathbf{q}\mathbf{V}_0 + i\nu}{q_\parallel}\sqrt{\frac{M}{2T}}. \tag{39}$$

In the case when $q_\perp u_\perp/\Omega \sim q_\perp\sqrt{T/M}/\Omega \gg 1$, we can use the asymptotic expressions for J_0^2:

$$J_0^2(x) \approx 2\cos^2\left(x - \frac{\pi}{4}\right)\Big/\pi x = \left[1 + \cos\left(2x - \frac{\pi}{2}\right)\right]\Big/\pi x.$$

The second term $[\cos(2x - \pi/2)]$ makes a small contribution to the integral and can be omitted. Taking (37) into account we obtain ultimately

$$n_q = \frac{\Omega}{2q_\perp \pi}\int \frac{I(\mathbf{u})}{i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu}\frac{d^3u}{u_\perp} \qquad \left(\frac{q_\perp}{\Omega}\sqrt{\frac{T}{M}} \gg 1\right). \tag{40}$$

In the second limiting case $q_\perp\sqrt{T/M}/\Omega \ll 1$, we obtain after simple calculations

$$n_q = \int I(\mathbf{u})\, d^3u/(-i\mathbf{q}\mathbf{V}_0 - \nu q_\perp^2 T/2M\Omega^2). \tag{41}$$

In the next section we shall compare formulas (40) and (41) with the formulas obtained from equations with exact collision integrals.

4. SOLUTION OF EQUATIONS WITH EXACT COLLISION INTEGRALS IN DIFFERENT LIMITING CASES

In the present section we derive formulas for n_q with the aid of equations with exact collision integrals (11), (12), and (13). In this case the answer can be obtained in explicit form for limiting cases, given by inequalities (9) and (10).

Assume initially that inequality (10) is satisfied. We consider first the case of collisions between ions and neutral particles, that is, the collision integral (13). Substituting it into equation (30), we obtain

$$i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) g + g\int W(\mathbf{u}, \mathbf{u}_1; \mathbf{u}', \mathbf{u}_1') f_{M0}(\mathbf{u}_1)\, d^3u' d^3u_1 d^3u_1' \frac{d\beta}{2\pi} - \frac{1}{2\pi}\int e^{-i\gamma\sin\beta} d\beta \int W(\mathbf{u}, \mathbf{u}_1; \mathbf{u}', \mathbf{u}_1') f_{M0}(\mathbf{u}_1') e^{i\gamma'\sin\beta'} g \times (u_\parallel', \mathbf{u}_\perp')\, d^3u' d^3u_1 d^3u_1' = J_0(\gamma) I(\mathbf{u}). \tag{42}$$

[2)]To evaluate the integrals with respect to $du_\perp$ it is necessary here to employ the formula

$$\int_0^\infty e^{-x^2} J_0^2(\alpha x)\, x\, dx = \frac{1}{2} e^{-\alpha^2/2} I_0\left(\frac{\alpha^2}{2}\right),$$

where I_0 is the Bessel function of imaginary argument.

We now recognize that under condition (10) the function $e^{i\gamma\sin\beta}$ is rapidly oscillating. This allows us to neglect the term in the right half under the integral sign, so that (42) reduces to

$$[i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu_{1M}] g = J_0(\gamma) I(\mathbf{u}), \tag{43}$$

where the velocity-dependent effective number of collisions is given by the formula

$$\nu_{1M}(\mathbf{u}) = \int W(\mathbf{u}, \mathbf{u}_1; \mathbf{u}', \mathbf{u}_1') f_{M0}(\mathbf{u}_1)\, d^3u' d^3u_1 d^3u_1' \frac{d\beta}{2\pi}. \tag{44}$$

Solving (43) and replacing J_0 by its asymptotic expression, we obtain an expression for n_q:

$$n_q = \frac{1}{2\pi}\frac{\Omega}{q_\perp}\int \frac{I(\mathbf{u})}{i(q_\parallel u_\parallel - \mathbf{q}\mathbf{V}_0) + \nu_{1M}(\mathbf{u})}\frac{d^3u}{u_\perp}. \tag{45}$$

Expression (44) is analogous to (40), except that now ν depends on the velocity.

We now change over to the case of ion collisions, that is, to the collision integral (11). It is clear beforehand that the term in Y in which the unknown function is under the integral sign is again small, owing to the oscillating factor. We can therefore write down immediately $Y^{(ii)}$ in the form

$$Y^{(ii)} = -\frac{1}{M}\frac{\partial j_k^{(ii)}}{\partial u_k}, \quad j_k^{(ii)} = -\frac{\pi e^4 L}{M}\left[\frac{\partial f_q(\mathbf{u})}{\partial u_l} + \frac{M u_l}{T} f_q(\mathbf{u})\right] A_{lk}(\mathbf{u});$$

$$A_{lk}(\mathbf{u}) = \int f_0(\mathbf{u}') \frac{w^2\delta_{lk} - w_l w_k}{w^3} d^3u'. \tag{46}$$

The integral in A_{lk} can be expressed in terms of the probability integral. Leaving out these transformations, we write out the answer immediately

$$A_{lk} = n_0 (T/2\pi M)^{1/2} [a(u^2) u_l u_k/u^2 + b(u^2)\delta_{lk}],$$

$$a(u^2) = [(Mu^2/T - 3)\sqrt{2T/Mu^2}\,\Phi(\sqrt{Mu^2/2T}) + 3e^{-Mu^2/2T}]\, u^{-2},$$

$$b(u^2) = [(Mu^2/T + 1)\sqrt{2T/Mu^2}\,\Phi(\sqrt{Mu^2/2T}) - e^{-Mu^2/2T}]\, u^{-2},$$

$$\Phi(x) = \int_0^x e^{-x^2} dx. \tag{47}$$

Returning now to (30), we note that

$$q_\perp u_\perp \sin\beta = \mathbf{q}_1\mathbf{u}, \tag{48}$$

where $\mathbf{q}_1$ is a vector perpendicular to $\mathbf{q}_\perp$ and **H**, with $|\mathbf{q}_1| = |\mathbf{q}_\perp|$. Substituting (48) and (47) in (30), we take account of the fact that if $q_\perp\sqrt{T/M}/\Omega$ is large we should differentiate with respect to **u** only the exponential. The integration with respect to $d\beta$ reduces to averaging over all directions of the vector $\mathbf{q}_1$ in the plane perpendicular to **H**. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta e^{-i\gamma\sin\beta} Y^{(ii)}\left[e^{i\gamma'\sin\beta'}g\right] = \frac{\pi e^4 L}{M\Omega^2}\, g\overline{q_{1l}q_{1k}}A_{lk}$$

$$= \frac{\pi e^4 L}{M\Omega^2}\, q_\perp^2\left(b + \frac{1}{2}a\frac{u_\perp^2}{u^2}\right) n_0\left(\frac{T}{2\pi M}\right)^{1/2} = \frac{q_\perp^2 T}{M\Omega^2}\nu_{1i}(\mathbf{u})\, g. \quad (49)$$

Substituting (49) in (30) and solving the equation we obtain

$$n_{\mathbf{q}} = \frac{1}{2\pi}\frac{\Omega}{q_\perp}\int \frac{I(\mathbf{u})}{i(q_\parallel u_\parallel - \mathbf{qV}_0) + q_\perp^2 T\nu_{1i}(\mathbf{u})/M\Omega^2}\,\frac{d^3u}{u_\perp}. \quad (50)$$

(Collisions between the ions and the electrons make a small contribution in this case.) Formula (50) differs from (40) not only in the dependence of ν on $\mathbf{u}$, but also in the large factor $q_\perp^2 T/M\Omega$ ahead of ν. This means that in the region where the true collision integral has the form (10), the interpolation formula (39) yields too high values of $n_{\mathbf{q}}$ when $q_\perp^2 T/M\Omega^2 \gg 1$.

We now go to the other limiting case (10)

$$q_\perp\sqrt{T/M} \ll \Omega.$$

We see that in this case the collisions are significant only if

$$q_\parallel\sqrt{T/M} \sim \mathbf{qV}_0 \sim q_\perp^2 T\nu/M\Omega^2 \ll \nu. \quad (51)$$

We introduce again the vector $\mathbf{q}_1$ [see (48)] and expand the left half of (30) in terms of $\mathbf{q}_1\cdot\mathbf{u}$. We obtain

$$i(q_\parallel u_\parallel - \mathbf{qV}_0)g - \frac{1}{2\pi}\int_0^{2\pi} d\beta\left(1 - \frac{i\mathbf{q}_1\mathbf{u}}{\Omega}\right)Y\left[\left(1 + \frac{i\mathbf{q}_1\mathbf{u}'}{\Omega}\right)g\right] = I(\mathbf{u}). \quad (52)$$

To estimate the function g, we integrate (52) with respect to d^3u. Then the main term containing Y drops out because of the ion-number conservation law. Therefore

$$\int g d^3u \sim n_{\mathbf{q}} \sim \int I(\mathbf{u})\, d^3u/\mathbf{qV}_0, \qquad g \sim \frac{I}{\mathbf{qV}_0}. \quad (53)$$

If we substitute a quantity of the same order in (44), there arises, however, a large term Y(g) $\sim \nu I/\mathbf{q}\cdot\mathbf{V}_0 \gg 1$. To cause this term to vanish, we must seek g in the form

$$g = \mathrm{const}\cdot f_0(\mathbf{u}) = n_{\mathbf{q}} f_0(\mathbf{u})/n_0 \quad (54)$$

(We have taken directly into account Eq. (34), according to which $n_{\mathbf{q}} = \int g d^3u$ in this case.)

Substituting (54) in (52) and integrating with respect to d^3u, we get

$$n_{\mathbf{q}} = \int I(\mathbf{u})\, d^3u \Big/ \left[-i\mathbf{qV}_0 + \nu_2\frac{q_\perp^2 T}{M\Omega^2}\right]; \quad (55)$$

$$\nu_2 = -\frac{M}{3Tn_0}\int d^3u\, \mathbf{u}Y[\mathbf{u}' f_0(\mathbf{u}')]. \quad (56)$$

Formula (55) has the same form as formula (41), obtained with the aid of the effective number of collisions.

Substituting in (56) the value of Y from (13), we get ν^2 for collisions with molecules:

$$\nu_2 = \nu_{2M} = \frac{M}{3Tn_0}\int W(\mathbf{u}, \mathbf{u}_1; \mathbf{u}', \mathbf{u}_1')\,(u^2 - \mathbf{uu}')f_0(\mathbf{u}) \times f_{M0}(\mathbf{u}_1)\, d^3u d^3u' d^3u_1 d^3u_1'. \quad (57)$$

The situation is more complicated in the region where the principal role is played by collisions of charged particles. The point is that the integral for collisions between ions and ions satisfies the momentum conservation law, by virtue of which

$$\int \mathbf{p}Y^{(ii)}d^3u = M\int \mathbf{u}Y^{(ii)}d^3u \equiv 0. \quad (58)$$

It therefore is necessary to take Y in (56) to mean the ion-electron collision integral $Y^{(ie)}$. Recognizing that $m \ll M$, we can reduce $Y^{(ie)}$ to the form

$$Y^{(ie)} = -\frac{1}{M}\frac{\partial j_k^{(ie)}}{\partial u_k},$$

$$j_k^{(ie)} = -\frac{\pi e^4 L}{M}\left[\frac{\partial f_{\mathbf{q}}}{\partial u_l} + \frac{Mu_l}{T}f_{\mathbf{q}}\right]B_{lk},$$

$$B_{lk} = \int f_{0e}(\mathbf{u}')\frac{w^2\delta_{lk} - w_l w_k}{w^3}\, d^3u' \approx \int f_{0e}(\mathbf{u}')\frac{u'^2\delta_{lk} - u_l' u_k'}{u'^3}\, d^3u'$$

$$= \frac{2}{3}\delta_{lk}\int f_{0e}(\mathbf{u})\frac{d^3u}{u} = \frac{4}{3}n_0\sqrt{\frac{m}{2\pi T}}\,\delta_{lk}. \quad (59)$$

(We have allowed for the fact that $\mathbf{w} = \mathbf{u} - \mathbf{u}' \approx -\mathbf{u}'$, since the electron velocity is much larger than the ion velocity.)

Substituting (69) in (56) we obtain

$$\nu_2 = \nu_{2e} = \frac{2\sqrt{2\pi}}{3\sqrt{M}}\frac{e^4 L}{T^{3/2}}\sqrt{\frac{m}{M}}\, n_0. \quad (60)$$

This expression is $\sqrt{m/M}$ times smaller in order of magnitude than the effective number of collisions ν. Thus, whereas formula (6) gives an exaggerated value of $n_{\mathbf{q}}$ for the collisions between charged particles when $q_\perp\sqrt{T/M}/\Omega$ is large, it undervalues $n_{\mathbf{q}}$ in the region $q_\perp\sqrt{T/M}/\Omega \ll 1$.

We note in conclusion that the assumption that I(u) does not depend on β is not essential. It is easy to generalize formulas (45), (50), and (54) to the case of arbitrary I(u).

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