ON THE ANOMALOUS PENETRATION OF ELECTROMAGNETIC FIELDS INTO METALS

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The penetration of an alternating field into a metal is considered. It is shown that, under anomalous-skin-effect conditions in a slightly inclined magnetic field, sharp field and current spikes arise in the interior of the metal at $\xi = nD_0$ (n is an integer and D_0 is the orbit diameter). The amplitude of the spikes decreases very slowly over distances of the order $D_0^3/\delta_{\text{eff}}^2$ (δ_{eff} is the effective skin-layer depth). A pronounced periodic variation of the effective decay length of the spikes at cyclotron resonance in a slightly inclined field is predicted. With relation to Gantmakher's experiments, anomalies in the field penetration when **H** is parallel to the surface are considered. It is shown that in the general case the spikes decay rapidly, but that particular dispersion laws can lead to very slow decay. Various effects caused by the presence of field spikes in a metal are considered.

1. FIELD AND CURRENT SPIKES IN THE INTERIOR OF A METAL

An electromagnetic wave is usually attenuated very rapidly in metals. The effective penetration depth $\delta = c/(2\pi\omega\sigma)^{1/2}$ is very small—of order 10^{-4} to 10^{-5} cm—in good metals in the radio-frequency range. However, at low temperatures in an external magnetic field when the conditions for the anomalous skin-effect are satisfied $\delta \ll D, l$ (D is the typical diameter of an electronic orbit in the magnetic field, and l is the mean free path), the way the field penetrates into the metal can differ greatly from the usual concepts of the normal skin effect. Azbel' has shown [1] that under cyclotron resonance conditions at very high frequencies in an accurately parallel magnetic field, sharp spikes of field and current should exist at depths significantly greater than the skin-layer depth. The amplitudes of the first spikes are, in order of magnitude, the same as that close to the metal surface, and the amplitude decreases extremely slowly with distance. The physical cause for the appearance of these spikes of current and field is related to the resonance condition and to the marked departure of the dispersion law for the electrons from the quadratic $\epsilon(p) = p^2/2m$.

The cyclotron frequency Ω for conduction electrons in metals depends essentially on p_z—the projection of the quasi-momentum on the direction of **H**. Therefore, not all electrons participate in cyclotron resonance, but only those whose cyclotron frequency Ω is close to $\Omega_{\text{ext}}(p_z)$ the extremal value with respect to p_z—in particular those electrons close the central section $p_z = 0$ of the Fermi surface $\epsilon(p) = \epsilon_F$ [2]. Owing to the averaging over p_z, the relative fraction of "resonant" electrons is of order $(\Omega t_0)^{-1/2}$ (t_0 is the time between collisions, $(\Omega t_0)^{-1}$ is the relative width of the resonance for electrons with the given p_z). On the other hand the diameter of the electronic orbit, D, [which, for example, for a quadratic isotropic dispersion law is $D(p_z) = (2c/eH)(p_F^2 - p_z^2)^{1/2}$] also changes when p_z changes. Close to the central section the value of $D(p_z)$ is also extremal. For "resonant" electrons the indeterminacy of the diameter is $\Delta D \sim D_0(\Delta p_z/p_F)^2 \sim D_0/\Omega t_0$. If the uncertainty ΔD is small in comparison with the effective skin-layer depth δ, all the resonant electrons reinforce in phase at the depth $\xi = D_0$ (see Fig. 1), and in the interval $\Delta\xi \sim \delta$ provide a current spike, which in order of magnitude is the same as the current close to the surface. The "skin-layer" at $\xi = D_0$ is the source for the production of the subsequent spike at $\xi = 2D_0$ and so on. Because the remaining electrons do not participate in the resonance, their contribution to the current is small, and they can, in general, be neglected. Therefore, the distance over which the spikes de-

FIG. 1

cay is very large—much greater than D_0. Analysis [1] shows that decay occurs in distances of the order $D_0^2/\delta \gg D_0$.

Thus, for the existence of slowly decaying field spikes two conditions must be satisfied: first, we require cyclotron resonance ($\omega = n\Omega_{ext}$), and second, we require resonance to be sufficiently "sharp" so that the indeterminacy of the diameter ΔD for the resonant electrons is small in comparison with δ. The latter condition is very stringent, and necessitates the use of very high frequencies and magnetic fields ($\Omega t_0 \gg D_0/\delta$). If the inequality $\Omega t_0 \gg D_0/\delta$ is not satisfied, the spikes decay rapidly. [1]

It is shown in the present paper that similar very slowly decaying field and current spikes should exist in metals under conditions much less stringent and more favorable from the point of view of experimental observation.

In fact, for field spikes to exist at large distances from the surface, we require some mechanism of selecting electrons according to diameter: electrons with a definite value of $D(p_z)$ must play a dominant role in creating the high frequency current. In a parallel field, cyclotron resonance provides such a mechanism in metals with an essentially non-quadratic dispersion law (Ω depends on p_z) at high frequencies ($\omega = n\Omega = D_0/t_0\delta$). However, if the field **H** is slightly inclined to the surface, the natural drift of the electrons along the field, and, consequently, into the depth of the metal, will cause the electrons with small drift velocity $\bar{v}_z$ (close to the central section $p_z = 0$) to experience high frequency field conditions completely different from all the remaining electrons. Electrons with $\bar{v}_z \sim v_F$ are in the skin-layer at best once, and their contribution to the conductivity will be small compared with the electrons of the central section which return repeatedly to the metal surface.

In fact, for electrons with $v_z \sim v_F$ the effective mean free path in a significant field is of order δ, and the contribution to the conductivity is of order $\sigma(\delta/l)$ (σ is the static conductivity). The relative number of electrons close to the central section, which in the course of time t_0 return to the skin-layer, is small (of order $\delta/l\varphi$, where φ is the angle of inclination of **H** to the surface). However, their contribution to the conductivity is proportional to the number of rotations $N \sim \Omega t_0$, and is of order $\sigma(\delta/l)(\delta/D\varphi)$. In the range of angles $\delta/l \ll \varphi \ll \delta/D$ the conductivity will be determined, therefore, only by a small group of electrons close to the central section with $|\Delta p_z/p_F| \lesssim \delta/l\varphi$. The field spikes arising with the aid of the orbit chains shown in Fig. 2 will decay slowly if the indeter-

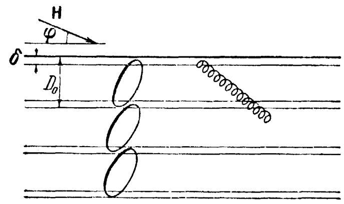

FIG. 2

minacy of the diameters $\Delta D \sim D_0(\Delta p_z/p_F)^2 \sim D_0(\delta/l\varphi)^2$ is small compared with δ. This leads to the following condition for the angle φ:

$$\varphi^2 \gg \delta D_0/l^2, \tag{1.1}$$

which, together with the inequality $\varphi \ll \delta/D$, defines the range of magnetic fields in which this effect should be observed:

$$D_0^3 \ll \delta l^2 \quad \text{or} \quad (\Omega t_0)^2 \gg D_0/\delta. \tag{1.2}$$

It is clear that inequality (1.2) is considerably laxer than the requirement, $\Omega t_0 \gg D/\delta$, for the existence of spikes in a parallel field at cyclotron resonance. [1] Apart from this, cyclotron resonance $\omega = n\Omega$ is not required: the effect should be observed even at low frequencies—of the order of several megacycles. The anomalous-skin-effect condition $\delta \ll D$ is sufficient. This circumstance is all the more favorable in that resonance effects (periodic in the inverse field) will not mask the experimental observation of spikes.

The presence of field spikes, in conformity with Azbel''s results [1], leads to a whole series of new effects: impedance jumps, periodic in the magnetic field, should be observed for a massive plate, the thickness of which is large compared with not only D but also with l; the selective transparency of films; a spatial "echo" similar to a spin echo, etc. A similar anomalous penetration of field into a metal occurs in cases when any mechanism selects electrons according to their velocities. It is also of interest to study the features of the surface impedance and the field structure in a metal when there is no such mechanism, for example, in the case of a parallel magnetic field at low (non-resonant) frequencies. It appears that spikes then also occur in the interior of the metal, the amplitude of which will, however, decay comparatively rapidly (in distances of the order D_0). Despite such a rapid decay of the spikes, Gantmakher [3] has observed impedance jumps in a tin plate in a parallel field. The jumps were periodic as a function of H, and were defined by the condition $D_{ext} = d/n$ ($H_n = nH_1$).

The existence of a dimensional effect for $n = 1$ ($D_{ext} = d$) was first pointed out by the author. [4] The physical cause for the appearance of impedance anomalies in films with $D_{ext} = d$ is related to the fact that when $D_{ext} > d$ some of the electrons collide with both surfaces of the plate at each rotation and "are out of the game," and their contribution to the current is small compared with that from electrons which do not collide with the surface. This phenomenon was first experimentally observed by Khaĭkin [5] and Gantmakher [8]. Based on it, a method was proposed for determining very accurately the extremal diameters of the Fermi surface by "cutting off the resonant and non-resonant orbits." [5]

The impedance singularities at n = 2, 3, 4, ... etc., are associated with the rapidly decaying field spikes. They arise because of the incomplete interference of the currents at $\xi = D_{ext}$, $2D_{ext}$, etc. In fact, through points at a depth ξ smaller than, for example, the maximum diameter D_{max}, electrons pass with diameters both less and greater than ξ. Owing to the averaging over p_z, the current is exponentially small. For $\xi > D_{max}$ the averaging is one-sided, $D(p_z) \leq D_{max}$, the interference is incomplete, the current grows and is of order $(\delta/D)^{1/2}$ times the value of the current close to the surface. Further, this spike "propagates" with corresponding attenuation by means of the orbit chains into the depth of the metal (Fig. 1).

It is clear that in the half-space the rapidly decaying spikes cause an insignificantly small correction to the impedance $[\sim(\delta/D)^{1/2}]$. When a second surface of the plate is present the appearance (or exclusion) of successive spikes in the orbit chain causes a jump in the field at the surface, and, consequently, impedance anomalies.

2. THE FIELD DISTRIBUTION IN A SEMI-INFINITE METAL AT LOW FREQUENCIES

The high-frequency field in a metal is determined by Maxwell's equations:

$$d^2E_\mu(\xi)/d\xi^2 = 4\pi i\omega c^{-2} j_\mu(\xi); \qquad \mu = \zeta,\, y. \qquad (2.1)$$

The coordinate system (Fig. 3) is chosen so that **H** is parallel to O_z, the O_x axis lies in the plane through **H** and the internal normal to the metal surface $O\xi$, and the direction of the $O\zeta$ axis is that of the projection of **H** on the plane $\xi = 0$. We shall consider only the field component E_y, because the dominant role is played by the electrons of the central section $p_z = 0$, which provide a current in this direction. The current density $j_y(\xi)$ is

$$j_y(\xi) = \frac{2e}{h^3}\int v_y f d^3\mathbf{p} = \frac{2e}{h^3}\int f m v_y \, d\varepsilon d\tau dp_z;$$

f is the non-equilibrium addition to the distribu-

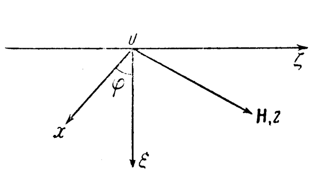

FIG. 3

tion function, $\epsilon(\mathbf{p})$ is the energy, $\mathbf{p}$ is the quasi-momentum, $m = (2\pi)^{-1}\,\partial S(\epsilon, p_z)/\partial\epsilon$ is the effective mass, e is the electronic charge, and τ is the dimensionless time describing the motion along the orbit in the magnetic field.

The function f should be found from the kinetic equation under the condition of diffuse reflection of the electrons from the metal surface: [2]

$$i\omega f + v_\xi \partial f/\partial\xi + \Omega\partial f/\partial\tau + \nu f = -e\mathbf{E}(\xi)\,\mathbf{v}\partial f_0/\partial\varepsilon,$$
$$f(\xi = 0,\, v_\xi > 0) = 0.$$

Here $f_0(\epsilon)$ is the equilibrium Fermi function, and $-\partial f_0/\partial\epsilon = \delta(\epsilon - \epsilon_F)$. The collision integral in the anomalous skin effect can lead to a relaxation time. [2]

If we continue $j_y(\xi)$ and $E_y(\xi)$ in an even way into the region $\xi < 0$, it can be shown that, to within an unimportant factor of order unity, the boundary condition for the distribution function can be ignored (see also [2,1]). Then equation (2.1) is rapidly solved with the aid of a Fourier cosine-transformation. [2] After straightforward manipulation we obtain the following system of equations:

$$-k^2\mathscr{E}_y(\mathrm{k}) - 2E_y'(0) = 4\pi i\omega c^{-2}\sigma_{yy}(\mathrm{k})\,\mathscr{E}_y(\mathrm{k}),$$

$$\sigma_{yy}(\mathrm{k}) = \frac{2e^2}{h^3}\int\limits_{\varepsilon(p)=\varepsilon_F} dp_z \frac{m}{\Omega}\int\limits_0^{2\pi} d\tau v_y(\tau)$$
$$\times \int\limits_{-\infty}^{\tau} d\tau' v_y(\tau')\, e^{\gamma(\tau'-\tau)} \cos\left(\frac{\mathbf{k}}{\Omega}\int\limits_{\tau}^{\tau'} \mathbf{v}\, d\tau''\right),$$

$$\mathscr{E}_y(k) = 2\int\limits_0^\infty E(\xi)\cos k\xi\, d\xi, \qquad E(\xi) = \frac{1}{\pi}\int\limits_0^\infty \mathscr{E}(k)\cos k\xi\, dk, \qquad (2.2)$$

where $\gamma = (i\omega + \nu)/\Omega$.

The vector **k** is directed along the normal to the surface $\xi = 0$ ($k_x = k\sin\varphi \approx k\varphi$, $k_z = k\cos\varphi \approx k$). Purely for simplicity all calculations are first performed for an isotropic quadratic dispersion law $\epsilon(\mathbf{p}) = p^2/2m$, where the effective mass m does not depend on p_z. In the non-resonant region the dispersion law does not generally play an essential part, and when the cyclotron resonance region is considered we shall point out what changes arise by taking into account a non-quadratic dispersion law. Introducing polar coordinates in velocity space $[\mathbf{v} = (v_F \sin\theta\cos\tau,\, v_F\sin\theta\sin\tau,\, v_F\cos\theta)]$, we obtain

$$\sigma(\mathbf{k}) = \frac{3\pi ne^2}{m\Omega}\int_0^\pi \frac{d\theta \sin^3\theta}{1-\exp[-2\pi\gamma - 2\pi i k_z v_F \Omega^{-1}\cos\theta]}\frac{1}{(2\pi)^2}\int_0^{2\pi} d\tau \sin\tau \times \int_{\tau-2\pi}^{\tau} d\tau' \sin\tau' \exp\Big[\frac{ik_x v_F \sin\theta}{\Omega}(\sin\tau - \sin\tau') + \gamma(\tau'-\tau)\Big]. \qquad (2.3)$$

In the argument of the exponential in the integrals with respect to τ and τ' we have neglected the small terms $k_z v_F \Omega^{-1}\cos\theta(\tau' - \tau)$. In the low frequency range, when $\gamma \ll 1$, and also close to resonance, the factor $\exp[\gamma(\tau' - \tau)]$ which is non-periodic in τ can be replaced by unity. If we use the expansion

$$[1-\exp(-2\pi\gamma - 2\pi i x)]^{-1} = \sum_{\mu=-\infty}^{\infty}(2\pi[\gamma + i(x-\mu)])^{-1}, \qquad (2.4)$$

then $\sigma(\mathbf{k})$ assumes the form [1)]

$$\sigma(\mathbf{k}) = \frac{3}{2}\frac{ne^2}{m}\sum_{\mu=-\infty}^{\infty}\int_0^\pi \frac{d\theta \sin^3\theta J_{\mu+1}^2(k_x r\sin\theta)}{\nu + i(\omega - \mu\Omega + k_z v_F\cos\theta)}, \qquad (2.5)$$

where $J_\mu(x)$ is a Bessel function, and $r = mv_F c/eH$.

For an arbitrary dispersion law (2.5) is replaced by

$$\sigma(\mathbf{k}) = \frac{4\pi e^2}{h^3}\sum_{\mu=-\infty}^{\infty}\int \frac{m dp_z}{\nu + i(\omega - \mu\Omega + k_z \bar{v}_z)}\overline{|J_{\mu+1}(k_x r)|^2}, \qquad (2.6)$$

where the bar signifies averaging with respect to τ $[2\pi\bar{\psi} = \int_0^{2\pi} d\tau\psi(\tau)]$, and

$$J_{\mu+1} = \frac{1}{2\pi}\int_0^{2\pi} d\tau\, v_y(\tau)\exp\Big[-\frac{ik_x p_y(\tau, p_z)}{eH} + i\mu\tau\Big], \qquad (2.7)$$

because

$$\frac{1}{\Omega}\int^{\tau} v_x\, d\tau = \frac{cp_y(\tau)}{eH}.$$

The field distribution in the metal has the form

$$E(\xi) = -\frac{2}{\pi}E'(0)\int_0^\infty \frac{\cos k\xi\, dk}{k^2 + 4\pi i\omega c^{-2}\sigma(k)}. \qquad (2.8)$$

We consider the limiting case when $\omega \lesssim \nu \ll k_z v_F \ll \Omega$ (low frequencies, strong field) and when the principal term in the sum (2.5) is that with $\mu = 0$. The remaining terms in the sum are small in the ratio $(k_z v_F/\Omega)^2$. We have

$$\sigma(k) = \frac{3}{2}\frac{ne^2}{m}\int_0^\pi d\theta \sin^3\theta J_1^2(k_x r\sin\theta)\frac{\nu}{\nu^2 + (k_z v_F\cos\theta)^2}. \qquad (2.9)$$

Two rapidly changing functions lie in the integral in (2.9):

$$J_1^2 \approx (\pi k_x r\sin\theta)^{-1}[1 - \sin(2k_x r\sin\theta)],$$

$$\nu[\nu^2 + (k_z v_F\cos\theta)^2]^{-1}.$$

The form of the asymptotic representation of $\sigma(\mathbf{k})$ depends on the relative rate with which they change. Both functions have sharp maxima at $\theta = \pi/2$ ($v_\xi = 0$). The rate of change of $\sin(k_x D_0 \sin\theta)$ is determined by the interval $\Delta\theta \sim (k_x D_0)^{-1/2}$ and the denominator changes significantly in the interval $\Delta\theta \sim |k_z l|^{-1}$. The case when the denominator is the "sharpest" function is of most interest (slowly decaying spikes then arise), i.e.,

$$|k_z l| \gg (k_x D_0)^{1/2} \quad \text{or} \quad \varphi \gg (D_0\delta/l^2)^{1/2}. \qquad (2.10)$$

Then $\nu/[\nu^2 + (k_z v_F\cos\theta)^2]$ can be replaced by $\pi\delta(k_z v_F\cos\theta)$, and we have

$$\sigma(k) = \frac{3\pi}{2}\frac{ne^2}{m|k_z v_F|}J_1^2(k_x r_0) = \frac{3ne^2(1 - \sin|k_x D_0|)}{p_F k^2 D_0\varphi}. \qquad (2.11)$$

For an arbitrary dispersion law (2.11) assumes the form

$$\sigma(\mathbf{k}) = \frac{2\pi e^2}{h^3 k^2\varphi}(1 - \sin|k_x D_0|)\int d^3p\,\delta(\varepsilon - \varepsilon_F)\,\delta\Big(\frac{v_x}{\Omega}\Big)\,\delta(\bar{v}_z)\,v_y^2. \qquad (2.11a)$$

It is clear from (2.11a) that the contribution to the conductivity is determined only by the electrons close to the central section, when they move parallel to the surface $v_x = \bar{v}_z = 0$. This confirms the assumption originally made that the electrons of the central section play the principal role. In the opposite limiting case $k_z l \ll (k_x D_0)^{1/2}$

$$\sigma(\mathbf{k}) = \frac{3ne^2}{p_F k_x k_z D_0}\Big[1 + \Big(\frac{2k_z^2 l^2}{\pi k_x D_0}\Big)^{1/2}\cos\Big(k_x D_0 + \frac{\pi}{4}\Big)\Big]. \qquad (2.12)$$

The amplitude of the oscillating component in $\sigma(\mathbf{k})$ is in this case small compared with unity.

We now show that an asymptotic representation of type (2.11), which oscillates with respect to k, leads to very slowly attenuating field spikes in the bulk of the metal, whilst for small "modulation depth" (2.12) the spikes decay rapidly (as a power of the "modulation parameter"). Substituting (2.11) into (2.8) we obtain, after the change of variables $kD_0 = k_x D_0 = x$,

$$E(\xi) = -\frac{2}{\pi}E'(0)\,D_0 I\Big(\frac{\xi}{D_0}\Big);$$

[1)]In order to include correctly the boundary condition of diffuse reflection, it is sufficient to introduce the factor ½ in (2.5). Then we obtain a correct expression for the impedance to within an accuracy of several percent.[2]

$$I(t)=\int_0^\infty \frac{x\cos xt\,dx}{x^3+{}^1\!/_2 iM^4\pi J_1^2(x/2)},\quad M^4=\frac{2D_0^3}{\delta_0^3\varphi},\qquad \delta_0=\left(\frac{c^2l}{6\pi\omega\sigma}\right)^{1/3},$$

$$\delta_{eff}\sim(\delta_0^3 D_0\varphi)^{1/4}, \tag{2.13}$$

where δ_0 is of the order of the effective penetration depth of the field in the anomalous skin effect with H = 0.

For $|t-n|\ll M^{-1}$ (i.e., $|\xi-nD_0|\ll\delta_{eff}$, n is an integer) the function $J_1^2(x/2)$ can be replaced by the asymptotic form $2(1-\sin x)/\pi x$ and

$$I(t)=\int_0^\infty \frac{x^2\cos xt\;dx}{x^4+iM^4(1-\sin x)}.$$

We transform I(t) in the following way:

$$I(t)=\sum_{\mu=0}^{\infty}\int_{2\pi\mu}^{2\pi(\mu+1)}dx\ldots=\sum_{\mu=0}^{\infty}\int_0^{2\pi}dx\,\frac{(2\pi\mu+x)^2\cos(2\pi\mu t+xt)}{(2\pi\mu+x)^4+iM^4(1-\sin x)}.$$

Because $M|t-n|\ll 1$ we can replace t in the argument of the cosine by n. In the sum over μ, large values of μ (of order M) are important, and the sum can, therefore, be replaced by an integral. As a result,

$$I(n)=\frac{1}{2\pi M}\int_0^\infty d\xi\cdot\xi^2\int_0^{2\pi}\frac{dx\cos xn}{\xi^4+i(1-\sin x)}$$

$$=\frac{\pi^{1/2}}{2^{7/4}M}e^{-\pi i/8}(-1)^n\cos\frac{n\pi}{2}\,\frac{\Gamma(n+{}^1\!/_4)}{\Gamma(n+{}^3\!/_4)}. \tag{2.14}$$

It follows from (2.14) that I(n) = 0 for all odd n, and is non-zero for even n. For large n the asymptotic representation of I(n) has the form

$$I(n)\sim\frac{\pi^{1/2}}{2^{7/4}Mn^{1/2}}e^{-\pi i/8}(-1)^n\cos\frac{n\pi}{2}=I(0)\frac{2\pi}{\Gamma^2({}^1\!/_4)}\frac{\cos(n\pi/2)}{\sqrt{n}}. \tag{2.15}$$

Thus, for large n the height of the spikes falls as $n^{-1/2}$, where the signs of neighboring maxima alternate. The quantity I(0), which is related to the surface impedance, is

$$I(0)=\frac{\Gamma^2({}^1\!/_4)}{2^{9/4}\pi^{1/2}M}e^{-\pi i/8}. \tag{2.16}$$

The quantity I(0) is a factor of $\Gamma^2(1/4)/2^{3/4}\pi^{3/2}\approx 1.4$ greater than the value of I(0) obtained when the oscillating component in the asymptotic representation of conductivity is neglected.

In order to obtain the field between the spikes ($|t-n|\gg M^{-1}$), we express (2.13) as

$$I(t)=\frac{1}{2}\int_0^\infty\frac{x^2(e^{itx}+e^{-itx})}{x^4+iM^4(1-\sin x)}dx. \tag{2.17}$$

In the integral containing exp (itx) the contour of integration is turned to Im x > 0, and in the second integral to Im x < 0. Then we obtain the corresponding sum of the residues and the integral along the imaginary axis. It is not difficult to show that the latter is at most a factor of M^3 smaller than the sum of the residues, which is

$$I(t)=\frac{1}{iM^2}\sum_{n=1}^{\infty}\cos x_n t\cdot\exp\left[-(1-i)\frac{x_n^2 t}{M^2}\right],$$

$$x_n=\left(2n+\frac{1}{2}\right)\pi. \tag{2.18}$$

For $|t-m|\sim 1$ only $n\sim 1$ is important in the sum, the integral I(t) oscillates in distances of the order unity, and decays extremely slowly over distances of the order $t\sim M^2$. The effective decay length is

$$D_0M^2\approx D_0^3/\delta_{eff}^2,\qquad \delta_{eff}=(\delta_0^3D_0\varphi)^{1/4}. \tag{2.19}$$

It is obvious from (2.18) and (2.14) that the field at the maxima of a spike exceeds the field between spikes by a factor of $M\sim D_0/\delta_{eff}$. The general picture of the field distribution in the metal is shown in Fig. 4. For odd n there are two sharp maxima of opposite sign, and for even n there are single peaks. The signs of two neighboring even spikes are opposite. Such a picture of the field distribution applies for $\varphi\gg(\delta D_0)^{1/2}/l$.

We now turn to consider the other limiting case when $\varphi\ll(\delta D_0)^{1/2}/l$ (where $\varphi\gg\delta/l$) and the amplitude for the oscillating component in the conductivity $\sigma(k)$ is small:

$$2a=(2k_z^2l^2/\pi kD_0)^{1/2}=(2M/\pi)^{1/2}l\varphi/D_0\ll 1. \tag{2.20}$$

The field in the metal will also display jumps; however, their amplitude decays rapidly with increasing number. In fact, expanding I(t) in power series in a, we obtain

$$I(t)=\int_0^\infty dx\,\frac{x^2\cos xt}{x^4+iM^4[1+2a\cos(x+\pi/4)]}$$

$$=\sum_{\mu=0}^{\infty}(-2iM^4a)^\mu\int_0^\infty dx\,\frac{x^2\cos xt\cos^\mu(x+\pi/4)}{(x^4+iM^4)^{\mu+1}}. \tag{2.21}$$

The field distribution close to t = n has the form

$$\frac{I(t)}{I(0)}=\frac{4}{\sqrt{2\pi}}e^{i\pi/8}(-ia)^n\int_0^\infty\frac{dx\cdot x^2\cos[M(t-n)x-n\pi/4]}{(x^4+i)^{n+1}}. \tag{2.22}$$

For $|t-n|\ll M^{-1}$

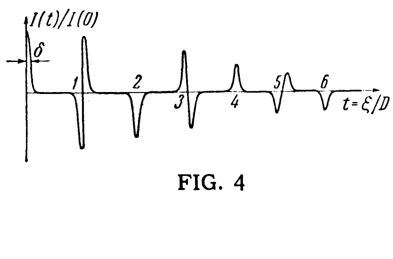

FIG. 4

$$\frac{I(n)}{I(0)} = \frac{\Gamma(3/4)}{\pi\sqrt{2}}(-1)^n \cos\frac{n\pi}{4} a^n \frac{\Gamma(n+1/4)}{\Gamma(n+1)}.$$

The amplitude of the spikes for large n decreases as $a^n/n^{3/4}$. The field distribution in this case is shown schematically in Fig. 5.

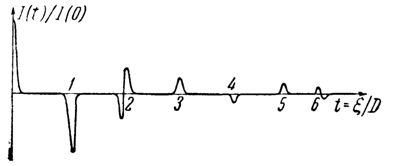

FIG. 5

3. SPIKES OF THE HIGH FREQUENCY FIELD AT CYCLOTRON RESONANCE IN AN INCLINED MAGNETIC FIELD

We now turn to considering the resonance region close to $\omega = \mu\Omega$. In formulae (2.5) and (2.6) we can retain, with accuracy up to non-resonant components of order $|k_z r| \ll 1$, only a single resonant term. In the simplest case of a quadratic dispersion law (Ω does not depend on p_z) the field distribution picture in the metal will be exactly the same as at low frequencies ($\omega \lesssim \nu$). For $\varphi \gg (\delta/D_0)^{1/2}$ the effective attenuation length of the spikes is large. As the resonance is "detuned" the repeated return of the electrons to the skin layer becomes less effective, and the field spikes start to weaken. The following condition is the criterion:

$$|\omega - \mu\Omega| \lesssim k_z v_F/(kD_0)^{1/2} \sim \varphi v_F/(\delta D_0)^{1/2}.$$

In terms of the magnetic field this corresponds to the "resonance width" $\Delta H/H \sim (\varphi/\mu)(D_0/\delta)^{1/2} \ll 1$. On further departure from resonance conditions, the amplitude of the spikes falls rapidly, and when $|\omega - \mu\Omega| \lesssim \omega$ the relative diminution of successive spikes is determined by the small parameter $(\delta/D_0)^{1/2}$, i.e., roughly speaking, the spikes decay in a length D_0.

Thus, under cyclotron resonance conditions in a slightly inclined magnetic field, a curious phenomenon should be observed—the resonance change of the effective attenuation decay length of spikes from the value D_0^3/δ^2 close to resonance to the value D_0 far from it. It is clear from the physical picture of the phenomenon that the departure of the dispersion law from quadratic, which causes an additional spreading of the resonance due to the variability of $\Omega(p_z)$, plays no special role. The corresponding condition allowing us to ignore the variation of Ω on p_z is

$$|\partial^2\Omega/\partial p_z^2|\, \Delta p_z^2 \lesssim \nu. \qquad (3.1)$$

Since $|\Delta p_z| \sim p_F/(k_z l)$, it follows that $\Omega t_0 \lesssim (k_z l)^2$ or $D_0 l \varphi^2 \gtrsim \delta^2$.

The analogous phenomenon of a sharp periodic change of the decay length of field spikes should also, of course, occur in the case considered by Azbel',[1] i.e., at cyclotron resonance at high frequencies in a parallel field.

For large angles of inclination of the field **H** to the surface ($\sin\varphi \gg \delta/D_0$) a comparatively slowly decaying field component also exists in the metal which is associated with electrons drifting along **H**. For $\xi \lesssim l$ this part of the field is an oscillating function of **H**. The features of the field penetration in this case will be the subject of a separate communication.

4. FIELD SPIKES IN A METAL IN A PARALLEL MAGNETIC FIELD

When there is no mechanism which selects only electrons with a definite value of $D(p_z)$ the field spikes will decay rapidly. Nevertheless, the study of such spikes is of definite interest, since, apparently, it is this case which occurs in Gantmakher's experiments.[3]

The asymptotic form of the conductivity $\sigma(k)$ can be obtained directly from (2.3), (2.5), and (2.6), by putting $k_z = 0$. For a quadratic dispersion law

$$\sigma(k) = \frac{3\pi n e^2}{2 p_F k} \frac{1 + 2(2/\pi k D_0)^{1/2}\cos(kD_0 + \pi/4)}{1 - \exp[-2\pi(i\omega + \nu)/\Omega]}. \qquad (4.1)$$

For a non-quadratic dispersion law the general formula has a somewhat more complicated form, owing to the variation of Ω with p_z:

$$\sigma(k) = \frac{4\pi e^2}{h^3}\int \frac{dS}{v}\,\delta(kv_x)\, v_y^2 \frac{1 - \sin kD(p_z)}{1 - \exp[-2\pi(i\omega + \nu)/\Omega(p_z)]}. \qquad (4.2)$$

Here $D = cd/eH$, $d(p_z) = p_{y\,max} - p_{y\,min}$ is the diameter of the Fermi surface in the Oy direction (p_y is a function of p_z and τ, and is the extremum with respect to τ). At low frequencies $\omega \ll \Omega$

$$\sigma(k) = \frac{2e^2}{h^3 k(\nu + i\omega)}\left\{\oint \frac{d\chi\, \Omega n_y^2}{K(\chi)} + \frac{2\Omega_0}{K_0}\left(\frac{2\pi}{|kD_0''|}\right)^{1/2}\cos\left(kD_0 + \frac{\pi}{4}\right)\right\}. \qquad (4.3)$$

Here χ is the polar angle in velocity space with axis along the Ox axis, $dS = do/K$, K is the absolute value of the Gaussian curvature of the Fermi surface, do is the element of solid angle in velocity space, $D_0'' = \partial^2 D(\chi_0)/\partial\chi^2$, $\partial D(\chi_0)/\partial\chi = 0$. For a quadratic dispersion law, equations (4.1) and (4.3) are the same ($K = p_F^{-2}$, $n_y = \sin\chi$, $D(\chi) = D_0 \sin(\chi)$,

$\chi_0 = \pi/2$]. The relative amplitude a for the oscillating factor is of order $(\delta/D_0)^{1/2}$.

In the cyclotron resonance region the "modulation depth" of the conductivity $\sigma(k)$ is for the case of non-quadratic dispersion essentially different from the case $\Omega(p_z)$ = const. This is related to the fact that not all the electrons participate in the resonance, but only that small fraction [$\sim (\nu/\Omega)^{1/2}$] for which the effective mass is extremal with respect to p_z. The corresponding formula close to resonance at the central section (for $\Omega t_0 \ll D_0/\delta$) has the form

$$\sigma(k) = \frac{4\sqrt{2\pi}e^2}{h^3 k K\mu}\Big\{\left|\frac{m}{m''}\right|^{1/2}(\gamma_0^2+\Delta^2)^{-1/4} \times \exp\left(-is\operatorname{arc\,tg}\left[\left(1+\frac{\Delta^2}{\gamma_0^2}\right)^{1/2} - \frac{s\Delta}{\gamma_0}\right]\right) + (\gamma_0 + i\Delta)^{-1}\,|\pi k D_0''|^{-1/2}\cos\left(kD_0+\frac{\pi}{4}\right)\Big\}. \tag{4.4*}$$

Here $\gamma_0 = \nu/\omega$; $\Delta = (1 - \mu\Omega/\omega)$; $|\Delta|, \gamma_0 \ll 1$; $s = \operatorname{sign} m''(\chi_0)$. The value of all the functions is taken at the central section. For $\Delta = 0$

$$\sigma(k) = \frac{4\pi e^2}{h^3 k K\mu}\left|\frac{2m}{m''\gamma_0}\right|^{1/2}\left\{e^{-is\pi/4} + \left|\frac{m''}{\pi m k D_0''\gamma_0}\right|^{1/2}\cos\left(kD_0+\frac{\pi}{4}\right)\right\}. \tag{4.5}$$

The field distribution in the bulk is determined by formulae analogous to (2.13):

$$E(\xi) = -\frac{2E'(0)}{\pi} I\left(\frac{\xi}{D_0}\right), \tag{4.6}$$

where

$$I(t) = \int_0^\infty \frac{x\cos xt\,dx}{x^3 + iM^3(1 + 2a\cos(x+\pi/4))}. \tag{4.7}$$

Here $|a| \ll 1$ always. At low frequencies ($\omega \lesssim \nu$) and in the case of cyclotron resonance with quadratic dispersion

$$M^3 = \frac{6\pi\omega\sigma D_0^2}{c^2} = \frac{D_0^2 l}{\delta_0^3}; \qquad a \approx \left(\frac{2}{\pi M}\right)^{1/2} \ll 1. \tag{4.8}$$

We obtain for non-quadratic dispersion at resonance

$$M^3 \sim D_0^2 l \delta_0^{-3}\gamma_0^{1/2} \gg 1; \qquad a \sim (M\gamma_0)^{-1/2} \ll 1. \tag{4.9}$$

Close to the n-th spike the field distribution is determined by the following expression:

$$\frac{I(t)}{I(0)} = a^n \frac{3\sqrt{3}}{2\pi}\exp\left(\frac{\pi i}{6} - \frac{\pi i n}{2}\right)\int_0^\infty \frac{x\cos[M(t-n)x - n\pi/4]\,dx}{(x^3+i)^{n+1}},$$
$$I(0) = 2\pi\cdot 3^{-3/2}M^{-1}e^{-\pi i/6}. \tag{4.10}$$

For $t = n$

$$\frac{I(n)}{I(0)} = (-1)^n\Gamma\left(\frac{2}{3}\right)a^n\cos\frac{n\pi}{2}\,\frac{\Gamma(n+1/3)}{\Gamma(n+1)}. \tag{4.11}$$

The type of field distribution in the metal is in this case the same as in an inclined field for $\varphi \ll (\delta D_0)^{1/2}/l$ (see Fig. 5) and differs only in the values of a and M [see (2.20) and (2.23)].

If there is not a single extremal diameter, but several, it is quite obvious that field spikes should be observed, not only for $\xi = nD_0$, but also for linear combinations of the type $\xi = \sum_i n_i D_i$. In other words, the chain of electron orbits, with the aid of which the field extends into the depth of the metal, can consist of different "links." The amplitude of the spikes is, in this case, determined by (4.11) where a^n will be replaced by

$$\prod_i a_i^{n_i}, \qquad a_i \approx (\delta/D_i)^{1/2}.$$

We note that at cyclotron resonance the effective decay length of these field spikes also changes periodically due to the oscillations of the quantities a and M.

5. THE EFFECT OF FIELD SPIKES ON THE SURFACE IMPEDANCE

We shall calculate only the principal value of the total surface impedance tensor $Z_{\alpha\alpha} = Z$ that corresponds to the polarization of the electric field $\mathbf{E}_\alpha$ along $\mathbf{v}_0$—the tangential velocity of the electrons of the central section. For an isotropic dispersion law $\mathbf{v}_0$ is perpendicular to $\mathbf{H}$. The remaining components $Z_{\mu\nu}$ are much greater than Z, i.e., the effect of the anomalous penetration is sharply anisotropic relative to the angle between $\mathbf{H}$ and the high frequency current:

$$Z = -4\pi i\omega c^{-2}E_\alpha(0)/E_\alpha'(0). \tag{5.1}$$

We consider the impedance of the half-space in the case of low frequencies ($\omega \lesssim \nu$) and strong magnetic fields ($\Omega \gg \nu$) as a function of the angle φ between $\mathbf{H}$ and the metal surface. In a parallel field ($\varphi \lesssim \delta/l$) the field spikes decay comparatively rapidly [as $(\delta/D_0)^{n/2}$] and lead to only a trivially small correction of the order $(\delta/D)^{1/2}$:

$$Z = \frac{16\pi}{3\sqrt{3}c}\exp\left(\frac{\pi i}{3}\right)\frac{\delta_0}{\lambda\!\!\!^-}\left(\frac{D_0}{l}\right)^{1/3} \sim \omega^{2/3}H^{-1/3}, \tag{5.2}$$

where δ_0 is the effective depth of the skin layer for $H = 0$; $\lambda\!\!\!^- = c/\omega$. In the angular range $\delta/l \ll \varphi \ll (\delta D_0)^{1/2}/l$ the amplitude of the spikes and the corresponding correction to Z are also small:

$$Z = \frac{2^{5/4}\pi}{c}\exp\left(\frac{3\pi i}{8}\right)\frac{\delta_0}{\lambda\!\!\!^-}\left(\frac{D_0\varphi}{\delta_0}\right)^{1/4} \sim \omega^{3/4}H^{-1/4}. \tag{5.3}$$

The impedance defined by (5.3) is $(l\varphi/\delta_{\text{eff}})^{1/3}$ times greater than for $\varphi = 0$ [$\delta_{\text{eff}} \sim (\delta_0^3 D_0\varphi)^{1/4}$].

*arctg = $\tan^{-1}$.

For $(\delta D_0)^{1/2}/l \ll \varphi \ll \delta/D_0$, the impedance increases sharply:

$$Z = \Gamma^2\left(\frac{1}{4}\right)\left(\frac{2}{\pi}\right)^{1/2}\exp\left(\frac{3\pi i}{8}\right)\frac{1}{c}\left(\frac{\delta_0}{\lambda}\right)\left(\frac{D_0\varphi}{\delta_0}\right)^{1/4} \sim \omega^{3/4}H^{-1/4}. \quad (5.4)$$

Digressing from the increase of Z due to the increase of φ, we note that the appearance of the additional factor $\Gamma^2(1/4)\,2^{-3/4}\pi^{-3/2} \approx 1.4$ in formula (5.4) compared with (5.2) is explained by the field and current spikes in the bulk of the metal. Finally, for angles $\varphi \gtrsim \delta/D_0$, the magnitude of Z becomes equal to the impedance of a massive metal in zero magnetic field. [7]

The field spikes display themselves much more strongly in the variation of the impedance of the plate with the magnetic field. In order to determine Z for a plate of thickness d, we write down the equation for the Fourier components of the field in the plate:

$$-k^2\mathscr{E}(k) - 2E'(0) + 2E(d)\,k\sin kd + 2E'(d)\cos kd = (4\pi i\omega\sigma(k)/c^2)\mathscr{E}(k). \quad (5.5)$$

[the continuation of the field is $E(-\xi) = E(\xi)$, $E(\xi) = 0$ for $|\xi| > d$]. We have neglected the effect of the finite thickness of the plate on the conductivity operator σ. For relatively small spike amplitudes this is always valid if $\delta \ll d$, and for slowly decaying spikes for $D \ll d$. The term in $E'(d)$ in (5.5) can usually be neglected, since it leads to contributions of order δ/λ. [2)]

Substituting for E(d) its value in the infinite metal, we easily obtain a formula for the impedance of the plate:

$$Z(d) = Z_\infty - (8i\omega D_0/\pi c^2)\,dI^2(\alpha)/d\alpha, \quad (5.6)$$

where $Z_\infty = 8i\omega D_0 I(0)c^{-2}$ is the impedance of the half-space and

$$I(\alpha) = \int_0^\infty \frac{dx\cos\alpha x}{x^2 + 4\pi i\omega D_0^2 c^{-2}\sigma(x/D_0)}; \qquad \alpha = \frac{d}{D_0}. \quad (5.7)$$

In this way the surface impedance of the plate is related to the function I(t) that describes the field distribution in the bulk metal.

Figures 6 and 7 show schematically the form of the variation of the real and imaginary parts of $Z - Z_\infty$ respectively, as functions of $d/D_0 \sim Hd$. The impedance spikes are periodic in the field. The structure of the spikes is quite complicated, particularly if the quantity measured directly is not the impedance, but its derivative with respect to the magnetic field, as is often the case. [3,5,6]

[2)]From the boundary condition at the second surface of the plate from which only one wave escapes, it follows that $E'(d) = (i\lambda)^{-1}E(d)$.

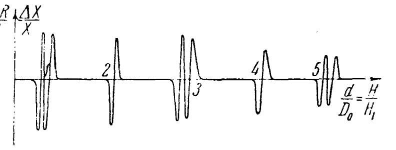

FIG. 6

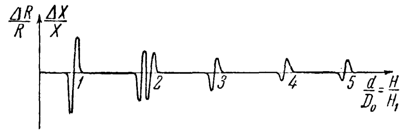

FIG. 7

The width of the spikes increases as H increases ($\sim \delta_{eff}/D$).

The appearance and disappearance of impedance spikes can be observed not only as a function of H, but also for a fixed value of H as the plate is rotated about the normal to its surface. [1] It is obvious that, due to the anisotropy of the diameter at the central section, for certain angles of rotation of the plate, its thickness d becomes a multiple of D_0.

The selective transparency of plates for $d = nD_0$ is one of the effects produced by field spikes in a metal. The power transmission coefficient is

$$T = |I(\alpha)/I(0)|^2\,|cR_\infty/2\pi|^2 \approx (2\delta_{eff}/\lambda)^2\,|I(\alpha)/I(0)|^2.$$

Under usual conditions the ratio $I(\alpha)/I(0)$ is exponentially small for $\alpha \gg \delta_{eff}$. When slowly decaying spikes are present $I(\alpha)/I(0)$ is many orders larger, and for small n can even by of the order unity.

It is apparent that the study of impedance jumps in plates gives another convenient method of determining the anisotropy of the central diameters of the Fermi surface, and thereby allows its shape to be established.

6. COMPARISON WITH EXPERIMENT

Gantmakher [3] observed impedance spikes periodic in H in a plate of very pure tin (d = 0.4 mm, l ~ 1-3 mm) at a frequency of 1-5 Mc. In these experiments the magnetic field was parallel to the surface. The specimen was placed in the coil of a resonant circuit, and the derivative with respect to the field of the reactive part of the impedance, dX/dH, was measured by a modulation method. In agreement with theory a complicated structure of spikes was discovered, for which the widths increased and the amplitudes quite rapidly decreased on increasing the field. The study of the anisotropy of the effect also agreed with the conclusions of the theory: the amplitude of the spikes was greatest close to $\mathbf{E} \perp \mathbf{H}$. Spikes were clearly seen associated with the various extremal diameters of the electrons (a "chain" of orbits with various "links").

The anomalously large amplitude of the spikes and their comparatively slow decrease for electrons of the fourth hole zone attracts attention. This feature, as Gantmakher correctly remarked, is related to the fact that, for that orientation of $\mathbf{H}$ relative to the axes of the single crystal, the Fermi surface in this zone is very nearly a cylinder, the diameter of which is almost independent of p_z. Therefore, the field spikes are, in fact, determined by all the electrons on the Fermi surface. Owing to such a peculiarity of the dispersion law, no additional selection of electrons with respect to diameters is necessary, and the field distribution is described not by formula (4.7) with $|a| \ll 1$, but by the function

$$I(t) = \int_0^\infty \frac{x \cos xt\, dx}{x^3 + iM^3(1 - \sin x)}, \qquad M^3 = \frac{D^2 l}{\delta_0^3}. \tag{6.1}$$

The field spikes decay slowly (see Fig. 4) ($l_{\text{dec}} \sim D_0 M^{3/2}$). Close to $t = n$

$$I(n) = I(0)(-1)^n \frac{3^{1/2}\Gamma(1/3)}{2^{2/3}\pi} \cos\frac{n\pi}{2} \frac{\Gamma(n + 1/3)}{\Gamma(n + 2/3)}; \qquad I(0) = \frac{\Gamma^2(1/6)\exp(-\pi i/6)}{M3\sqrt{3}\Gamma(1/3)}. \tag{6.2}$$

For large n

$$I(n)/I(0) \sim 3^{1/2} 2^{-2/3} \pi^{-1} \Gamma(1/3)(-1)^n n^{-1/3} \cos n\pi/2. \tag{6.3}$$

The impedance spikes of the plates are similar to those shown in Fig. 6. The fact that in the experiments Gantmakher observed up to five impedance spikes confirms their very slow decay.

I am very grateful to I. M. Lifshitz and M. I. Kaganov for discussing the results of this work.

Note added in proof (February 12, 1963): On satisfying inequality (2.10) there should exist in the metal a peculiar resonance effect due to the inclusion of non-diagonal elements of the conductivity tensor $\sigma_{y\zeta}$.[8] The physical nature of this resonance is related to the excitation in the metal of weakly decaying electromagnetic waves with a discrete spectrum. The formulae obtained in the given work are valid far from the region of resonance excitation of similar waves.

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