

OPEN RESONATORS FOR LASERS

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A theory of natural vibrations is developed for resonators consisting of sections of circular or plane waveguide or formed by plane parallel mirrors of rectangular or circular shape. This theory is based on a rigorous theory of the diffraction at the open end of a waveguide and leads to simple and graphic relations whose accuracy increases with increase of the frequency and with decrease of the radiative damping of the oscillations. Resonators of these types are of interest for lasers, and also for the physics and technology of millimeter and submillimeter waves.

INTRODUCTION

THE quantum light generator (laser), many forms of which have now been developed, has as its oscillating system an open resonator, which in the simplest case consists of two plane parallel mirrors placed opposite to one another. Open resonators of this type have been considered by a number of authors,^[1-6] but a quantitative theory of resonators with plane mirrors was first given in a paper by Fox and Li,^[7] and this theory lacks intuitive clarity and requires computations with fast computing machines; these calculations are based essentially on a modelling (with some simplifying assumptions) of the process of the establishing of oscillations in an open resonator.

A remarkable feature of open resonators is that all of their dimensions are much larger than the wavelength, and the spectrum of natural frequencies is sparse as compared with that of a closed resonating volume. Therefore open resonators should find wide application in the physics and technology of millimeter and submillimeter waves (and perhaps of longer waves as well).

We give here a theory of open resonators formed by plane mirrors, and also of resonators consisting of sections of waveguide with open ends, and we show that all of the characteristics of such resonators can be calculated rather simply, at least for oscillations with small radiation losses. This theory also gives an intuitive physical idea of the principle of operation of open resonators.

The basis of the theory is the following idea (stated by Suchkin^[6]): the smallness of the radiation loss of an open resonator is due to the fact that the wave reaching the edge or end of the reso-

nator is a guided wave of frequency only slightly higher than the critical frequency; such a wave, as has been shown in a number of papers^[8-12] and in a book by the present writer,^[13] scarcely emerges as radiation at all, but is reflected back with a reflection coefficient of absolute value close to unity. Suchkin did not succeed, however, in developing this idea in a sufficiently convincing way, nor in making quantitative calculations for open resonating systems, nor in particular in finding the main characteristics of their natural oscillations (frequency, radiative damping, field and current distributions).

As will be seen from what follows, this idea actually allows us to construct a theory of open resonators, and thus this theory is implicitly contained in the theory of diffraction at an open end of a waveguide.

1. DIFFRACTION AT AN OPEN END OF A WAVEGUIDE

Let us consider the diffraction at the end of a plane waveguide (Fig. 1) from a point of view rather different from that used in previous treatments.^[8,9,13] Taking the time dependence in the form $e^{-i\omega t}$, we shall assume that the wave number $k = \omega/c$ is connected with the distance $2a$ between the plates of the waveguide by the following relation

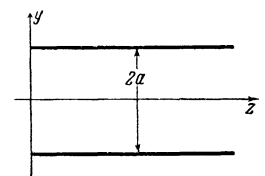


FIG. 1. Waveguide with open end.

$$ka = \pi(q/2 + p), \quad (1)$$

where q is a large integer and $|p| < 1/2$. Let there arrive at the open end of the guide a wave H_{0q} or E_{0q} with field independent of the coordinate x and frequency which by Eq. (1) is close to the critical frequency (these frequencies are equal for $p = 0$). The diffraction field of this wave can be expressed simply in terms of a function $F(w)$, which is connected with the surface density of current on the upper plate $y = a$ by the relation

$$f(z) = \int_C e^{i\omega z} F(w) dw, \quad (2)$$

where the path C goes mainly along the real axis and dips below the point $w = -w_0$ which corresponds to the wave arriving at the end. The function $F(w)$ is determined from the functional equations (cf. [8] or Chapter I of [13])

$$\int_C e^{i\omega z} F(w) dw = 0 \text{ for } z < 0, \\ \int_C e^{i\omega z} L(w) F(w) dw = 0 \text{ for } z > 0, \quad (3)$$

where for a wave H_{0q}

$$L(w) = \frac{k}{v} [1 - (-1)^q e^{2iva}], \quad (4)$$

and for a wave E_{0q}

$$L(w) = \frac{v}{k} [1 - (-1)^q e^{2iva}]. \quad (5)$$

Under the conditions

$$|\omega| \ll k, \quad 2ak^{-3} |\omega|^4 \ll 1 \quad (6)$$

the functions (4) and (5) can be replaced by the simpler (integral) function

$$L(w) = 1 - \exp\{i(2\pi p - w^2 a/k)\}, \quad (7)$$

since in the exponent we can set

$$v \equiv \sqrt{k^2 - \omega^2} = k - \omega^2/2k, \quad (8)$$

and in the coefficient of the brackets we can set $v = k$.

The function (7) vanishes for $w = \pm w_j$, where

$$w_j = \sqrt{k/2a s_j}, \quad s_j = \sqrt{4\pi(j+p)}. \quad (9)$$

It is easy to verify that w_j is the wave number of the wave H_{0q+2j} or E_{0q+2j} in the plane waveguide [approximately, under the conditions (1) and (6)]. The values $j = 1, 2, \dots$ in Eq. (9) give propagating waves; the values $j = -1, -2, \dots$ give damped waves {for which $s_j = i|s_j| = i[4\pi(-j-p)]^{1/2}$ }; and the value $j = 0$ corresponds to the arriving wave H_{0q} or E_{0q} , which propagates for $p > 0$ and is damped for $p < 0$.

We shall solve the equations (3) with the integral function (7), so that at the very beginning we introduce into the diffraction problem an approximation which depends on the condition (6). The function (7) breaks up into factors

$$L(w) = L_1(w) L_2(w), \quad L_2(w) = L_1(-w), \quad (10)$$

where $L_1(w)$ is a function holomorphic in the upper half-plane $\text{Im } w \geq 0$, which satisfies in that half-plane the condition $L_1(w) \rightarrow 1$ for $|w| \rightarrow \infty$ and is given by the formulas

$$L_1(w) = e^{U(s,p)}, \quad s = \sqrt{\frac{2a}{k}} w, \\ U(s,p) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln(1 - e^{i2\pi p - t^2/2}) \frac{dt}{t - se^{i\pi/4}}. \quad (11)$$

The function $U(s,p)$ was introduced in [9] and afterwards was studied in detail and tabulated in a book by the writer. [13] When the equations (3) are solved the functions $F(w)$ and $f(z)$ are obtained in the forms

$$F(w) = \frac{A}{2\pi i} \frac{L_1(w_0)}{(w + w_0) L_2(w)}, \quad (12)$$

$$f(z) = A \left(e^{-i\omega z} + \sum_j R_{0,j} e^{i\omega_j z} \right), \quad (13)$$

where A is the current amplitude of the arriving wave,

$$R_{0,j} = - \frac{i \exp\{U(s_0, p) + U(s_j, p)\}}{(s_0 + s_j) s_j} \quad (14)$$

is (for $j \neq 0$) the coefficient of transformation of the incident wave into the wave with the index j , and $R_{0,0}$ is the reflection coefficient of the incident wave with respect to current. Analogous formulas are obtained in cases of incident waves which do not have the index $j = 0$, but indices $j = 1, j = -1$, and so on.

Figure 2 shows the absolute values of the coefficients $R_{0,0}$, $R_{0,1}$, $R_{1,1}$, and $R_{1,0}$ as functions of p for $-1/2 < p < 1/2$. In virtue of the relation

$$U(is, p) = U^*(s, -p) \quad (15)$$

we also have

$$|R_{-1,-1}(p)| = |R_{1,1}(-p)|, \quad |R_{-1,0}(p)| = |R_{1,0}(-p)|, \\ |R_{0,-1}(p)| = |R_{0,1}(-p)|. \quad (16)$$

Thus for small values of p the wave with the index $j = 0$ undergoes strong reflection, with almost no transformation into the waves with the indices $j = \pm 1$. For $|p| < 1/2$ the latter waves are weakly reflected from the end ($|R_{1,1}|$ and $|R_{-1,-1}|$ are smaller than 0.1) and cannot give rise to oscillations with small losses to radiation. Therefore the only quantity of importance for what follows is

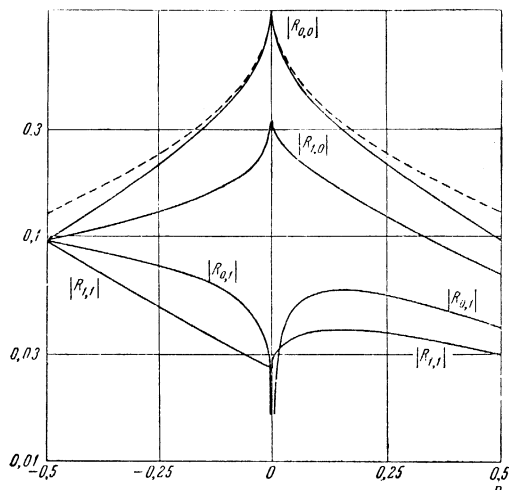


FIG. 2. Absolute values of reflection and transformation coefficients for current.

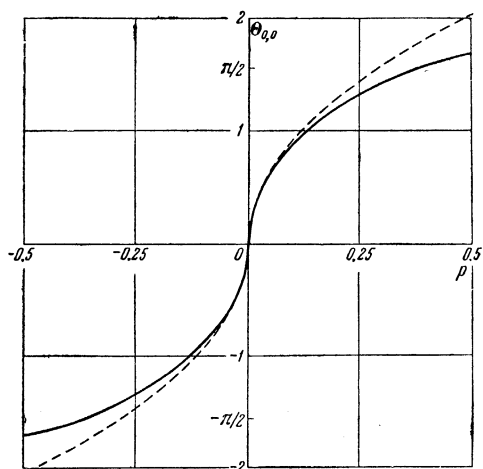


FIG. 3. Phase of the reflection coefficient $R_{0,0}$.

the reflection coefficient

$$R_{0,0} = -|R_{0,0}| \exp(i\theta_{0,0}), \quad (17)$$

whose phase $\theta_{0,0}$ is shown in Fig. 3; by Eq. (15) it is an odd function of p , whereas $|R_{0,0}|$ is an even function. For small p the coefficients (17) can be represented in the form

$$R_{0,0} = -\exp\{i\beta(1+i)s_0\}, \quad (18)$$

where

$$\beta = -\zeta(1/2)/\sqrt{\pi} = 0.824, \quad s_0 = \sqrt{4\pi p}, \quad (19)$$

and $\zeta(z)$ is the zeta function of Riemann. The dashed lines in Figs. 2 and 3 show the functions $|R_{0,0}|$ and $\theta_{0,0}$ calculated from the approximate formula (18); we see that it gives agreement within the accuracy of plotting for $|p| < 0.05$, and is qualitatively useful right up to $|p| \approx 0.5$.

The physical meaning of these results is as follows. As is well known, a wave propagating in a

plane waveguide can be represented as the sum of two plane waves, i.e., as two beams of parallel rays. If the frequency of the wave is close to the critical frequency, these rays make a small angle ϵ with the normal to the walls (Fig. 4) and therefore can easily be turned, owing to diffraction,

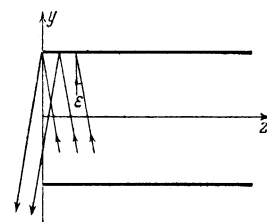


FIG. 4. For the interpretation of Eq. (18).

through the angle 2ϵ , which results in the formation of a reflected wave. There is practically no transformation into waves with other indices, since in such a transformation the rays have to be turned through much larger angles. The reflection coefficient depends not on the angle ϵ alone, but on the parameter $s_0 = \epsilon(2ka)^{1/2}$ which characterizes the diffraction creep (transverse diffusion) of the beam of rays reflected from each semiinfinite wall to the distance $2a$, where the other wall is located, which forms the reflected wave from the turned rays.

Let us now consider a circular waveguide of radius a . Suppose that at its open end there arrives a wave E_{mn} with frequency close to its critical frequency, i.e., with

$$ka = \nu_{mn} + \pi p, \quad J_m(\nu_{mn}) = 0, \quad (20)$$

or else a wave H_{mn} with the analogous condition

$$ka = \mu_{mn} + \pi p, \quad J'_m(\mu_{mn}) = 0, \quad (21)$$

where $|p| < 1/2$, $m = 0, 1, 2, \dots$ is the azimuthal index, and $n \gg 1$ is the number of the root. In the general case of the reflection of a wave E_{mn} ($m \neq 0$) from the open end, in addition to the longitudinal component

$$j_z = f(z) \sin(m\varphi + \varphi_0) \quad (22)$$

of the current density there is an azimuthal component j_φ , and the functional equations are complicated (cf. [10,11] or Chapter IV in [13]). They become simpler under the conditions

$$m \ll ka, \quad |\omega| \ll k, \quad (23)$$

which follow from the condition (20), and go over into the simpler equations (3) with the function (7).

Under the conditions (23) a wave H_{mn} excites an azimuthal component of current density

$$j_\varphi = f(z) \cos(m\varphi + \varphi_0) \quad (24)$$

and a negligibly small component j_z . The new function $F(w)$, connected with $f(z)$ by the relation (2), also satisfies the equations (3) with the function (7), but now the parameter p is determined by Eq. (21), not by Eq. (20).

It follows from this that the formulas (12)–(19), Figs. 2 and 3, and all of our previous conclusions apply to waves E_{mn} and H_{mn} in a circular waveguide with an open end. We note that the vector character of electromagnetic waves does not manifest itself near the critical frequency: the same results are obtained (cf. [12] or Chapter III of [13]) for scalar (sound) waves; waves satisfying the condition $\Phi = 0$ at the wall behave like waves E_{mn} , and those satisfying the condition $\partial\Phi/\partial r = 0$ behave like waves H_{mn} .

Having obtained the asymptotic laws relating to the reflection of waves of high index near their critical frequencies (i.e., for $ka \gg 1$), we may naturally ask: when do these laws come into effect? An inspection of the graphs given in [8–13] shows that only the wave H_{11} in a circular waveguide (among waves of the guided type) has qualitatively different properties, and that already for the E_{03} and H_{03} waves in a plane waveguide ($ka \approx 3\pi/2$) and for E_{m2} and H_{m2} waves in a circular waveguide ($ka \approx \nu_{m2}$ and $ka \approx \mu_{m2}$) the asymptotic laws give satisfactory accuracy, which improves rapidly as ka increases, i.e., as the number of the wave increases.

We note that the asymptotic laws are not affected by the behavior of the current near a sharp edge, which is different for waves of different polarizations, for example for H_{0q} and E_{0q} waves in a plane waveguide. Moreover, our solution (13) does not involve the current flowing on the outer surfaces of the walls [since the function (7) is an integral function]. Therefore the asymptotic laws are valid for waveguides with flanges, with walls of finite thickness (not necessarily metallic), and so on.

2. OPEN CYLINDRICAL RESONATOR (OPEN TUBE)

As is well known, an open tube (under the condition $ka \ll 1$) is an acoustical resonator with high figure of merit. The design of such a resonator is based on the theory of diffraction at the open end of the tube (cf. [12] or Chapter III of [13]). In the opposite case $ka \gg 1$ the open tube also has good resonance properties for both electromagnetic and acoustical oscillations. The calculation for this case is easily made by means of the formulas derived above.

Let $2a$ be the diameter of the tube and $2l$ its length. We choose the origin of coordinates at the center of the tube, so that the inner surface of the tube is given by $r = a$, $-l < z < l$. We shall look for the value of the complex wave number $k = \omega/c$ of the oscillation E_{mnq} ($q = 1, 2, \dots$) in the cylindrical resonator without special terminations in the form (20) with unknown but small complex parameter p . Then the function $f(z)$, which in Eq. (22) determines the current distribution on the surface of the tube, will be of the form

$$f(z) = \cos \omega_0 z \quad \text{for } q = 1, 3, \dots, \quad (25)$$

$$f(z) = \sin \omega_0 z \quad \text{for } q = 2, 4, \dots, \quad (26)$$

where ω_0 is connected with p by Eq. (9) with $j = 0$. In fact, because of the symmetry of the system the function $f(z)$ is either even or odd; since the vibration E_{mnq} arises as the result of successive reflections of the wave E_{mn} from the two ends, and since these produce practically no other waves, we get the formulas (25) and (26). The former can be written

$$f(z) = \frac{1}{2} e^{i\omega_0 l} [e^{i\omega_0(z-l)} + e^{-2i\omega_0 l} e^{-i\omega_0(z-l)}]. \quad (27)$$

The first term in the brackets is the wave arriving at the open end $z = l$, and the second term is the reflected wave, which by Eq. (13) must be equal to $R_{0,0} \exp\{-i\omega_0(z-l)\}$. Equating $R_{0,0}$ to the coefficient which appears in Eq. (27), we get the characteristic equation

$$R_{0,0} = e^{-2i\omega_0 l} \quad (q = 1, 3, \dots). \quad (28)$$

The formula (26) gives the characteristic equation

$$R_{0,0} = -e^{-2i\omega_0 l} \quad (q = 2, 4, \dots). \quad (29)$$

Using the fact that according to Eq. (9)

$$2\omega_0 l = Ms_0, \quad M = \sqrt{2kl^2/a}, \quad (30)$$

we can rewrite the equations (28) and (29) in the form

$$e^{l(M+\beta+i\beta)s_0} = (-1)^q, \quad q = 1, 2, 3, \dots \quad (31)$$

with the obvious solution

$$s_0 = \sqrt{4\pi p} = \pi q / (M + \beta + i\beta), \quad q = 1, 2, 3, \dots \quad (32)$$

The required complex parameter p is given by

$$p = \pi q^2 / 4(M + \beta + i\beta)^2 \quad (33)$$

or in more detail

$$p = p' - ip'', \quad p' = \frac{\pi q^2}{4} \frac{M(M+2\beta)}{[(M+\beta)^2 + \beta^2]^2}, \quad (34)$$

$$p'' = \frac{\pi q^2}{2} \frac{\beta(M+\beta)}{[(M+\beta)^2 + \beta^2]^2}.$$

The frequency $\omega = \omega' - i\omega''$ of the oscillation E_{mnq} is complex. By the relation $\exp(-i\omega t) = \exp(-i\omega't) \exp(-\omega''t)$ the quantities p' and p'' have the following meanings: $\Delta = 2\pi p'$ is the additional increase of the phase in the time $\tau = 2a/c$ owing to the fact that the frequency of the oscillations is not equal to the critical frequency, and

$$\Lambda = 1 - \exp(-4\pi p'') \approx 4\pi p'' \quad (35)$$

is the fractional decrease of the energy of the oscillations during the same time τ . Because p' is small, the quality of the oscillation E_{mnq} is given by

$$Q = \nu_{mn}/2\pi p'' \quad (36)$$

The formulas (32)–(36) are approximate. They give good accuracy if Eq. (33) gives $|p| < 0.05$, and are roughly correct right up to $|p| \sim 0.5$ (see Section 1). For large values of M

$$p' \approx \pi q^2/4M^2, \quad p'' \approx \pi\beta q^2/2M^3, \quad (37)$$

and we get the series of resonance shown schematically in Fig. 5. The width of the q -th resonance curve is proportional to q^2 , and the distance between adjacent curves is proportional to $2q \pm 1$; for $q \sim M$ the resonance curves begin to overlap, and then our approximate formulas are no longer good.

Setting $q = 0$ in Eqs. (32)–(34) we get a formal solution of the equation (31) which has no physical meaning; it corresponds to undamped oscillations at the critical frequency, which are impossible in open systems.

When we use the relations (30) and (32) the functions (25) and (26) take the forms

$$f(z) = \cos \frac{\pi qz}{2l[1 + \beta(1+i)/M]}, \quad q = 1, 3, \dots; \quad (38)$$

$$f(z) = \sin \frac{\pi qz}{2l[1 + \beta(1+i)/M]}, \quad q = 2, 4, \dots, \quad (39)$$

and for small values of p the formulas (29) and (33) can be rewritten in the form

$$k_{mnq} = \left\{ \left(\frac{\nu_{mn}}{a} \right)^2 + \left[\frac{\pi q}{2l(1 + \beta(1+i)/M)} \right]^2 \right\}^{1/2}. \quad (40)$$

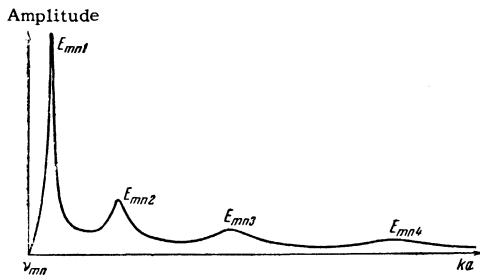


FIG. 5. Resonance curve of open tube.

The formulas (38) and (39) show that for $M \gg 1$ the current distribution of the oscillation E_{mnq} in an open cylindrical resonator has a node near each end $z = \pm l$, whereas in a closed cylindrical resonator there are antinodes of the current at the end walls $z = \pm l$. For this reason there is an oscillation E_{mn0} in the closed resonator which is impossible in an open one, but the natural frequencies of the other oscillations E_{mnq} are nearly the same if the quantity M is sufficiently large. The radiation losses in open resonators make their spectra more widely space (Sec. 6).

In this system the same formulas (25)–(40) hold for the oscillations H_{mnq} , except that by Eq. (21) we must replace ν_{mn} by μ_{mn} . The function $f(z)$ is now used in Eq. (24) to give the distribution of the azimuthal current. It is interesting to note that for $M \gg 1$ the vibrations H_{mnq} in open and closed resonators of the same dimensions have nearly the same current and field distributions; the only difference is that in a closed resonator the current falls to zero at $z = \pm l$, whereas in the open resonator it falls to small values of the order of $1/M$ (see Figs. 8 and 9 below). The drop of the current to small values at the boundaries is a characteristic effect in open resonators; owing to it the radiation loss is kept to a minimum.

In conclusion let us consider a "semi-open" resonator of length l (reflecting piston at $z = 0$, open end at $z = l$). It is easy to show that in such a resonator there will be oscillations E_{mnq} with odd indices q and oscillations H_{mnq} with even indices q , which are the same as for the open resonator of length $2l$ examined above.

3. TWO-DIMENSIONAL RESONATOR FORMED BY PLANE MIRRORS

Figure 6 shows a resonator formed by two parallel mirrors of width $2a$ and of infinite length, separated by the distance $2l$. Let us introduce coordinates x, y, z as shown in Fig. 6, and consider two-dimensional oscillations in the resonator with fields independent of y . In the new coordinate system x, y, z these oscillations can be denoted by $E_{m0q}^{(x)}$ and $H_{m0q}^{(x)}$; the index x shows that the classification is based on the x component of the field, and not the z component as in the usual notation (the oscillation $H_{m0q}^{(x)}$ can also be labelled $E_{m0q}^{(y)}$, cf. Sec. 4).

The properties of these oscillations can be derived without difficulty from the relations obtained in Sec. 2, since this resonator is a section of plane waveguide, and in Sec. 2 we considered a section of circular waveguide. The oscillations $E_{m0q}^{(x)}$ occur

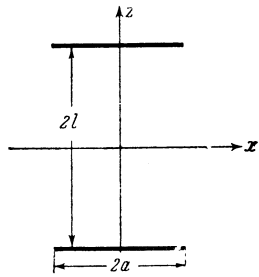


FIG. 6. Two-dimensional resonator with plane mirrors.

at a frequency $\omega = ck$ such that

$$kl = \pi (q/2 + p), \tag{41}$$

where q is a large integer (for practical purposes $q > 3$) and p is a small correction, which is found to be given by

$$p = \pi m^2 / 4(M + \beta + i\beta)^2, \tag{42}$$

where

$$M = \sqrt{2ka^2/l}. \tag{43}$$

Equations (41) and (42) can be combined into a single formula

$$k_{mq} = \frac{\pi}{2} \left\{ \left[\frac{m}{a(1 + \beta \frac{1+i}{M})} \right]^2 + \left(\frac{q}{l} \right)^2 \right\}^{1/2}. \tag{44}$$

The current distribution (longitudinal, i.e., directed along the x axis) is given by the functions

$$f(x) = \cos \frac{m\pi x}{2a[1 + \beta(1+i)/M]} \text{ for } m = 1, 3, \dots, \tag{45}$$

$$f(x) = \sin \frac{m\pi x}{2a[1 + \beta(1+i)/M]} \text{ for } m = 2, 4, \dots \tag{46}$$

The same formulas also apply to the oscillations $H_{m0q}^{(x)}$, for which the function $f(x)$ gives the distribution of transverse current—directed along the y axis. Thus for this resonator there is a characteristic polarization degeneracy—the oscillations $E_{m0q}^{(x)}$ and $H_{m0q}^{(x)}$ have the same frequency (41). The degeneracy disappears if we go over to the semi-open resonator shown in Fig. 7; in this resonator, owing to the presence of the ideally conducting wall at $x = 0$ the only oscillations are $E_{m0q}^{(x)}$ with odd indices m and $H_{m0q}^{(x)}$ with even m , which are the same as the corresponding oscillations of the open resonator of Fig. 6.

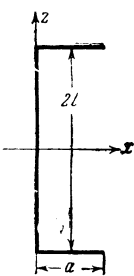


FIG. 7. Semi-open resonator.

The two-dimensional resonator with plane (“ribbon”) mirrors has been studied in detail by Fox and Li,^[7] who obtained from elaborate calculations curves for the quantities which we denoted in Section 2 by Λ (fractional energy loss during the time $\tau = 2l/c$) and Δ (additional phase change in the time τ), for $m = 1$ and $m = 2$. Oscillations with higher indices were not investigated in [7] and could scarcely be dealt with in the framework of the method used in that paper.

In our theory Λ and Δ are given by the simple formulas

$$\Lambda = 4\pi\rho'' = 2\pi^2 m^2 \frac{\beta(M + \beta)}{[(M + \beta)^2 + \beta^2]^2}, \tag{47}$$

$$\Delta = 2\pi\rho' = \frac{\pi^2 m^2}{2} \frac{M(M + 2\beta)}{[(M + \beta)^2 + \beta^2]^2}, \tag{48}$$

where the quantity M is connected with the “Fresnel zone number”

$$N = a^2/2l\lambda = ka^2/4\pi l, \tag{49}$$

introduced in [7] by the simple relation

$$M = \sqrt{8\pi N}. \tag{50}$$

If we compare the formulas (48) with the curves given in [7] for $m = 1$ and $m = 2$, it is found that they agree within the accuracy of reading the curves.

Figures 8 and 9 show the current distributions on the plates of the resonator for the oscillations $m = 1$ and $m = 2$, for the values of N for which the corresponding curves are given in [7]. When we compare these latter curves with Figs. 8 and 9, we see that they are different: the curves of Fox and Li are irregular and wavy, and in general do not run the same as our curves. The difference is due to the fact that we have not taken into account the waves of other numbers ($j = 1, j = -1$, etc.) which arise, though indeed with small ampli-

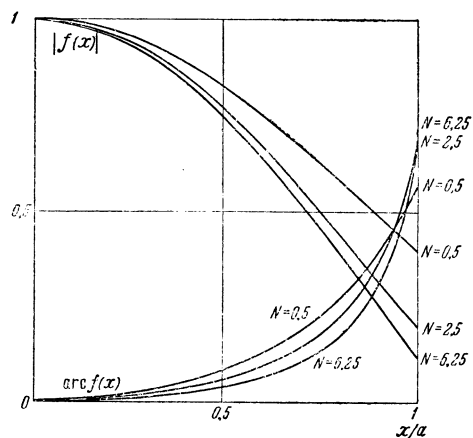


FIG. 8. The function (45) for $m = 1$.

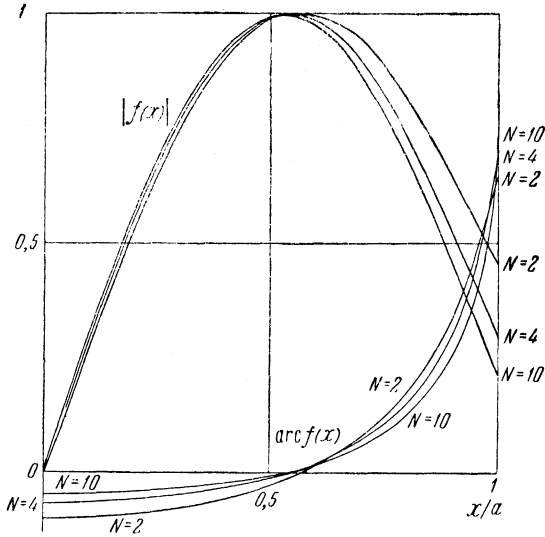


FIG. 9. The function (46) for $m = 2$.

tudes, when the waves are reflected from the edge. These "fine details" of the current distributions are unimportant for the calculation of the natural frequencies, just as they are unimportant for the calculation of the Joule and other losses caused by the fact that the walls of the open resonator are not ideally reflecting.

4. OPEN RESONATOR FORMED BY RECTANGULAR MIRRORS

Let an open resonator be formed by parallel rectangular mirrors

$$-a < x < a, \quad -b < y < b, \quad z = \pm l. \quad (51)$$

In the analysis of the oscillations in this system we shall start from the scalar wave equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0 \quad (52)$$

and the boundary condition $\Phi = 0$ at the mirrors. The solution of Eq. (52) can be written in the form

$$\Phi(x, y, z) = \frac{2\pi i}{c} \iint_{-\infty}^{\infty} e^{i(\omega_x x + \omega_y y)} \frac{1}{v} [e^{i v |z-l|} - (-1)^q e^{i v |z+l|}] F(\omega_x, \omega_y) d\omega_x d\omega_y, \quad (53)$$

where

$$v = \sqrt{k^2 - \omega_x^2 - \omega_y^2}. \quad (54)$$

Writing

$$L(\omega_x, \omega_y) = (k/v) [1 - (-1)^q e^{2i v l}], \quad (55)$$

we get as the equations for the unknown function $F(w)$:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega_x x + \omega_y y)} F(\omega_x, \omega_y) d\omega_x d\omega_y &= 0 \quad \text{for } |x| > a, |y| > b, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega_x x + \omega_y y)} L(\omega_x, \omega_y) F(\omega_x, \omega_y) d\omega_x d\omega_y &= 0 \quad \text{for } |x| < a, |y| < b, \end{aligned} \quad (56)$$

which cannot be solved exactly. For $\omega_x \ll k, \omega_y \ll k$ the function (55) takes the form

$$\begin{aligned} L(\omega_x, \omega_y) &= 1 - \exp \left\{ i \left(2\pi \rho - \frac{\omega_x^2 + \omega_y^2}{k} l \right) \right\} \\ &= 1 - \exp \left\{ i \left(2\pi \rho_a - \frac{\omega_x^2}{k} l \right) \right\} \exp \left\{ i \left(2\pi \rho_b - \frac{\omega_y^2}{k} l \right) \right\}, \end{aligned} \quad (57)$$

where we have set

$$kl = \pi (q/2 + p), \quad p = \rho_a + \rho_b. \quad (58)$$

Substituting the function (57) in the equations (56), we see that they have the solution

$$F(\omega_x, \omega_y) = F_a(\omega_x) F_b(\omega_y), \quad (59)$$

where the function $F_a(w)$ is a solution of the equations

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i w x} F_a(w) dw &= 0 \quad \text{for } |x| > a, \\ \int_{-\infty}^{\infty} e^{i w x} \left[1 - \exp \left\{ i \left(2\pi \rho_a - \frac{w^2}{k} l \right) \right\} \right] F_a(w) dw &= 0 \quad \text{for } |x| < a, \end{aligned} \quad (60)$$

which correspond to the two-dimensional oscillations of the resonator considered in Sec. 3, and the function $F_b(w)$ satisfies the same equations with a replaced by b . The function

$$f(x, y) = \iint_{-\infty}^{\infty} e^{i(\omega_x x + \omega_y y)} F(\omega_x, \omega_y) d\omega_x d\omega_y \quad (61)$$

can also be put in the form of a product

$$f(x, y) = f_a(x) f_b(y). \quad (62)$$

where, according to Sec. 3,

$$\begin{aligned} f_a(x) &= \cos \frac{m\pi x}{2a(1 + \beta(1+i)/M_a)}, \\ f_b(y) &= \cos \frac{n\pi y}{2b(1 + \beta(1+i)/M_b)} \quad \text{for } m, n = 1, 3, \dots, \\ f_a(x) &= \sin \frac{m\pi x}{2a(1 + \beta(1+i)/M_a)}, \\ f_b(y) &= \sin \frac{n\pi y}{2b(1 + \beta(1+i)/M_b)} \quad \text{for } m, n = 2, 4, \dots, \\ M_a &= \sqrt{2ka^2/l}, \quad M_b = \sqrt{2kb^2/l}. \end{aligned} \quad (63)$$

The parameters ρ_a and ρ_b are given by

$$\rho_a = \pi m^2/4 (M_a + \beta + i\beta)^2, \quad \rho_b = \pi n^2/4 (M_b + \beta + i\beta)^2, \quad (64)$$

so that

$$k_{mnq} = \left\{ \left[\frac{m}{a(1 + \beta(1 + i)/M_a)} \right]^2 + \left[\frac{n}{b(1 + \beta(1 + i)/M_b)} \right]^2 + \left(\frac{q}{l} \right)^2 \right\}^{1/2}. \quad (66)$$

The formulas (56)–(66) are also obtained when the boundary condition is $\partial\Phi/\partial z = 0$. To go over to the case of the vector electromagnetic field we use the formulas

$$\mathbf{E} = ik^{-1} (\text{grad div } \mathbf{A} + k^2 \mathbf{A}), \quad \mathbf{H} = \text{rot } \mathbf{A} \quad (67)^*$$

and set Φ equal to one of the rectangular components of the vector potential \mathbf{A} . Then the expression (62) must be proportional to the current density on the mirror—only then are the formulas (63) valid; in their derivation one uses the expression for the current reflection coefficient. We can set

$$\Phi = A_x \text{ and } \Phi = A_y \quad (68)$$

but we cannot take Φ to be A_z or the z component of the magnetic vector potential.

With $\Phi = A_x$ we get an oscillation $E_{mnq}^{(x)}$ polarized along the x axis, for which $j_x = f$ and $j_y = 0$; with $\Phi = A_y$ we get an oscillation $E_{mnq}^{(y)}$, for which $j_x = 0$ and $j_y = f$ (see Fig. 10). The oscillations $E_{mnq}^{(x)}$ and $E_{mnq}^{(y)}$ have the same frequency (66); this is the polarization degeneracy, which is removed in the semi-open resonator with three walls

$$\begin{aligned} 0 < x < a, & \quad -b < y < b, & \quad z = \pm l; \\ x = 0, & \quad -b < y < b, & \quad -l < z < l, \end{aligned} \quad (69)$$

for which the only oscillations are $E_{mnq}^{(x)}$ with odd m and $E_{mnq}^{(y)}$ with even m , which are the same as in the open oscillator. By means of other walls one can obtain other semi-open resonators from the open resonator (for example, a section of a rectangular waveguide), and there is also no difficulty in making the calculations for these.

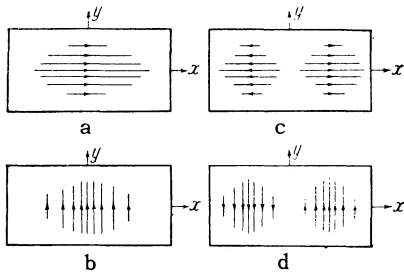


FIG. 10. Current distributions on rectangular mirror for the oscillations E_{11q} and E_{21q} : a—oscillation $E_{11q}^{(x)}$; b—oscillation $E_{11q}^{(y)}$; c—oscillation $E_{21q}^{(x)}$; d—oscillation $E_{21q}^{(y)}$.

*rot = curl.

5. OPEN RESONATOR FORMED BY CIRCULAR MIRRORS

When we consider a resonator formed by parallel circular mirrors given, in cylindrical coordinates r, φ, z by the equations

$$0 < r < a, \quad z = \pm l, \quad (70)$$

the solutions of Eq. (52) are of the form

$$\Phi(r, \varphi, z) = \Psi(r, z) \cos m\varphi, \quad (71)$$

so that

$$f(r, \varphi) = \frac{c}{4\pi} \frac{\partial\Phi}{\partial z}(r, \varphi, l=0) = g(r) \cos m\varphi. \quad (72)$$

Under the condition (41) we can write the function $g(r)$ in the form

$$g(r) = J_m(\omega_0 r), \quad (73)$$

since when the wave is reflected from the edge $r = a$ there is practically no production of other waves. We shall assume that the reflection coefficient for the cylindrical wave at the edge $r = a$ is approximately equal to the coefficient (18) derived for a plane wave, and introduce functions $A_m(x)$ and $\Omega_m(x)$ —the amplitude and phase of the cylindrical wave—with the formulas

$$H_m^{(1)}(x) = -iA_m(x)e^{i\Omega_m(x)}, \quad J_m(x) = A_m(x) \sin \Omega_m(x). \quad (74)$$

If we require that the function (73) have the form (cf. Sec. 2)

$$\begin{aligned} (r) = \frac{1}{2i} A_m(\omega_0 r) e^{i\Omega_m(\omega_0 a)} \{ e^{i[\Omega_m(\omega_0 r) - \Omega_m(\omega_0 a)]} \\ + R_{0,0} e^{-i[\Omega_m(\omega_0 r) - \Omega_m(\omega_0 a)]} \}, \end{aligned} \quad (75)$$

we get the characteristic equation

$$R_{0,0} = -e^{-2i\Omega_m(\omega_0 a)}. \quad (76)$$

As in Eq. (20), let us denote the n -th zero of the function $J_m(x)$ by ν_{mn} . Using the relations

$$\Omega_m(\nu_{mn}) = n\pi, \quad \Omega_m(x) = \Omega_m(\nu_{mn}) + \Omega'_m(\nu_{mn})(x - \nu_{mn}) \quad (77)$$

and the fact that in Eq. (18) we must replace s_0 by $s_0 \Omega'(\nu_{mn})$, we put Eq. (76) in the form

$$e^{i(M + \beta + i\beta) s_0 \Omega'(\nu_{mn})} = e^{2i\nu_{mn} \Omega'(\nu_{mn})}, \quad M = \sqrt{2ka^2/l}, \quad (78)$$

from which we have

$$s_0 = 2\nu_{mn}/(M + \beta + i\beta), \quad \rho = \nu_{mn}^2/\pi(M + \beta + i\beta)^2. \quad (79)$$

For the resonator with circular mirrors the quantities Λ and Δ are given by

$$\Lambda = 4\pi\rho'' = 8\nu_{mn}^2 \frac{\beta(M+\beta)}{[(M+\beta)^2 + \beta^2]^2},$$

$$\Delta = 2\pi\rho' = 2\nu_{mn}^2 \frac{M(M+2\beta)}{[(M+\beta)^2 + \beta^2]^2}; \quad (80)$$

and the quality figure and complex frequency can be calculated from the formulas

$$Q = \frac{q}{4\rho''}, \quad k_{mnq} = \left\{ \left[\frac{\nu_{mn}}{a(1+\beta\frac{1+i}{M})} \right]^2 + \left(\frac{\pi q}{2l} \right)^2 \right\}^{1/2}, \quad (81)$$

while the function (73) takes the form

$$g(r) = J_m \left(\frac{\nu_{mn} r/a}{1+\beta(1+i)/M} \right). \quad (82)$$

When we connect the function Φ with the electromagnetic field with the formulas (67) and (68), we obtain oscillations $E_{mnq}^{(x)}$ ($j_x = f, j_y = 0$) and $E_{mnq}^{(y)}$ ($j_x = 0, j_y = f$) in the resonator with circular mirrors. For $m = 1, 2, \dots$ each of the oscillations $E_{mnq}^{(x)}$ and $E_{mnq}^{(y)}$ has a rotational degeneracy; replacement of $\cos m\varphi$ by $\sin m\varphi$ gives two new oscillations, so that for $m \neq 0$ the formulas (80)–(82) apply to four oscillations. By introducing an additional metal plane at $x = 0$ and keeping only half of each mirror ($x \geq 0$) one can destroy the twofold (polarization, see end of Sec. 4) degeneracy of the symmetric oscillations with the index $m = 0$.

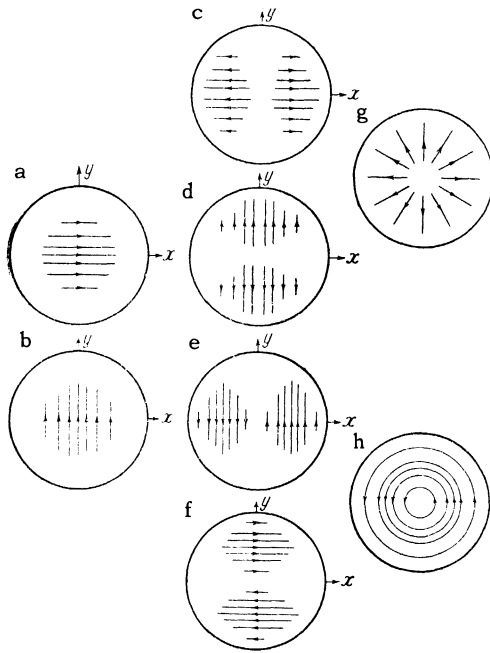


FIG. 11. Current distributions on circular mirror for oscillations E_{01q} and E_{11q} : a – oscillation $E_{01q}^{(x)}$, $A_x = \Psi$; b – oscillation $E_{01q}^{(y)}$, $A_y = \Psi$; c – oscillation $E_{11q}^{(x)}$, $A_x = \Psi \cos \varphi$; d – oscillation $E_{11q}^{(y)}$, $A_y = \Psi \sin \varphi$; e – oscillation $E_{11q}^{(y)}$, $A_y = \Psi \cos \varphi$; f – oscillation $E_{11q}^{(x)}$, $A_x = \Psi \sin \varphi$; g – oscillation $11q$, $A_r = \Psi$; h – oscillation $11q$, $A_\varphi = \Psi$.

The first and second columns in Fig. 11 show schematically the current distributions on a mirror for the oscillations $E_{01q}^{(x,y)}$ and $E_{11q}^{(x,y)}$. By combining oscillations $E_{11q}^{(x)}$ and $E_{11q}^{(y)}$ one can get oscillations with radial or azimuthal currents which have axial symmetry (third column of Fig. 11). The current distributions are shown in more detail in Figs. 12 and 13, where we have plotted the absolute values and phases of the functions (84) for $m = 0, n = 1$ ($\nu_{01} = 2.405$) and $m = n = 1$ ($\nu_{11} = 3.832$), and given to the number N connected with M by the relation (50) the values 2, 5, and 10. The calculations were made by means of tables.^[14]

In the paper by Fox and Li^[7] curves are given for Λ and Δ with $m = 0$ and 1, $n = 1$, which are practically equivalent to the formulas (80); the paper also gives curves for the function $g(r)$, which have a wavy character, but agree “on the average” with the shapes of the curves shown in Figs. 12 and 13 (cf. end of Sec. 3).

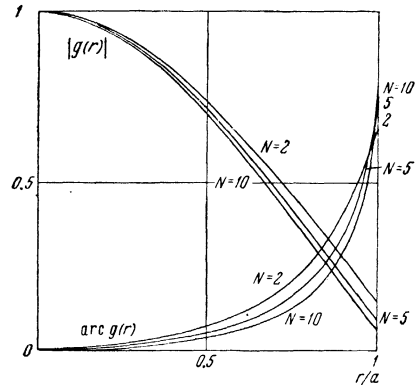


FIG. 12. The function (84) for $m = 0, n = 1$.

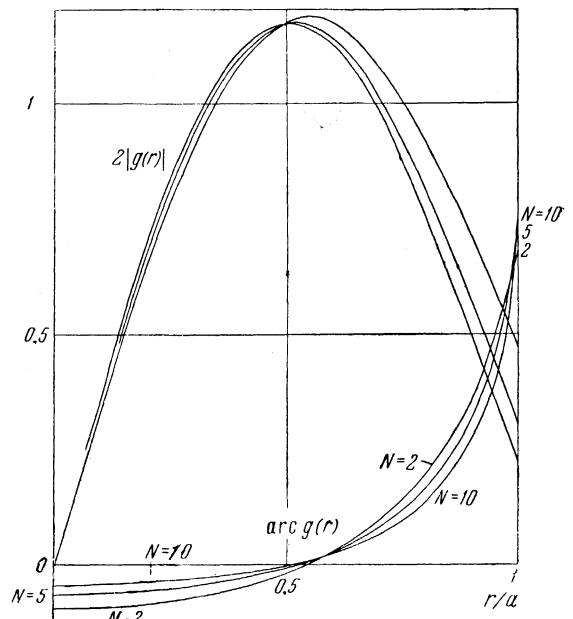


FIG. 13. The function (84) for $m = n = 1$.

We note that in the rigorous theory of diffraction by a disk the electromagnetic field cannot be expressed in terms of a single scalar function, since the condition at the sharp edge cannot be satisfied in this way (cf. e.g., [15]). The condition at the sharp edge is of no importance, however, for the asymptotic laws in which we are interested (cf. Sec. 1), and accordingly we can manage with the single function introduced in Eq. (68); the results obtained apply to resonators with walls of any thickness.

6. THE SPECTRUM OF NATURAL FREQUENCIES

As is well known (cf. e.g., [16] and [17]), for closed resonating volumes the spectrum of natural frequencies becomes denser as we go to higher frequencies; the number N of oscillations belonging to the frequency range $\Delta\omega$ is given by

$$\Delta N = (V/2\pi^2c^3) \omega^2 \Delta\omega, \quad (83)$$

where V is the volume of the resonator. The damping coefficient of the oscillations, caused by Joule losses in the walls, is proportional to $\omega^{1/2}$ for constant wall resistance; therefore at sufficiently high frequencies the resonance curves of a resonator of fixed dimensions overlap, and the resonance properties are lost.

In the open resonators considered above the spectrum of the natural frequencies is sparser than for closed resonators. To examine this matter, let us plot the natural frequencies ω_{mnq} of a resonator with square or circular mirrors as points in the complex plane of the variable ω (Fig. 14). It follows from Eqs. (58), (65), and (79) that these points lie on rays which begin at the points $\omega_q = \pi c q / 2l$ on the real axis and make with this axis angles ψ given by

$$\psi = 2 \tan^{-1} \frac{\beta}{M + \beta} \approx \frac{2\beta}{M} \text{ for } M \gg 1, \quad (84)$$

where the value of M must be taken for $\omega = \omega_q$.

It can be seen from Fig. 14 that as the indices m and n increase there is an increase of the radiative damping of the oscillations—the points get farther from the real axis (actually the increase of the damping is more rapid, since for finite $|p|$ the formula (18) understates the radiation loss, cf. Fig. 2). Moreover, the ordinates of the points increase in magnitude much more rapidly than the

differences of abscissas of adjacent points. Since when

$$\omega'_+ - \omega'_- < (\omega''_+ + \omega''_-)/2 \quad (85)$$

the resonance curves corresponding to two adjacent natural frequencies $\omega_- = \omega'_- - i\omega''_-$ and $\omega_+ = \omega'_+ - i\omega''_+$ overlap, near each frequency ω_q there remain only a finite number of natural frequencies which appear when one works with an open resonator. If there remains just the one oscillation with the smallest indices ($m = n = 1$ for rectangular mirrors and $m = 0, n = 1$ for circular mirrors), we get a spectrum with practically equally spaced frequencies and no bunching at all—the same as for the ideal case of a one-dimensional resonating system.

For a closed resonator $\psi = 0$ and the rays in Fig. 14 fall on the axis of abscissas, so that there is bunching of levels in the spectrum.

Open resonators formed by sections of waveguide (Sec. 2) also have more open spectra than closed resonators of the same dimensions. In open waveguides, however, the spectrum in the best case takes the form characteristic of two-dimensional regions (cf. [17]), and instead of Eq. (83) the formula is

$$\Delta N = (S/\pi c^2) \omega \Delta\omega \quad (86)$$

(S is the area of a transverse section of the waveguide). Therefore such resonators can be used only in the region of longer wavelengths, where the increasing density of the spectrum (86) is not intolerable.

The excitation of open resonators differs from that of closed resonators because a considerable fraction of the power supplied can be taken up by oscillations with large radiative damping. These radiative losses are unavoidable (it is owing to them that the spectrum is more open than for closed systems), but with suitable excitation they can be reduced to a minimum.

SUMMARY

We have given here a theory of the simplest sort for the natural oscillations in open resonators; by its use one can easily calculate for each oscillation the natural frequency and the radiative damping, and also the current distribution on the walls, from which one can find the additional damping

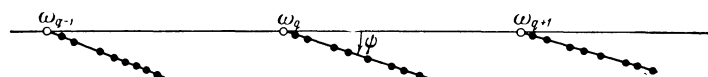


FIG. 14. Spectrum of an open resonator.

caused by the Joule losses or the partial transparency of the walls. Also it is not hard to construct the electric and magnetic field distributions in the volume of the resonator.

The formulas obtained are simple and present no difficulties in the calculations, and also they admit of an intuitive physical interpretation. They are approximate, with higher accuracy for oscillations with higher quality figures.

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Translated by W. H. Furry

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