RESTRICTIONS ON THE VALUES OF COUPLING CONSTANTS IN QUANTUM FIELD THEORY. I

B. V. GESHKENBEĬN and B. L. IOFFE

Institute of Theoretical and Experimental Physics

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Starting from the Lehmann-Källén expansion for the Green's function and using the dispersion relation for the vertex part $\Gamma(\kappa^2)$, we show that with prescribed masses there must be an upper bound on the coupling constant g^2. A concrete expression is found for the upper bound on the coupling constant, which shows, for example, that in the case of the deuteron $g^2 < 16m_d^2(\Delta/m)^{1/2}$ (m_d, m are the masses of the deuteron and nucleon, and Δ is the binding energy), and in the case of the interaction between π mesons and nucleons $f^2 < 0.47$.

INTRODUCTION

In the usual formulation of quantum field theory values are independently prescribed for the masses of the particles and the coupling constants of the various fields. Meanwhile, in a number of papers [1-5] it has been shown that with given values of the masses there are definite upper bounds on the physical (renormalized) coupling constants, and that these bounds are functions of the masses of the particles. In these papers, however, the treatment either was based on the use of models, or else [4] depended on the hypothesis that the interaction of elementary particles is characterized by an effective radius which depends only on the masses of the particles and does not increase with an increase of the coupling constant. In the present paper we obtain restrictions on the sizes of the coupling constants for prescribed masses, on the basis of general principles of quantum field theory, without any additional assumptions and without resorting to the use of models.

We base our treatment on the representation of the Green's function in the form of a Lehmann-Källén expansion [6]; that is to say, we shall assume that all of the conditions for the validity of this expansion are satisfied (existence of a complete system of functions in Hilbert space, definite metric, Lorentz invariance). Starting from the Lehmann-Källén expansion we shall obtain a relation, Eq. (10), which restricts the maximum possible value of the coupling constant g^2 of three fields.

In the boson case this relation involves the vertex part $\Gamma(\kappa^2)$. As a second postulate we assume that the vertex part $\Gamma(\kappa^2)$ is an analytic function of κ^2 with a cut beginning at the square of the sum of the masses of the nearest two particles, and that a dispersion relation can be written for $\Gamma(\kappa^2)$. The number of subtractions in this dispersion relation is uniquely determined from the same relation, Eq. (10). The maximum value of g^2 will be obtained by Minimizing the expression (10) with respect to all possible functions $\Gamma(\kappa^2)$ having the given analytic properties. Variation of the function $\Gamma(\kappa^2)$ leads to an integral equation for Im $\Gamma(\kappa^2)$, which can be solved and gives restrictions on the coupling constant which depend on the masses of the particles. In the case in which the mass of one of the interacting particles is close to the sum of the masses of the other two, the limit so obtained is twice the limit which follows from nonrelativistic quantum mechanics for an interaction with an infinitely small radius. [2,3,5]

The treatment for the fermion case is similar in principle but is somewhat complicated by the presence of two functions ρ_1 and ρ_2 which are involved in the Lehmann-Källén expansion, and by the fact that the expression for the vertex part is more complicated. Nevertheless, here also we are able, within the framework of the same hypotheses, to get an explicit expression for the limit on the coupling constant. In particular, the coupling constant f^2 for the interaction between π mesons and nucleons must be smaller than 0.47, and if Σ and Λ have opposite parities the constant g^2 for the $\Lambda\Sigma\pi$ interaction must be smaller than 3.2.

BOSONS

The Lehmann-Källén representation for the

Green's function of a boson with spin zero and mass m_a is[6]:

$$D(\varkappa^2) = \frac{1}{\varkappa^2 - m_a^2} - \int\limits_{(m_b+m_c)^2}^{\infty} \frac{\rho(\varkappa'^2)}{\varkappa'^2 - \varkappa^2 - i\delta} d\varkappa'^2, \qquad \rho(\varkappa^2) \geqslant 0, \tag{1}$$

where m_b and m_c are the masses of the nearest (in sum of masses) particles into which particle a can be converted in a transition.

We are interested in the value of the coupling constant g^2 of the three fields a, b, c. To get restrictions on the value of g^2, we shall first proceed by a method analogous to that of a paper by Lehmann, Symanzik, and Zimmermann.[7] It can easily be seen from the relation (1) that the function $-D(\kappa^2)$ is an R-function in the complex plane of κ^2, i.e., that the sign of the imaginary part of $-D(\kappa^2)$ is the same as the sign of the imaginary part of κ^2. It follows from this that the function $D^{-1}(\kappa^2)$ is also an R-function, and consequently the most general expression for $D^{-1}(\kappa^2)$ is of the form[1]

$$D^{-1}(\varkappa^2) = \frac{1}{\pi} \int\limits_{(m_b+m_c)^2}^{\infty} \operatorname{Im} D^{-1}(\varkappa'^2) \left[\frac{1}{\varkappa'^2 - \varkappa^2} - \frac{1}{\varkappa'^2 - m_a^2}\right] d\varkappa'^2 + \sum_n R_n \left(\frac{1}{\varkappa_n^2 - \varkappa^2} - \frac{1}{\varkappa_n^2 - m_a^2}\right) + \alpha(\varkappa^2 - m_a^2), \tag{2}$$

where the constants R_n, κ_n^2, and α are real and positive, and

$$m_a^2 < \varkappa_n^2 < (m_b + m_c)^2, \quad D^{-1}(m_a^2) = 0,$$

$$\operatorname{Im} D^{-1}(\varkappa^2) = \pi\rho(\varkappa^2) / |D(\varkappa^2)|^2. \tag{2'}$$

The relation (2) is written with one subtraction (at the point m_a^2). We shall show that this is sufficient.

It can be seen from the Lehmann-Källén expansion that for large κ^2 the function $D^{-1}(\kappa^2)$ cannot increase more rapidly than κ^2. Therefore the function $D^{-1}(\kappa^2)/(\kappa^2 - m_a^2)^2$ can be expressed as a Cauchy integral over the usual path:

$$D^{-1}(\varkappa^2) = [D^{-1}(m_a^2)]'(\varkappa^2 - m_a^2) + \frac{(\varkappa^2 - m_a^2)^2}{\pi} \int\limits_{(m_b+m_c)^2}^{\infty} \frac{\operatorname{Im} D^{-1}(\varkappa'^2)}{(\varkappa'^2 - \varkappa^2)(\varkappa'^2 - m_a^2)^2} d\varkappa'^2 + R(\varkappa^2), \tag{3}$$

where $R(\kappa^2)$ is the contribution from the poles of $D^{-1}(\kappa^2)$. It follows from Eq. (3) that the integral

$$\int \frac{\operatorname{Im} D^{-1}(\varkappa^2)}{(\varkappa^2)^2} d\varkappa^2$$

converges, because otherwise for large κ^2 the function $D^{-1}(\kappa^2)$ would increase more rapidly than κ^2, which is impossible. Thus the integral in the formula (2) converges, and one subtraction is enough.

Let us differentiate Eq. (2) with respect to κ^2 and set $\kappa^2 = m_a^2$. Then, using the equation $[D^{-1}(\kappa^2)]'_{\kappa^2 = m_a^2} = 1$, we get

$$\int\limits_{(m_b+m_c)^2}^{\infty} \frac{\rho(\varkappa^2)}{|D(\varkappa^2)|^2} \frac{d\varkappa^2}{(\varkappa^2 - m_a^2)^2} + \sum_n R_n \frac{1}{(\varkappa_n^2 - m_a^2)^2} + \alpha = 1. \tag{4}$$

Dropping the two positive terms in the left member of Eq. (4), we arrive at the inequality

$$\int\limits_{(m_b+m_c)^2}^{\infty} \frac{\rho(\varkappa^2)}{|D(\varkappa^2)|^2} \frac{1}{(\varkappa^2 - m_a^2)^2} d\varkappa^2 \leqslant 1. \tag{5}$$

As is shown in Lehmann's paper,[6] $\rho(\kappa^2)$ can be written in the form

$$\rho(\varkappa^2) = (2\pi)^3 \sum_n |a_{0n}|^2, \tag{6}$$

where

$$a_{0n} = \langle 0 | A(0) | \Phi_n \rangle, \tag{6'}$$

$A(x)$ is the field of particle a, and Φ_n is an arbitrary state of the complete system of functions with four-momentum k such that $k^2 = \kappa^2$. It is clear from Eq. (6) that the inequality (5) is only strengthened if in the summation over n in Eq. (6) we take into account only two-particle states of particles b and c.

The contribution to $\rho(\kappa^2)$ owing to the two-particle states can be expressed in terms of the Green's function $D(\kappa^2)$ and the vertex part $\Gamma(\kappa^2)$. We shall suppose that particles b and c are scalar particles and that the product of the parities of all three particles a, b, and c is unity. Then from the definition of the vertex part

$$g\Gamma(x, y, z) = \int d^4x'\, d^4y'\, d^4z' D_a^{-1}(z - z')\, D_b^{-1}(x - x')\, D_c^{-1}(y - y') \times \langle 0 | T\{A(z'), \varphi_b(x'), \varphi_c(y')\} | 0 \rangle \tag{7}$$

we can obtain a connection between the matrix element $\langle 0 | A(0) | \Phi_{k_b,k_c} \rangle$ (where Φ_{k_b,k_c} is the two-particle state of particles b and c with the momenta k_b, k_c, and $(k_b + k_c)^2 = \kappa^2$, $k_b^2 = m_b^2$, $k_c^2 = m_c^2$) and the vertex part $\Gamma(m_b^2, m_c^2, \kappa^2) \equiv \Gamma(\kappa^2)$:

$$\langle 0 | A(0) | \Phi_{k_b,k_c} \rangle = g D_a(\varkappa^2)\, \Gamma(\varkappa^2) / \sqrt{4\pi\omega_b\omega_c} \tag{8}$$

(ω_b and ω_c are the energies of particles b and c). Substituting Eq. (8) in Eq. (6), we find the contribution to $\rho(\kappa^2)$ owing to two-particle states:

$$\rho_{\text{two-particle}} = (1/2\pi)\, g^2 |D(\varkappa^2)|^2 |\Gamma(\varkappa^2)|^2 q(\varkappa^2)/\varkappa, \tag{9}$$

$$q(\varkappa^2) = \sqrt{[\varkappa^2 - (m_b + m_c)^2][\varkappa^2 - (m_b - m_c)^2]} / 2\varkappa. \tag{9'}$$

After substituting Eq. (9) in Eq. (5) we arrive at an inequality which restricts the possible values of the renormalized coupling constant g^2 of the three fields a, b, c[1]:

$$\frac{g^2}{2\pi}\int\limits_{(m_b+m_c)^2}^{\infty}\frac{|\Gamma(\varkappa^2)|^2}{(\varkappa^2-m_a^2)^2}\frac{q(\varkappa^2)}{\varkappa}d\varkappa^2<1. \tag{10}$$

In order to get from the inequality (10) definite restrictions on g^2 for prescribed masses m_a, m_b, m_c it is necessary that the value of the integral

$$\Phi=2(m_b+m_c)^2\int\limits_{(m_b+m_c)^2}^{\infty}\frac{|\Gamma(\varkappa^2)|^2}{(\varkappa^2-m_a^2)^2}\frac{q(\varkappa^2)}{\varkappa}d\varkappa^2 \tag{11}$$

have a lower bound.

We shall look for the minimum of Φ over the class of functions $\Gamma(\kappa^2)$ that have the following properties:

1) $\Gamma(\kappa^2)$ is a holomorphic function of κ^2 in the complex plane of κ^2 with a cut along the real axis which begins at the point $\kappa^2=(m_b+m_c)^2$.[2] On the real axis to the left of the point $\kappa^2=(m_b+m_c)^2$ the function $\Gamma(\kappa^2)$ is real;

2) the increase of $\Gamma(\kappa^2)$ at infinity is not faster than a power law;

3) the value at the point $\kappa^2=m_a^2$ is $\Gamma(m_a^2)=1$.

We assume that $\Gamma(\kappa^2)$ has no poles in the complex plane of κ^2.[3] In principle there could be poles of $\Gamma(\kappa^2)$ on the real axis in the interval $m_a^2<\kappa^2<(m_b+m_c)^2$, at the points κ_n^2 at which the Green's function $D(\kappa^2)$ is equal to zero, while $\Gamma(\kappa^2)D(\kappa^2)\to \text{const}$ for $\kappa^2\to\kappa_n^2$. [The last fact follows, for example, from the Schwinger equation for the Green's function $D(\kappa^2)$]. These poles of $\Gamma(\kappa\ )$ correspond to true bound states of the particles b and c, and owing to the terms $g^2\Gamma^2(\kappa^2)$ $D(\kappa^2)$ they will lead to poles in the scattering amplitude of particles b and c. [This situation occures, for example, in the theory of superconductivity, where a pole of $\Gamma(\kappa^2)$ corresponds to a bound state of a Cooper pair.] Thus our assumption means that there are no particles with the same quantum numbers as particle a and lying between m_a and m_b+m_c.

The function $\Gamma(\kappa^2)$ will automatically have the properties 1)—3) if we write the dispersion relation

$$\Gamma(\varkappa^2)=1+\frac{\varkappa^2-m_a^2}{\pi}\int\limits_{(m_b+m_c)^2}^{\infty}d\varkappa'^2\frac{\operatorname{Im}\Gamma(\varkappa'^2)}{(\varkappa'^2-\varkappa^2-i\delta)(\varkappa'^2-m_a^2)}, \tag{12}$$

with an arbitrary function $\operatorname{Im}\Gamma(\kappa^2)$ with one subtraction. It follows from the inequality (10) that on the real axis $|\Gamma(\kappa^2)|$ increases more slowly than $(\kappa^2)^{1/2}$. By using the condition (2) we can show from this that one subtraction is surely enough in the dispersion relation for Γ.

We can now find the minimum of the functional $\Phi[\Gamma(\kappa^2)]$ for the class of functions $\Gamma(\kappa^2)$ with which we are concerned. To do so we substitute Eq. (12) in Eq. (11) and perform the integration over κ^2. After changing to the dimensionless variable $x=\kappa^2/(m_b+m_c)^2$ we get

$$\begin{aligned}\Phi=&\frac{1}{\alpha^2}\Big[-\alpha+l+\frac{1}{2}\frac{\alpha(1+\lambda)-2\lambda}{(1-\alpha)(\alpha-\lambda)}\beta\Big]\\&-\frac{2}{\pi}\int\limits_1^{\infty}\Big[\frac{l}{\alpha x}+\frac{\beta}{\alpha(x-\alpha)}-\frac{L(x)}{x(x-\alpha)}\Big]\varphi(x)\,dx\\&+2\int\limits_1^{\infty}\frac{\sqrt{(x-1)(x-\lambda)}}{x}\varphi^2(x)\,dx\\&+\frac{1}{\pi^2}\int\limits_1^{\infty}dx\int\limits_1^{\infty}dy\Big\{\frac{l}{xy}+\frac{1}{y-x}\Big[\frac{L(x)}{x}-\frac{L(y)}{y}\Big]\Big\}\varphi(x)\varphi(y),\end{aligned} \tag{13}$$

where

$$\alpha=\frac{m_a^2}{(m_b+m_c)^2},\qquad \lambda=\frac{(m_b-m_c)^2}{(m_b+m_c)^2},$$

$$\varphi(x)=\frac{\operatorname{Im}\Gamma(x)}{x-\alpha},\qquad l=\sqrt{\lambda}\ln\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}},$$

$$\beta=\sqrt{(1-\alpha)(\alpha-\lambda)}\Big[\frac{\pi}{2}+\arcsin\frac{2\alpha-1-\lambda}{1-\lambda}\Big],$$

$$L(x)=\sqrt{(x-1)(x-\lambda)}\ln\frac{(\sqrt{x-\lambda}+\sqrt{x-1})^2}{1-\lambda}. \tag{13'}$$

[The formula (13) is written for the case $\alpha\ge\lambda$, i.e., $m_a\ge|m_b-m_c|$.) Finding the variation of Φ with respect to φ, we get the equation satisfied by the function $\varphi(x)$ which minimizes the integral Φ:

$$\begin{aligned}&\frac{\sqrt{(x-1)(x-\lambda)}}{x}\varphi(x)+\frac{1}{2\pi^2}\int\limits_1^{\infty}\Big\{\frac{l}{xy}+\frac{1}{y-x}\Big[\frac{L(x)}{x}-\frac{L(y)}{y}\Big]\Big\}\varphi(y)\,dy\\&-\frac{1}{2\pi}\Big[\frac{l}{\alpha x}+\frac{\beta}{\alpha(x-\alpha)}-\frac{L(x)}{x(x-\alpha)}\Big]=0.\end{aligned} \tag{14}$$

When we use Eq. (14) the minimum value of Φ can be written

$$\begin{aligned}\Phi_{min}=&\frac{1}{\alpha^2}\Big[-\alpha+l+\frac{1}{2}\frac{\alpha(1+\lambda)-2\lambda}{(1-\alpha)(\alpha-\lambda)}\beta\Big]\\&-\frac{1}{\pi}\int\limits_1^{\infty}dx\varphi(x)\Big[\frac{l}{\alpha x}+\frac{\beta}{\alpha(x-\alpha)}-\frac{L(x)}{\alpha(x-\alpha)}\Big].\end{aligned} \tag{15}$$

[1]We use the Gaussian system of units, so that, for example, in our notation $e^2=1/137$.

[2]In the case considered (with b and c the lightest possible particles) there can be no anomalous thresholds in $\Gamma(\varkappa^2)$.

[3]The fact that $\Gamma(\varkappa^2)$ can have poles on the real axis in the complex plane of $\varkappa^2$ has been called to our attention by V. Ya. Faĭnberg and E. S. Fradkin, to whom we express our deep gratitude.

Equation (14) is a Fredholm integral equation of the second kind, which, as is well known, has a unique solution (provided the parameter $1/2\pi^2$ is not equal to an eigenvalue of the homogeneous equation, and in our case there is no reason for this to be so). In the general case the equation (14) can be solved numerically, but some properties of the solution [and consequently of the $\Gamma(\kappa^2)$ which gives the minimum value of Φ] can be seen directly from the equation.

For x close to unity and $x > 1$, i.e., near the threshold for production of particles b and c, it is not hard to see that $\varphi(x)$ is proportional to $(x-1)^{-1/2}$, so that $\Gamma(x)$ is of the form

$$\Gamma(x) = iA/\sqrt{x-1} + B, \tag{16}$$

where A and B are constants and A is real. Using Eqs. (2), (2′), and (9), one can easily show that if $\Gamma(x)$ near the threshold is given by Eq. (16) and goes to infinity for $x \to 1$, then the Green's function $D(x)$ will go to zero for $x \to 1$ and near $x = 1$ will be given by

$$D(x) = \left[i\frac{g^2}{4\pi}A^2\frac{\sqrt{1-\lambda}}{\sqrt{x-1}} + C\right]^{-1}, \tag{17}$$

where C is a constant. It is interesting to see whether such forms of Γ and D may not lead to an unphysical behavior of the scattering amplitude of particles b and c near the threshold. To calculate the s-wave scattering amplitude $a_0(x)$ for particles b and c we can use the relation

$$\mathrm{Im}\,[\Gamma(x)D(x)] = \Gamma(x)D(x)a_0^*(x)q(x). \tag{18}$$

Substituting Eqs. (16) and (17) in Eq. (18), we find that for $x \to 1$

$$a_0 q = e^{i\delta}\sin\delta = \gamma\sqrt{x-1} \equiv \gamma' q \tag{19}$$

(γ, γ' are real), i.e., we get the effective-range approximation for the s-wave scattering amplitude.

It must be noted that although when we calculate the scattering amplitude of particles b and c near threshold in this way we get a quite reasonable expression, we nevertheless cannot be sure that the $\Gamma(x)$ determined from Eq. (14) can correspond to an actual physical situation. In fact, if we use the $\Gamma(x)$ of Eq. (15) and the $D(x)$ of Eq. (16) to calculate the pole term in the scattering amplitude $a_0(x)$ of particles b and c near threshold, $(g^2/2\pi)\Gamma^2(x)D(x)$, it will be proportional to $(x-1)^{-1/2}$, and consequently in order for the effective-range approximation to hold it is necessary that the pole term be compensated by some sort of more complicated diagrams, and this seems very strange.[4)]

Thus when we find the minimum of the functional Φ by means of the function $\varphi(x)$ determined from Eq. (14) we may possibly be getting too low a value for this minimum (and consequently too high a limit on g^2), in comparison with what can in principle be realized in actual physical problems.

Let us now consider the limiting case of small values of $1-\alpha \ll 1$, i.e., the case $\Delta = m_b + m_c - m_a \ll m_a$. This case corresponds to the nonrelativistic approximation; particle a can be interpreted as a bound state of particles b and c, and Δ is the binding energy.

It is not hard to see that when $1-\alpha$ is small the values of x that are important for the integral in Eq. (15) are those close to unity, $x-1 \sim 1-\alpha$. Then we can neglect the integral term in Eq. (14), and[5)]

$$\varphi(x) = \sqrt{1-\alpha}/2(x-\alpha)\sqrt{x-1}. \tag{20}$$

For Φ_{min} we get

$$\Phi_{min} = \frac{\pi}{4}\sqrt{\frac{1-\lambda}{1-\alpha}} = \frac{\pi}{4}\sqrt{\frac{2\mu}{\Delta}}, \tag{21}$$

where $\mu = m_b m_c/(m_b + m_c)$ is the reduced mass of particles b and c. For $\Gamma(x) \equiv 1$ we would have had $\Phi = (\pi/2)(2\mu/\Delta)^{1/2}$, i.e., the value of the integral Φ cannot be decreased by more than a factor two because $\Gamma(x)$ is different from unity. We get as the result the following upper limit on the coupling constant g^2:

$$g^2 < 16m_a^2\sqrt{\Delta/2\mu}. \tag{22}$$

A physical example of such a situation is the case of the deuteron; the deuteron is particle a, and particles b and c are the neutron and proton.

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4)If there were no such cancellation it would mean that the scattering amplitude has a resonance at zero energy.

5)It can easily be shown from Eq. (14) that in the nonrelativistic case the vertex part Γ that gives the minimum of Φ and the corresponding Green's function D are

$$\Gamma(\varepsilon) = \tfrac{1}{2}(1 + i\sqrt{\Delta/\varepsilon}), \qquad D(\varepsilon) = -i(8m_a/g^2\sqrt{2\mu})\sqrt{\varepsilon}/(\varepsilon+\Delta),$$

where ϵ is the kinetic energy of particles b and c; $\varkappa = m_b + m_c + \epsilon$. Calculating the scattering amplitude according to Eq. (18), we find $a_0 = i[(2\mu\epsilon)^{1/2} - i(2\mu\Delta)^{1/2}]^{-1}$ – the usual expression for the scattering amplitude when there is a bound state in a potential with an infinitely small range (the Bethe-Peierls formula for np scattering in the triplet state). It is important to point out that the usual expression for the scattering amplitude has been obtained here with $\Gamma(\epsilon) = \frac{1}{2}[1 + i(\Delta/\epsilon)^{1/2}]$, and not with $\Gamma(\epsilon) = 1$, as in the Bethe-Peierls theory. For this reason the bound on g^2 was twice the usual value.

The fact that the deuteron, neutron, and proton have spins is unimportant, since at small energies the interaction is that in the triplet state and the spin functions can be factored out.

The fact that at small values of the binding energy Δ the coupling constant g^2 decreases in proportion to $\Delta^{1/2}$ has been derived in a number of papers, [2,3,5] which treated the problem on the basis of nonrelativistic theory and with the range of the nuclear forces set equal to zero. With these assumptions the limit obtained on g^2 is smaller than ours by a factor two (i.e., is the same as would be obtained with $\Gamma = 1$). In the paper of Gribov, Zel'dovich, and Perelomov [3] it was pointed out that inclusion of a finite range of the nuclear forces raises the limit on the coupling constant g^2. It follows from our treatment that the increase of the value of g^2 due to the finite range of nuclear forces cannot be by more than a factor two. From the experimental data on the scattering of neutrons by protons (cf. e.g., [8]) one gets

$$g^2 = 12m_D^2 \sqrt{\Delta/2\mu}.$$

These data show that for the case of the deuteron the estimate (22) is a very satisfactory one.

The numerical solution of Eq. (14) and the determination of $\operatorname{Im}\Gamma(x)$ and Φ_{min} from Eq. (15) have been carried out for a number of values of α and λ by A. S. Kronrod, G. M. Adel'son-Vel'skiĭ, F. M. Filler, and L. V. Il'kov with an electronic computing machine.

The minimum value of Φ can also be obtained analytically by an elegant method [9-11] pointed out to us by N. N. Meĭman. To find the minimum of the integral

$$\Phi = \int_1^\infty \frac{\sqrt{(x-1)(x-\lambda)}}{x(x-\alpha)^2} |\Gamma(x)|^2 dx \tag{23}$$

in the complex plane of x, one makes the conformal transformation

$$z = -(\sqrt{x-1} - i\sqrt{1-\alpha}) / (\sqrt{x-1} + i\sqrt{1-\alpha}), \tag{24}$$

which takes the two sides of the cut along the real axis from unity to infinity over into the unit circle. Then the entire cut plane of x goes over into the interior of the unit circle, and the point x into the center of the circle. The integral (23) is transformed into

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)|\Gamma(z)|^2 d\theta, \qquad z = e^{i\theta}, \tag{25}$$

$$f(\theta) = \frac{\pi}{\sqrt{1-\alpha}} \frac{u\sqrt{1-\lambda+(1-\alpha)u}}{[1+(1-\alpha)u](1+u)}, \qquad u = \tan^2\frac{\theta}{2}. \tag{26}$$

$\Gamma(z)$ is an analytic function inside the unit circle and $\Gamma(0) = 1$. The solution of the problem of finding the minimum of the integral (25) over the class of functions $\Gamma(z)$ that are analytic inside the circle and have $\Gamma(0) = 1$ is obtained (cf. [9-11]) by expanding $\Gamma(z)$ in terms of the system of polynomials orthogonal on the unit circle with the weight $f(\theta)$, and the answer is

$$\Phi_{min} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f(\theta)\, d\theta\right\}. \tag{27}$$

After an elementary integration we get

$$\Phi_{min} = \frac{\pi}{4}\frac{\sqrt{1-\lambda}+\sqrt{1-\alpha}}{\sqrt{1-\alpha}\,(1+\sqrt{1-\alpha})^2}. \tag{28}$$

[The results of calculation with the formula (28) naturally agree with the results of numerical computation with Eqs. (14) and (15)].

Thus in the case of the interaction of three boson fields the coupling constant g^2 is bounded by the inequality

$$g^2 < 16\,\frac{\sqrt{(m_b+m_c)^2-m_a^2}\left(m_b+m_c+\sqrt{(m_b+m_c)^2-m_a^2}\right)^2}{2\sqrt{m_b m_c}+\sqrt{(m_b+m_c)^2-m_a^2}}. \tag{29}$$

It is obvious that in the nonrelativistic case ($m_b + m_c - m_a \ll m_a$) the inequality (29) goes over into (22).

FERMIONS

If a particle a is a fermion, the Lehmann-Källén representation for its Green's function is of the form

$$G(\hat{p}) = \frac{1}{\hat{p}-m_a} - \int_{(m_b+m_c)^2}^{\infty} \frac{(\hat{p}+\varkappa)\rho_1(\varkappa^2)-\rho_2(\varkappa^2)}{\varkappa^2-p^2-i\delta}\, d\varkappa^2, \tag{30}$$

where the functions ρ_1 and ρ_2 are positive and $2\kappa\rho_1 - \rho_2 \geq 0$. Writing $G(\hat{p})$ in the form

$$G(\hat{p}) = \hat{p} f_1(p^2) + f_2(p^2), \tag{31}$$

we get for the functions $f_1(p^2)$ and $f_2(p^2)$ the relations

$$f_1(p^2) = \frac{1}{p^2-m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\rho_1(\varkappa^2)}{\varkappa^2-p^2-i\delta}\, d\varkappa^2, \tag{32}$$

$$f_2(p^2) = \frac{m_a}{p^2-m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\varkappa\rho_1(\varkappa^2)-\rho_2(\varkappa^2)}{\varkappa^2-p^2-i\delta}\, d\varkappa^2. \tag{33}$$

It follows from Eq. (32) that $f_1^{-1}(p^2)$ is an R-function. Then by arguments analogous to those used for the boson case in going from Eq. (1) to Eq. (5), we get

$$\int\limits_{(m_b+m_c)^2}^{\infty} \frac{\rho_1(\varkappa^2)}{|f_1(\varkappa^2)|^2} \frac{1}{(\varkappa^2-m_a^2)^2} d\varkappa^2 \leqslant 1. \tag{34}$$

Just as before, the inequality (34) is only strengthened if in $\rho_1(\kappa^2)$ we confine ourselves to including only the lowest two-particle states. We shall suppose that the lowest state is a boson with spin zero and a fermion with spin $1/2$. Then[6]

$$\rho_{1\,\text{two-particle}}(p^2) = \frac{1}{4p_0}(2\pi)^3 \sum_{n,r} C_\alpha^{(r)*} C_\alpha^{(r)}, \tag{35}$$

where

$$C_\alpha^{(r)} = \langle 0 | \psi_\alpha(0) | \Phi_{p_1,r;k} \rangle \tag{36}$$

and $\Phi_{p_1,r;k}$ is the two-particle state of a fermion with momentum p_1 and polarization r and a boson with momentum k, $(p_1+k)^2 = p^2$. From the definition of the vertex part $\Gamma(x, y; z)$,

$$g\Gamma(x, y; z) = \int d^4x' d^4y' d^4z' G_a^{-1}(x-x') D_c^{-1}(z-z')$$
$$\times \langle 0 | T\{\varphi(z'), \psi(x'), \bar{\psi}(y')\} | 0 \rangle G_b^{-1}(y-y'), \tag{37}$$

it is easy to show, when we go over to the momentum representation, that $C_\alpha^{(r)}$ is proportional to

$$g \frac{1}{\sqrt{k_0 p_0}} G_a(\hat{p}) \Gamma(\hat{p}) u_{p_1}^r, \tag{38}$$

where $u_{p_1}^r$ is a spinor which describes the free state of fermion b with four-momentum p_1 and polarization r. Substituting this expression for $C_\alpha^{(r)}$ in Eq. (35), we arrive at the following formula for $\rho_{1\,\text{two-particle}}$ [$q = q(p^2)$ is determined according to Eq. (9′)]:

$$4p_0\rho_{1\,\text{two-particle}}$$
$$= \frac{1}{2\pi} g^2 \frac{q(p^2)}{\sqrt{p^2}} \operatorname{Sp}\{\beta\Gamma^+(\hat{p}) G^+(\hat{p}) G(\hat{p}) \Gamma(\hat{p}) (\hat{p}_1 + m_b)\}. \tag{39}$$

Let us consider two cases: a) the scalar case, in which the parity of the boson c is the same as the product of the parities of the fermions a and b; b) the pseudoscalar case, in which the parity of the boson c is opposite to the product of the parities of the fermions a and b. In the scalar case the general expression for $\Gamma(\hat{p})$ is of the form

$$\Gamma_S(\hat{p}) = \Gamma_{1S}(p^2) + \hat{p}\Gamma_{2S}(p^2), \tag{40}$$

and in the pseudoscalar case

$$\Gamma_{PS}(\hat{p}) = \gamma_5\Gamma_{1PS}(p^2) + \hat{p}\gamma_5\Gamma_{2PS}(p^2). \tag{41}$$

[Particle b is a real particle, and therefore for it $\hat{p}_1 = m_b$, so that according to Eq. (38) the only spinor terms remaining in the expression for Γ are those indicated in Eqs. (40), (41).] Substituting Eqs. (31), (40), (41) in Eq. (39) and calculating the trace, we get the following expression for $\rho_{1\,\text{two-particle}}$:

$$\rho_{1\,\text{two-particle}}(p^2)$$
$$= (g^2/4\pi)\{|f_1p + f_2|^2 |\Gamma_1 + p\Gamma_2|^2 [(p \pm m_b)^2 - m_c^2]$$
$$+ |f_1p - f_2|^2 |\Gamma_1 - p\Gamma_2|^2 [(p \mp m_b)^2 - m_c^2]\} q(p^2)/2p^3, \tag{42}$$

where $p = (p^2)^{1/2}$, and the upper sign holds for the scalar case and the lower for the pseudoscalar case. Substituting Eq. (42) in Eq. (34) and introducing the notations

$$(f_1\Gamma_1 + f_2\Gamma_2)/2f_1 = F_1(p^2), \quad (f_1\Gamma_2p^2 + f_2\Gamma_1)/2f_1m_a = F_2(p^2), \tag{43}$$

we can write the inequality (34) in the form

$$(g^2/2\pi)\,\Phi < 1, \tag{44}$$

$$\Phi = \int\limits_{(m_b+m_c)^2}^{\infty} \frac{d\varkappa^2}{\varkappa^2(\varkappa^2-m_a^2)^2} \frac{q(\varkappa^2)}{\varkappa}$$
$$\times \{|F_1(\varkappa^2)\varkappa + F_2(\varkappa^2) m_a|^2 [(\varkappa \pm m_b)^2 - m_c^2]$$
$$+ |F_1(\varkappa^2)\varkappa - F_2(\varkappa^2) m_a|^2 [(\varkappa \mp m_b)^2 - m_c^2]\}. \tag{45}$$

Just as in the boson case, we assume that $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ are analytic functions of κ^2 in the entire complex plane of κ^2 with a cut along the real axis from $\kappa^2 = (m_b + m_c)^2$ to infinity, which are real on the real axis to the left of the point $\kappa^2 = (m_b + m_c)^2$ and do not increase at infinity more rapidly than a power of the argument. According to the definition of physical charge and the renormalized vertex part the following condition must hold:

$$\Gamma_1(m_a^2) + m_a\Gamma_2(m_a^2) = 1. \tag{46}$$

In analogy with the boson case we assume that the functions $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ have no poles anywhere in the complex plane, including the interval from m_a^2 to $(m_b + m_c)^2$ on the real axis. This means that the Green's function of the fermion a has no zeroes associated with poles of Γ_1 and Γ_2. In addition, we here make the stronger assumption that the Green's function $f_1(p^2)$ of the fermion a has no zeroes at all.[6)]

On these assumptions the functions $F_1(\kappa^2)$ and $F_2(\kappa^2)$ are analytic functions of κ^2 in the plane of κ^2 cut along the real axis from $\kappa^2 = (m_b + m_c)^2$ to infinity. On the real axis to the left of $(m_b + m_c)^2$ the functions $F_1(\kappa^2)$ and $F_2(\kappa^2)$ are real, and at the point $\kappa^2 = m_a^2$ they satisfy the conditions

[6)]It follows from Eq. (32) that if $f_1(\varkappa^2)$ has only one pole, at the point $\varkappa = m_a^2$, then in the most general case $f_1(\varkappa^2)$ can have not more than one zero, and this zero must lie on the real axis in the interval $m_a^2 < \varkappa^2 < (m_b + m_c)^2$. The restrictions on g^2 in the case in which $f_1(\varkappa^2)$ has one zero will be dealt with in a later paper.

$$F_1(m_a^2) = F_2(m_a^2) = {}^1/_2. \tag{47}$$

It will be convenient for us to transform the expression (45) for Φ, introducing instead of F_1 and F_2 the new functions $\overline{F}_1(\kappa^2)$ and $\overline{F}_2(\kappa^2)$:

$$F_1(\varkappa^2) = \frac{2m_b(m_b+m_c)}{\varkappa^2+m_b^2-m_c^2}[\mp\overline{F}_1(\varkappa^2)+\overline{F}_2(\varkappa^2)],$$

$$F_2(\varkappa^2) = \frac{m_b+m_c}{m_a}\overline{F}_1(\varkappa^2). \tag{48}$$

In terms of the new functions the formula for Φ is

$$\Phi = 8\int\limits_{(m_b+m_c)^2}^{\infty}\frac{d\varkappa^2}{(\varkappa^2+m_b^2-m_c^2)}\frac{(m_b+m_c)^2}{(\varkappa^2-m_a^2)^2}\frac{q(\varkappa^2)}{\varkappa}$$

$$\times[q^2(\varkappa^2)|\overline{F}_1(\varkappa^2)|^2+m_b^2|\overline{F}(\varkappa^2)|^2]. \tag{49}$$

It is obvious that the functions $\overline{F}_1(\kappa^2)$ and $\overline{F}_2(\kappa^2)$ have the same analyticity properties as F_1 and F_2. By the same arguments as used in the proof of Eq. (2) one can show that the behavior of $\overline{F}_1$ and $\overline{F}_2$ at infinity is such that for each of them there is a dispersion relation with one subtraction. At the point $\kappa^2 = m_a^2$

$$\overline{F}(m_a^2) = \frac{1}{2}\frac{m_a}{m_b+m_c},$$

$$\overline{F}_2(m_a^2) = \frac{1}{2}\left[\frac{m_a^2+m_b^2-m_c^2}{2m_b(m_b+m_c)}\pm\frac{m_a}{m_b+m_c}\right]. \tag{50}$$

From the requirement that $F_1(\kappa^2)$ be regular at the point $\kappa^2 = -m_b^2 + m_c^2$ there follows one further condition on the functions $\overline{F}_1$ and $\overline{F}_2$:

$$\mp\overline{F}_1(-m_b^2+m_c^2)+\overline{F}_2(-m_b^2+m_c^2) = 0. \tag{51}$$

Thus in order to determine the maximum possible value of g^2 we must find the minimum of the function Φ over the class of functions $\overline{F}_1$ and $\overline{F}_2$ having the analytic properties indicated above and satisfying the conditions (50) and (51). To solve this problem we substitute the dispersion relations for $\overline{F}_1(\kappa^2)$ and $\overline{F}_2(\kappa^2)$ [with one subtraction at the point $\kappa^2 = m_a^2$] in Eq. (49) and carry out the integration over κ^2. The result is

$$\Phi = C+\frac{1}{\pi}\int\limits_1^\infty dx\,\{R_1(x)\varphi_1(x)\sqrt{\alpha}$$

$$+R_2(x)[\sqrt{\alpha}(1+\sqrt{\lambda})\varphi_1(x)$$

$$-(1\pm\sqrt{\alpha})(\sqrt{\lambda}\pm\sqrt{\alpha})\varphi_2(x)](1+\sqrt{\lambda})\}$$

$$+2\int\limits_1^\infty dx\frac{\sqrt{(x-1)(x-\lambda)}}{x(x+\sqrt{\lambda})}$$

$$\times\left[\frac{(x-1)(x-\lambda)}{x}\varphi_1^2(x)+(1+\sqrt{\lambda})^2\varphi_2^2(x)\right]$$

$$+\frac{1}{\pi^2}\int\limits_1^\infty dx\int\limits_1^\infty dy\,\{K_1(x,y)\varphi_1(x)\varphi_1(y)$$

$$+K_2(x,y)[\varphi_2(x)\varphi_2(y)\,\varphi_1(x)\varphi_1(y)](1+\sqrt{\lambda})^2\}. \tag{52}$$

Here (we suppose $m_b \geq m_c$)

$$C = \frac{1}{4\alpha^2}\Big\{-\alpha(\alpha+2\sqrt{\lambda})+\left[2\lambda-\frac{\alpha}{2}(1-\sqrt{\lambda})^2\right]\frac{l}{\sqrt{\lambda}}$$

$$+\beta\left[2\sqrt{\lambda}+\alpha\frac{(\alpha-\sqrt{\lambda})(1+\sqrt{\lambda})^2}{2(1-\alpha)(\alpha-\lambda)}\right]\Big\}$$

$$+\frac{1}{4}\frac{1}{\alpha+\sqrt{\lambda}}[(1\pm\sqrt{\alpha})^2(\sqrt{\lambda}\pm\sqrt{\alpha})^2-\alpha(1+\sqrt{\lambda})^2]$$

$$\times\Big\{-\frac{1}{\alpha}+\left[\frac{2\alpha+\sqrt{\lambda}}{\alpha^2(\alpha+\sqrt{\lambda})}+\frac{2\alpha-1-\lambda}{2\alpha(1-\alpha)(\alpha-\lambda)}\right]\beta$$

$$+\frac{\alpha+\sqrt{\lambda}}{\alpha^2\sqrt{\lambda}}l-\frac{1+\sqrt{\lambda}}{\alpha+\sqrt{\lambda}}\frac{1}{\lambda^{1/4}}\ln\frac{1+\lambda^{1/4}}{1-\lambda^{1/4}}\Big\},$$

$$R_1(x) = \frac{\sqrt{\lambda}}{\alpha x}+\frac{1}{\alpha x}\left[-\frac{x+\alpha}{\alpha x}\lambda+\frac{1}{2}(1-\sqrt{\lambda})^2\right]\frac{1}{\sqrt{\lambda}}l$$

$$-\frac{\alpha+\sqrt{\lambda}}{\alpha^2(x-\alpha)}\beta+\frac{x+\sqrt{\lambda}}{x^2(x-\alpha)}L(x),$$

$$R_2(x) = \frac{1}{\alpha+\sqrt{\lambda}}\left[\frac{\alpha+\sqrt{\lambda}}{\alpha x\sqrt{\lambda}}l+\frac{1}{\alpha(x-\alpha)}\beta\right.$$

$$\left.-\frac{1+\sqrt{\lambda}}{x+\sqrt{\lambda}}\frac{1}{\lambda^{1/4}}\ln\frac{1+\lambda^{1/4}}{1-\lambda^{1/4}}-\frac{(\alpha+\sqrt{\lambda})L(x)}{(x+\sqrt{\lambda})x(x-\alpha)}\right],$$

$$K_1(x,y) = -\frac{\sqrt{\lambda}}{xy}+\frac{1}{xy}\left[\frac{x+y}{xy}\lambda-\frac{1}{2}(1-\sqrt{\lambda})^2\right]\frac{1}{\sqrt{\lambda}}l$$

$$-\frac{1}{x-y}\left[\frac{x+\sqrt{\lambda}}{x^2}L(x)-\frac{y+\sqrt{\lambda}}{y^2}L(y)\right],$$

$$K_2(x,y) = \frac{l}{\sqrt{\lambda}xy}-\frac{1+\sqrt{\lambda}}{(x+\sqrt{\lambda})(y+\sqrt{\lambda})}\frac{1}{\lambda^{1/4}}\ln\frac{1+\lambda^{1/4}}{1-\lambda^{1/4}}$$

$$-\frac{1}{x-y}\left[\frac{L(x)}{x(x+\sqrt{\lambda})}-\frac{L(y)}{y(y+\sqrt{\lambda})}\right],$$

$$\varphi_1(x) = \frac{\operatorname{Im}\overline{F}_1(x)}{(x-\alpha)},$$

$$\varphi_2(x) = \frac{\operatorname{Im}\overline{F}_2(x)}{(x-\alpha)}; \tag{53}$$

the remaining notation is the same as in the boson case [Eq. (13)]. The condition (51) can be written in the form

$$H = \frac{1}{2}\frac{1}{1+\sqrt{\lambda}}\pm\frac{1}{\pi}\int\limits_1^\infty\frac{\varphi_1(x)}{x+\sqrt{\lambda}}dx-\frac{1}{\pi}\int\limits_1^\infty\frac{\varphi_2(x)}{x+\sqrt{\lambda}}dx = 0. \tag{54}$$

When in the expression (52) we vary $\varphi_1(x)$ and $\varphi_2(x)$ subject to the condition (54) we get the following equations for the functions $\varphi_1(x)$ and $\varphi_2(x)$:

$$\frac{(x-1)^{3/2}(x-\lambda)^{3/2}}{x^2(x+\sqrt{\lambda})}\varphi_1(x)+\frac{\sqrt{\alpha}}{4\pi}[R_1(x)+(1+\sqrt{\lambda})^2R_2(x)]$$

$$+\frac{1}{2\pi^2}\int\limits_1^\infty[K_1(x,y)-(1+\sqrt{\lambda})^2K_2(x,y)]$$

$$\times\varphi_1(y)\,dy\mp\frac{\nu}{4\pi}\frac{1}{x+\sqrt{\lambda}} = 0, \tag{55}$$

$$\frac{(x-1)^{1/2}(x-\lambda)^{1/2}}{x(x+\sqrt{\lambda})}\varphi_2(x)-\frac{1}{4\pi}\frac{(1\pm\sqrt{\alpha})(\sqrt{\lambda}\pm\sqrt{\alpha})}{1+\sqrt{\lambda}}R_2(x)$$

$$+\frac{1}{2\pi^2}\int\limits_1^\infty K_2(x,y)\varphi_2(y)\,dy+\frac{\nu}{4\pi}\frac{1}{x+\sqrt{\lambda}} = 0. \tag{56}$$

The undetermined Lagrangian multiplier ν is to be found from Eq. (54). The minimum value of Φ is expressed in terms of the solution of the equations (54)—(56) in the following way:

$$\Phi_{min} = C + \frac{1}{2\pi}\int_1^\infty \{\sqrt{\alpha} R_1(x)\varphi_1(x) + R_2(x)(1+\sqrt{\lambda})[\sqrt{\alpha}(1+\sqrt{\lambda})\varphi_1(x) - (1\pm\sqrt{\alpha})(\sqrt{\lambda}\pm\sqrt{\alpha})\varphi_2(x)]\}\,dx. \quad (57)$$

The equations for the functions φ_1 and φ_2 have one important feature in which they differ from the corresponding equations in the boson case: for $x \to 1$ the coefficient of $\varphi_1(x)$ in Eq. (55) goes to zero as $(x-1)^{1/2}$. Because of this the equation (55) is singular and acquires meaning only if we suppose that the lower limit in the integral term in Eq. (55) is $1+\epsilon$, where ϵ is a small quantity. It can be shown that for $\epsilon \to 0$ the quantity Φ_{min} of Eq. (57) goes to a finite, nonzero limit, although the solution $\varphi_1(x)$ loses its meaning for $\epsilon = 0$.[7] This means that the minimum of Φ is realized through a sequence of functions, in which the limiting function, while satisfying all the requirements of analyticity, does not satisfy the dispersion relation. (That this is so can be seen below, when we calculate Φ_{min} by an analytic method.) Thus the question remains open as to whether the value of Φ_{min} which will be obtained corresponds to an actual physical situation, i.e., whether the minimum of Φ has not been set too low (and the maximum value of g^2 too high).

In the nonrelativistic (scalar) case $(1-\alpha \ll 1)$ the difficulties just pointed out are unimportant, and the problem can be solved easily. As follows from Eq. (57), for $1-\alpha \ll 1$ the function $\varphi_2(x)$ is the important one, and for it we have

$$\varphi_2(x) = \frac{\sqrt{1-\alpha}}{2(x-\alpha)\sqrt{x-1}}. \quad (58)$$

For Φ_{min} we get

$$\Phi_{min} = \frac{\pi}{4}\sqrt{\frac{1-\lambda}{1-\alpha}}(1+\sqrt{\lambda}), \quad (59)$$

which leads to the following restriction on g^2:

$$g^2 < 4\sqrt{\Delta/2\mu}\, m_a/m_b, \qquad \mu = m_b m_c/(m_b+m_c). \quad (60)$$

The formula (60) is equivalent to the formula (22) for the boson case, and goes over into that formula when we recall the fact that the normalization of the fermion wave functions differs from that of the boson wave functions by a factor $(2m)^{1/2}$.

We shall now calculate Φ_{min} by an analytical method.[9-11] We make the conformal transformation (24). Then the path of integration goes over into the unit circle and Eq. (49) can be written in the form

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\{f_1(\theta)|\bar{F}_1(z)|^2 + f_2(\theta)|\bar{F}_2(z)|^2\}, \qquad z = e^{i\theta}, \quad (61)$$

where $[u=\tan^2(\theta/2)]$

$$f_1(\theta) = \pi\sqrt{1-\alpha}\,\frac{u^2}{1+u}\,\frac{[1-\lambda+(1-\alpha)u]^{1/2}}{[1+(1-\alpha)u]^2[1+\sqrt{\lambda}+(1-\alpha)u]},$$

$$f_2(\theta) = \frac{\pi(1+\sqrt{\lambda})^2}{\sqrt{1-\alpha}}\,\frac{u}{1+u}\,\frac{[1-\lambda+(1-\alpha)u]^{1/2}}{[1+(1-\alpha)u][1+\sqrt{\lambda}+(1-\alpha)u]}. \quad (62)$$

The functions $\bar{F}_1(z)$ and $\bar{F}_2(z)$ can be expanded inside the unit circle $|z|<1$ in terms of systems of polynomials $\psi_n^{(1)}(z)$ and $\psi_n^{(2)}(z)$ which are orthogonal on the unit circle with the respective weight functions $f_1(\theta)$ and $f_2(\theta)$:

$$\bar{F}_1(z) = \sum_n a_n\psi_n^{(1)}(z), \qquad \bar{F}_2(z) = \sum_n b_n\psi_n^{(2)}(z). \quad (63)$$

[We note that the conditions for the expansions (63) to hold for $|z|=1$ are weaker than the conditions for existence of the dispersion relations for the functions $\bar{F}_1$ and $\bar{F}_2$ (cf. [10].] When Eq. (63) is substituted in Eq. (61) the integral Φ takes the form

$$\Phi = \sum_n (a_n^2 + b_n^2), \quad (64)$$

and the conditions (51), (50) can be written

$$\mp\bar{F}_1(z_0) + \bar{F}_2(z_0) = 0, \quad z_0 = -\frac{(1+\sqrt{\lambda})^{1/2}-(1-\alpha)^{1/2}}{(1+\sqrt{\lambda})^{1/2}+(1-\alpha)^{1/2}}, \quad (65)$$

$$\bar{F}_1(0) - \frac{\sqrt{\alpha}}{2} = 0, \quad \bar{F}_2(0) - \frac{1}{2}\left[\frac{\alpha+\sqrt{\lambda}}{1+\sqrt{\lambda}} \pm \sqrt{\alpha}\right] = 0. \quad (66)$$

When we look for the minimum of the expression (64) under the conditions (65), (66) we easily find

$$a_n = \tfrac{1}{2}[\nu_1\psi_n^{(1)}(0) \mp \nu_3\psi_n^{(1)}(z_0)],$$

$$b_n = \tfrac{1}{2}[\nu_2\psi_n^{(2)}(0) + \nu_3\psi_n^{(2)}(z_0)], \quad (67)$$

where ν_1, ν_2, ν_3 are Lagrangian undetermined multipliers. Substituting Eq. (67) in Eqs. (65), (66) and using the formula for calculating sums of orthogonal polynomials (see [10], page 310)

$$\sum_n \psi_n^*(z_1)\psi_n(z_2) = \frac{1}{1-z_1^* z_2}\,\frac{1}{D^*(z_1)D(z_2)}, \quad (68)$$

$$D(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln f(\theta)\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\theta\right\}, \quad (69)$$

we get the following system of equations for the determination of ν_1, ν_2, ν_3:

$$\nu_1 D_1^{-2}(0) \mp \nu_3 D_1^{-1}(0)D_1^{-1}(z_0) = \sqrt{\alpha},$$

$$\nu_2 D_2^{-2}(0) + \nu_3 D_2^{-1}(0)D_2^{-1}(z_0) = (\alpha+\sqrt{\lambda})/(1+\sqrt{\lambda}) \pm \sqrt{\alpha},$$

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[7]The authors are grateful to E. M. Landis for a discussion of this point.

$$\mp \nu_1 D_1^{-1}(0) D_1^{-1}(z_0) + \nu_2 D_2^{-1}(0) D_2^{-1}(z_0) + \nu_3 [D_1^{-2}(z_0) + D_2^{-2}(z_0)]/(1-z_0^2) = 0. \quad (70)$$

Substituting Eq. (67) in Eq. (64) and using Eq. (68), we find

$$\Phi_{min} = \tfrac{1}{4}[\nu_1^2 D_1^{-2}(0) \mp 2\nu_1\nu_3 D_1^{-1}(0) D_1^{-1}(z_0) + \nu_3^2 D_1^{-2}(z_0)/(1-z_0^2)] \tfrac{1}{4}[\nu_2^2 D_2^{-2}(0) + 2\nu_2\nu_3 D_2^{-1}(0) D_2^{-1}(z_0) + \nu_3^2 D_2^{-2}(z_0)/(1-z_0^2)]. \quad (71)$$

For the $f_1(\theta)$ and $f_2(\theta)$ found from Eq. (62) the functions $D_1(z)$ and $D_2(z)$ can be calculated easily, and are found to be $[v = (1-z)/(1+z)]$

$$D_1(z) = \frac{1}{2}\sqrt{\frac{\pi}{1-\alpha}\,\frac{(1-z)^2}{1+z}} \times \frac{[\sqrt{(1-\lambda)/(1-\alpha)} + v]^{3/2}}{[1/\sqrt{1-\alpha} + v]^2\,[((1+\sqrt{\lambda})/(1-\alpha))^{1/2} + v]},$$

$$D_2(z) = \frac{\sqrt{\pi}(1+\sqrt{\lambda})}{1-\alpha}\,\frac{1-z}{2} \times \frac{[\sqrt{(1-\lambda)/(1-\alpha)} + v]^{1/2}}{[1/\sqrt{1-\alpha} + v]\,[((1+\sqrt{\lambda})/(1-\alpha))^{1/2} + v]}. \quad (72)$$

In order to determine Φ_{min} we still have to solve the equations (70) for ν_1, ν_2, ν_3, substitute these solutions in Eq. (71), and express the functions D by using Eq. (72). The final result obtained for Φ_{min} is

$$\Phi_{min} = \frac{\pi}{16}\,\frac{(\sqrt{1-\alpha} + \sqrt{1-\lambda})^3}{(1+\sqrt{1-\alpha})^2\,[\sqrt{1-\alpha} + (1+\sqrt{\lambda})^{1/2}]^2}$$
$$\times \left\{ \frac{\alpha\sqrt{1-\alpha}}{(1+\sqrt{1-\alpha})^2} + \frac{(1\pm\sqrt{\alpha})^2(\sqrt{\lambda}\pm\sqrt{\alpha})^2}{\sqrt{1-\alpha}\,(\sqrt{1-\alpha}+\sqrt{1-\lambda})^2} \right.$$
$$+ 2\,\frac{[\sqrt{1-\lambda} + (1+\sqrt{\lambda})^{1/2}]^2\,[\sqrt{1-\alpha} + (1+\sqrt{\lambda})^{1/2}]^2}{(\alpha+\sqrt{\lambda})^2\,[\sqrt{1-\lambda} + 2(1+\sqrt{\lambda})^{1/2} + 1 + \sqrt{\lambda}]}$$
$$\times \left[\frac{\alpha+\sqrt{\lambda}}{\sqrt{1-\alpha}+\sqrt{1-\lambda}} \pm \sqrt{\alpha}\left(\frac{1+\sqrt{\lambda}}{\sqrt{1-\alpha}+\sqrt{1-\lambda}} \right.\right.$$
$$\left.\left.\left. - \frac{\sqrt{1-\alpha}\,(1+\sqrt{\lambda})^{1/2}\,[1+(1+\sqrt{\lambda})^{1/2}]}{[\sqrt{1-\lambda}+(1+\sqrt{\lambda})^{1/2}]\,(1+\sqrt{1-\alpha})} \right) \right]^2 \right\}. \quad (73)$$

In the scalar nonrelativistic case $(1-\alpha \ll 1)$ the expression (73) goes over into (59). When we substitute in Eq. (73) the numerical values of the masses for the most interesting case—the interaction of π mesons with nucleons (particles a and b are nucleons and c is a π meson)—we get $\Phi_{min} = 0.0245$. From this we get as the restriction on the pion-nucleon coupling constant (the coefficient $1/3$ is used on account of the isotopically symmetrical theory)

$$g^2 < 2\pi/3\Phi_{min} = 85 \quad (74)$$

or

$$f^2 = (\mu/2m)^2 g^2 < 0{,}47. \quad (75)$$

For the $\Sigma\Lambda\pi$ interaction (a is the fermion Σ, b is the fermion Λ, and c is the pion), if the parities of Σ and Λ are different we have $\Phi_{min} = 1.95$ and

$$g^2_{\Sigma\Lambda\pi} < 3.2. \quad (76)$$

For like parities of the Σ and Λ hyperons $\Phi_{min} = 0.0120$ and $g^2_{\Sigma\Lambda\pi} < 520$. Finally, for the $\Xi\Xi\pi$ interaction $\Phi_{min} = 0.0160$ and $g^2_{\Xi\Xi\pi} < 130$.

It is interesting to see how the restriction on the constant for the $\Sigma\Lambda\pi$ interaction is altered if we take Λ instead of Σ as fermion a. Calculating Φ_{min} we find that in this case $g^2_{\Sigma\Lambda\pi} < 7.5$ if Σ and Λ have opposite parities, and $g^2_{\Sigma\Lambda\pi} < 260$ if the parities are the same, i.e., as the binding energy increases the maximum value of g^2 increases for the scalar interaction and decreases for the pseudoscalar interaction.[8)]

CONCLUSION

We have found that for prescribed values of the masses the coupling constants in quantum field theory are bounded.[9)] The fact that the value of the coupling constant g^2 cannot exceed a certain value g^2_{max} means that the amplitudes of the quantum field theory, considered as functions of g^2, have a singular point at $g^2 = g^2_{max}$, and for $g^2 > g^2_{max}$ the theory will be physically contradictory.

It is interesting to discuss what may happen if g^2 is larger than g^2_{max}. One possibility is that the vertex part $\Gamma(\kappa^2)$ will have a pole $\kappa^2 = m_0^2$ which lies in the interval $m_a^2 < m_0^2 < (m_b + m_c)^2$, i.e., that a new physical state with the mass m_0^2 will appear.

Another possibility is that $D^{-1}(\kappa^2)$ will cease to be an R-function, i.e., for example one of the constants R or α in Eq. (2) will be negative. [When there is only one pole of $D(\kappa^2)$ only one of the constants R_n in Eq. (2) can be different from zero.] It is not hard to verify that in addition to the zero of $D^{-1}(\kappa^2)$ at $\kappa^2 = m_a^2$ there will be another zero at $\kappa^2 = \kappa_1^2$, with $\kappa_1^2 < m_a^2$ and $[D^{-1}(\kappa^2)]'_{\kappa^2=\kappa_1^2} = 0$. Consequently, in this case the Green's function $D(\kappa^2)$ has a pole at $\kappa^2 = \kappa_1^2$ with

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8)All of the numerical values of Φ_{min} given in the text have also been obtained by numerical solution of the equations (54)-(56) (by cutting the integrals off at the lower limit and afterward letting the cut-off limit approach zero) and calculation of Φ_{min} from Eq. (57).

9)In the problem of the interaction of neutral scalar mesons with a static nucleon, which, as is well known, has an exact solution and does not lead to difficulties of the zero-charge type, there has been an earlier (and rigorous) proof of the fact that the coupling constant is bounded (see [12]).

a negative residue, i.e., an unphysical state with negative norm (a "ghost") will appear.

The fact that the coupling constants in quantum field theory have upper bounds indicates that all attempts to approach the solution of problems by using expansions in powers of $1/g^2$ with fixed masses (strong-coupling theory) have extremely little chance of success.

We shall also make a comment on a paper by Landau.[13] In this paper Landau put forward the idea that the constant g^2 must be equal to the value determined for it for the case in which particle a is represented as a bound state of particles b and c with a point interaction. It follows from our treatment that in general this assertion cannot be correct, since all the other states besides the two-particle states make positive contributions to the relation (5).

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