ON THE PROPERTIES OF DISPLACEMENTS IN p-SPACE OF CONSTANT CURVATURE

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The displacements in constant-curvature space, which are among the basic elements of the generalized field theory developed by the author [2], are studied in detail. A characteristic of the displacements, the focusing property, is revealed, having no analogue in the ordinary Euclidean case. The field theory apparatus of [2] is refined. Deviations from the energy-momentum conservation law in scattering processes are considered; it is shown that the law remains valid in the generalized theory for the elastic scattering of fermions. The focusing property results in an interaction anomaly in a particle-antiparticle system. The possible formation of bound states of such systems during contact interactions with a small coupling constant is discussed briefly.

1. INTRODUCTION

IN earlier work [1,2] the present author constructed a field theory in which the pseudo-Euclidean momentum space was replaced with a p-space of constant curvature. In this theory the role of the ordinary addition of momenta is played by the displacement operator of p-space. In the present work the properties of the displacements are studied in detail. As in [2], we shall first consider an elliptic p-space having a positive-definite metric.[1)] Certain relations for displacements in the physical region (pseudo-elliptic space) will be considered in Sec. 6.

In investigating the system of displacements as a whole their parameterization by means of the vector k (k-parameterization) is not entirely suitable. The displacement equation in the case of k-parameterization is [2]

$$q = d_0(k)\,p = \frac{p\sqrt{1+k^2} + k\,[1 - pk/(1+\sqrt{1+k^2})]}{1 - pk} \qquad (1.1)$$

containing $\sqrt{1+k^2}$, which must undergo a change of signs as k^2 varies continuously.[2)] Elliptic p-space is a closed "one-sided space."[3] "Infinitely distant" points of this space forming a three-dimensional hyperplane are not isolated points and go over into "finite" points under displacements. The one-sidedness of the space consists in the fact the two infinitely distant points of a single ray are considered identical. More precisely, for each function of the vector $p = \beta n\,(n^2 = 1)$ we must obtain identical values when $\beta \to +\infty$ and $\beta \to -\infty$. This property will hereinafter be called the projective invariance of the theory. It is easily shown that (1.1) will be projectively invariant only if the sign of $\sqrt{1+k^2}$ is reversed when k^2 passes through infinity. The indeterminate sign of $\sqrt{1+k^2}$ is reversed when k^2 passes through infinity. The indeterminate sign of $\sqrt{1+k^2}$ creates certain inconveniences when the displacement equation (1.1) is used. In order to overcome this difficulty we introduce another, rational, parameterization of the displacement system (l-parameterization).

The displacement operation (1.1) satisfies the correspondence principle by going over into ordinary addition for small p and k. However, for large momenta the properties of displacements differ extremely from those of ordinary momentum addition. Specifically, when $q = -p$ we have the so-called focusing property. The present work consists essentially in an investigation of this property. From the physical point of view the focusing property signifies the appearance of an anomaly in the interaction between a particle and an antiparticle having identical momenta (Sec. 7).

2. l-PARAMETERIZATION OF DISPLACEMENTS

With each vector l of p-space we associate a displacement defined by

$$q = d_0(l)\,p = l + \frac{1 + l^2}{1 - l^2 - 2lp}(l + p) \qquad (2.1)$$

This equation is a rational function of the vectors p and l, and satisfies the condition of projective

[1)] The notation of [2] will be used unless specified otherwise.

[2)] I. E. Tamm pointed out the need to consider both signs of $\sqrt{1+k^2}$, without which the displacement equation (1.1) becomes incomplete.

invariance, as is easily shown. The advantage of (2.1) over (1.1) lies in the fact that the vector q is determined uniquely in terms of p and l. The parameterization of displacements represented by (2.1) will be called l-parameterization.

The relation between the two displacement formulas is given by

$$k = 2l/(1 - l^2). \qquad (2.2)$$

Substituting (2.2) in (1.1), we obtain (2.1). In virtue of (2.2) some vector k corresponds uniquely to each vector l. Fixing k and solving (2.2) for l, we obtain two values (positive roots):

$$l_1 = \frac{k}{1+\sqrt{1+k^2}}, \qquad l_2 = \frac{k}{1-\sqrt{1+k^2}}. \qquad (2.3)$$

For the scalar product of the vectors in (2.3) we have $l_1 l_2 = -1$, and therefore the distance [2]

$$s(l_1, l_2) = \pi/2. \qquad (2.4)$$

The vector k given by (2.2) is by definition the momentum transferred to a fermion at the vertex of a Feynman diagram. Because of the two values in (2.3) each momentum k corresponds to two essentially different displacements $d_0(l_1)$ and $d_0(l_2)$.

Let us consider the limiting case $k \to 0$. It is convenient to write $k = \beta n$, where $\beta \to 0$ and n is an arbitrary unit vector ($n^2 = 1$). From (2.3) we obtain

$$l_1 \approx \tfrac{1}{2}\beta n = \tfrac{1}{2} k \to 0, \qquad l_2 \approx -2n/\beta. \qquad (2.5)$$

Substituting (2.5) in (2.1), we obtain expressions for the displacements (with a fixed vector p):

$$q_1 = d_0(l_1)\, p \approx \beta n + p = k + p, \qquad (2.6)$$

$$q_2 = d_0(l_2)\, p \approx -p + 2(np)\, n.$$

The considered case of extremely small momenta k illustrates clearly the essential difference between the displacements $d_0(l_1)$ and $d_0(l_2)$. While the first displacement is of classical character (satisfying the correspondence principle), the second shift is the product of an inversion relative to a hyperplane perpendicular to the vector n and the simultaneous reflection of all four axes. The solution l_2 in (2.5) and the corresponding displacement $d_0(l_2)$ in (2.6), which have no analogues in the ordinary theory, will be called nonclassical. The nonclassical displacement $d_0(l_2)$ is an improper rotation of elliptic p-space (its determinant is −1). Kadyshevskiĭ [3] has discussed the possibility of continuous transitions from proper rotations to improper rotations in one-sided spaces of constant curvature.

The introduction of the l-parameterization of displacements (2.1) enables us to refine the meaning of the correspondence principle in the theory being developed. According to (2.5) and (2.6) correspondence is not achieved for small values of the transferred momentum k, but rather for small vectors of the vector l. The latter therefore acquires physical meaning, while k, which is given by (2.2), becomes a secondary quantity in a sense. The "classical region" in which the correspondence principle is satisfied will pertain to values $l^2 \ll 1$. Equation (2.5) shows that small momenta k also correspond to "nonclassical" values of l ($l^2 \gg 1$). The contribution of nonclassical displacements to real physical processes is considered in Sec. 5.

For a more detailed investigation of (2.1) it is convenient to employ a geometric construction. Having in mind the definition of the displacement operator for a scalar function of the vector p,

$$\hat{d}_0(l)\, f(p) = f(d_0(-l)\, p), \qquad (2.7)$$

we shall [in distinction from (2.1)] use q to denote the vector

$$q = d_0(-l)\, p = -l + \frac{1+l^2}{1-l^2+2lp}(p-l)$$
$$= \frac{1+l^2}{1-l^2+2lp}\, p - \frac{2(1+lp)}{1-l^2+2lp}\, l. \qquad (2.8)$$

The sum of the coefficients of the vectors (−p) and ($-l$) in the right-hand side of (2.8) equals unity. It follows that the three points (−p), ($-l$), and q lie on the same straight line (Fig. 1). From (2.8) we obtain

$$1 - ql = \frac{1+l^2}{1-l^2+2lp}(1+pl), \qquad (2.9)$$

$$1 + q^2 = \left(\frac{1+l^2}{1-l^2+2lp}\right)^2 (1+p^2). \qquad (2.10)$$

Combining (2.9) and (2.10), we obtain

$$|1 - ql|/\sqrt{1+q^2} = |1 + pl|/\sqrt{1+p^2}. \qquad (2.11)$$

From Eq. (1.2) in [2], representing the distance between two points of elliptic p-space, and (2.11) we obtain the following relations for the distance between the given points:

$$s(p, l) \equiv s(-p, -l) = s(-l, q). \qquad (2.12)$$

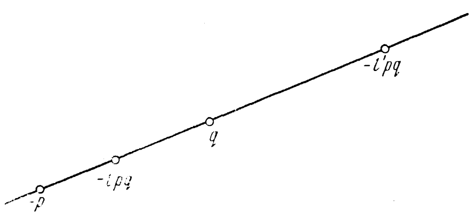

FIG. 1

The geometric interpretation of displacements is based on (2.12).

In investigating (2.12) we must take into account the facts that: 1) a straight line in elliptic space is closed (being equivalent topologically to a circle) with its total length equal to π, and 2) by definition the length of a segment of a straight line does not exceed $\pi/2$.

We introduce the abbreviated notation

$$\lambda = s(p, l) = s(-l, q), \qquad \rho = s(-p, q). \tag{2.13}$$

We easily find, using Fig. 1,

$$\rho = 2\lambda \qquad (0 \leqslant \lambda \leqslant \pi/4), \qquad \rho = \pi - 2\lambda \qquad (\pi/4 \leqslant \lambda \leqslant \pi/2). \tag{2.14}$$

The geometric construction (Fig. 1) enables us to establish graphically an extremely important property of displacements in elliptic space. Let us put

$$\lambda = s(p, l) = \pi/2. \tag{2.15}$$

Then, in virtue of (2.14), we have $\rho = s(-p, q) = 0$ and therefore $q = -p$, since the elliptic p-space has a positive-definite metric. The condition (2.15) is equivalent to the relation

$$pl = -1. \tag{2.16}$$

For a fixed value of p the vectors l satisfying (2.16) fill a three-dimensional hyperplane of p-space. We arrive at the conclusion that when p and l satisfy (2.15) [or, equivalently, (2.16)] the displaced vector

$$q = d_0(-l)\, p = -p \tag{2.17}$$

does not depend on the direction of the straight line passing through the points (−p) and ($-l$). This property of the displacements in elliptic space, for which there is no analogous property in the usual theory, will be called the focusing property; its physical interpretation will be considered in Sec. 7.

We shall now solve (2.8) for l with fixed values of p and q. In Fig. 1 the point ($-l$) must lie on a straight line passing through the points (−p) and q and must also satisfy (2.12). It is easily seen that two solutions exist:

$$l = l_{pq} \quad (0 \leqslant \lambda \leqslant \pi/4), \quad l = l'_{pq} \quad (\pi/4 \leqslant \lambda \leqslant \pi/2). \tag{2.18}$$

For $q = -p$ the construction no longer applies. In this case (2.8) has an infinite number of solutions; this corresponds to the focusing property. The solutions of (2.18) satisfy

$$s(l, l') = \pi/2. \tag{2.19}$$

In explicit analytic form the solutions of (2.18) are

$$l_{pq} = \frac{p\sqrt{1+q^2} - q\sigma\sqrt{1+p^2}}{\sqrt{1+q^2} + \sigma\sqrt{1+p^2}}, \quad l'_{pq} = \frac{p\sqrt{1+q^2} + q\sigma\sqrt{1+p^2}}{\sqrt{1+q^2} - \sigma\sqrt{1+p^2}}, \tag{2.20}$$

where

$$\sigma = \operatorname{sign}(1 - pq) \tag{2.21}$$

determines the selection of the branch of $\sqrt{1 + p^2}$.

When $q = -p$ an indeterminancy associated with the focusing property arises in (2.20). To investigate this indeterminacy we write $q = -p + \beta n$, where $\beta \to 0$ and $n^2 = 1$. In the limit $\beta = 0$, we obtain

$$l_{pq} = p, \qquad l'_{pq} = p - \frac{1+p^2}{(pn)}\, n. \tag{2.22}$$

Through the l-parameterization of displacements in p-space we are enabled to simplify the expression for the displacement operator in the spinor representation. In view of (2.2) the spinor displacement operator is easily represented by

$$\hat{d}_s(l) = (1 - \hat{l})/\sqrt{1 + l^2}. \tag{2.23}$$

The operator $\hat{d}_s(l)$ acts on the constant spinor ψ. As in [2], we define the total displacement operator $\hat{d}(l)$ acting on a spinor function $\psi(p)$ as the product of the operators (2.7) and (2.23):

$$\hat{d}(l) = \hat{d}_0(l)\, \hat{d}_s(l). \tag{2.24}$$

3. MATRIX ELEMENTS OF DISPLACEMENT OPERATORS

We shall now calculate explicitly the matrix elements of a displacement operator in the p-representation. Since the spinor displacement operator $\hat{d}_s(l)$ is given explicitly by (2.23) as a matrix, it is sufficient to calculate the matrix elements of $\hat{d}_0(l)$. We shall first consider the more general problem of determining the matrix elements of the operator

$$\hat{B} = \int d\Omega_l\, b(l)\, \hat{d}_0(l), \tag{3.1}$$

where $b(l)$ is an arbitrary function of l. Letting $\hat{B}$ operate on $f(p)$, we obtain

$$\hat{B}f(p) = \int d\Omega_l b(l)\, f(q), \tag{3.2}$$

where q, which depends on p and l, is given by (2.8). In (3.2) integration over $d\Omega_l$ must be replaced by integration over $d\Omega_q$. In order to obtain the relation between the respective volume elements we write the expression for a volume element of p-space in polar coordinates:

$$d\Omega_q = \sin^3\rho\, \sin^2\eta\, \sin\theta\, d\rho\, d\eta\, d\theta\, d\varphi, \tag{3.3}$$

where φ, θ, η, and ρ are the appropriate angle variables, which have the limits

$$0 \leqslant \varphi \leqslant 2\pi; \qquad 0 \leqslant \theta,\ \eta \leqslant \pi; \qquad 0 \leqslant \rho \leqslant \pi/2. \quad (3.3a)$$

We note that ρ represents geometrically the distance from some fixed point taken as the coordinate origin to an arbitrary point q of p-space. Using Fig. 1, taking the point (−p) as the coordinate origin, and considering that the points (−p), $(-l)$, and q lie on the same straight line and that $d\Omega_l = d\Omega_l$, we easily obtain the relation between the volume elements:

$$d\Omega_l = d\Omega_q/16\cos^3(\rho/2) \qquad (0 \leqslant \lambda \leqslant \pi/4),$$
$$d\Omega_l = d\Omega_q/16\sin^3(\rho/2) \qquad (\pi/4 \leqslant \lambda \leqslant \pi/2), \quad (3.4)$$

where λ and ρ are defined by (2.13). Using (3.4) for the change of variables in (3.2), we obtain

$$\hat{B}f(p) = \int d\Omega_q \left\{ \frac{b(l_{pq})}{16\cos^3(\rho/2)} + \frac{b(l'_{pq})}{16\sin^3(\rho/2)} \right\} f(q), \quad (3.4a)$$

whence, from the definition of a matrix element,

$$\langle p \mid \hat{B} \mid q \rangle = \frac{b(l_{pq})}{16\cos^3(\rho/2)} + \frac{b(l'_{pq})}{16\sin^3(\rho/2)}. \quad (3.5)$$

The quantities l_{pq} and l'_{pq} are given by (2.20). From the definition (3.1) of the operator $\hat{B}$ we obtain directly

$$\hat{d}_0(l) = \delta\hat{B}/\delta b(l). \quad (3.6)$$

The variational derivative of (3.5) yields the matrix element

$$\langle p \mid \hat{d}_0(l) \mid q \rangle = \frac{1}{16} \left\{ \frac{\delta(l, l_{pq})}{\cos^3(\rho/2)} + \frac{\delta(l, l'_{pq})}{\sin^3(\rho/2)} \right\}. \quad (3.7)$$

Using (2.14), this can be written in the form

$$\langle p \mid \hat{d}_0(l) \mid q \rangle = \frac{1}{16\cos^3\lambda} \{\delta(l, l_{pq}) + \delta(l, l'_{pq})\}, \quad (3.8)$$

where $\lambda = s(p, l)$.

The amplitudes of different processes are expressed in terms of the matrix elements for products of the displacement operators. Let us consider the diagonal matrix element for the product of two displacements that represents a polarization operator in a second-order perturbation.

The following general relation is convenient. Let $\hat{A}$ be an operator with $\langle p \mid \hat{A} \mid q \rangle$ as its matrix element. From the definition (2.7) of the displacement operator we have directly

$$\langle p \mid \hat{d}_0(l)\hat{A} \mid q \rangle = \langle d_0(-l)p \mid \hat{A} \mid q \rangle,$$
$$\langle p \mid \hat{A}\hat{d}_0(l) \mid q \rangle = \langle p \mid \hat{A} \mid d_0(l)q \rangle. \quad (3.9)$$

By relatively simple calculations we obtain

$$\langle p \mid \hat{d}_0(l)\hat{d}_0(-t) \mid p \rangle = \{\delta(l, t) + \delta(l, t')\}/16\cos^3\lambda. \quad (3.10)$$

Here l and t are arbitrary vectors of p-space, $\lambda = s(p, l)$, and t′ is a point lying on the straight line (p, l) and satisfying the condition $s(t, t') = \pi/2$. For $\lambda = \pi/2$, (3.10) has a singularity directly related to the focusing property of the displacement operation. This singularity results in divergence of the integral of (3.10) over $d\Omega_p$. Methods of eliminating this difficulty are discussed in the following section. It must be noted, however, that the divergence considered here has nothing in common with the divergences encountered in the usual field theory. This divergence is of extremely general mathematical character associated with the fact that the trace of any operator with a continuous spectrum is always infinite.

4. NORMALIZED DISPLACEMENTS

The appearance of a nonintegrable singularity in (3.10) leads to serious difficulties in calculating the matrix elements for diagrams containing closed loops. This situation is directly related to the existence of the focusing property in the displacement operators of p-space. This focusing property is an intrinsic property of elliptic p-space and from our point of view plays an extremely essential role in the construction of a physical theory of elementary particles (see Sec. 7). We shall therefore consider one of the possible ways of eliminating the indicated difficulty.

We suppose that the displacement operator (2.24) is not a good analogue of momentum addition in the theory based on a p-space of constant curvature. We introduce instead of (2.24) the so-called normalized displacement as an analogue of momentum addition. It must be emphasized that the introduction of a normalized displacement in the theory is a special hypothesis, the correctness of which can only be determined by the subsequent development of this theory.

We define the normalization operator $\hat{N}(l)$, which is diagonal in the p-representation, by its matrix element

$$\langle p \mid \hat{N}(l) \mid q \rangle = \cos^{3/2} s(p, l)\, \delta(p, q). \quad (4.1)$$

$\hat{N}(l)$ contains as its parameter a vector l of p-space, and satisfies the relation

$$\hat{N}(l)\hat{d}_0(l) = \hat{d}_0(l)\hat{N}(-l). \quad (4.2)$$

The validity of (4.2) follows from the fact that in virtue of (3.8) the matrix element $\langle p \mid \hat{d}_0(l) \mid q \rangle$ differs from zero only for $l = l_{pq}$ and $l = l'_{pq}$, and that for these values of l we have $s(p, l) = s(q, -l)$ according to (2.12). Therefore

$$\langle p|\hat{N}(l)\,\hat{d}_0(l)\,|q\rangle = \cos^{3/2} s(p, l)\,\langle p\,|\,\hat{d}_0(l)\,|\,q\rangle$$
$$= \langle|\,\hat{d}_0(l)\,|\,q\rangle \cos^{3/2} s(q, -l) = \langle p\,|\hat{d}_0(l)\hat{N}(-l)\,|\,q\rangle, \quad (4.2a)$$

which is equivalent to (4.2).

We define the normalized displacement operator for a scalar function by

$$\hat{d}_{N0}(l) = \hat{N}(l)\,\hat{d}_0(l). \quad (4.3)$$

The normalized displacement for a spinor field $\psi(p)$ is defined as a product:

$$\hat{d}_N(l) = \hat{d}_{N0}(l)\,\hat{d}_s(l) = \hat{N}(l)\,\hat{d}_0(l)\,\hat{d}_s(l). \quad (4.4)$$

The normalized displacement hypothesis signifies that the operator $\hat{d}_N(l)$ is taken to be a suitable analogue of momentum addition at vertices of Feynman diagrams. It is obvious that $\hat{d}_N(l)$, like the displacement operator $\hat{d}(l)$, satisfies all requirements of the correspondence principle in the classical region. Using the unitarity of the displacement operator (2.24) and the relation (4.2), we easily obtain the adjoint operator of (4.4):

$$\hat{d}_N^+(l) = \hat{d}_N(-l). \quad (4.5)$$

When in (3.10) the displacement $\hat{d}_0(l)$ is replaced with the normalized displacement $\hat{d}_{N0}(l)$, we obtain

$$\langle p\,|\,\hat{d}_{N0}(l)\,\hat{d}_{N0}(-t)\,|\,p\rangle = \tfrac{1}{16}\{\delta(l, t)$$
$$+ \operatorname{ctg}^{3/2} s(p, t)\,\delta(l, t')\}. \quad (4.6)^*$$

The matrix element (4.6) no longer contains a nonintegrable singularity. The same will evidently apply to all higher perturbation approximations. Thus the introduction of the normalized displacement operator (4.4) eliminates the difficulty mentioned at the end of the preceding section.

5. REFINEMENT OF THE FIELD THEORY

By introducing the l-parameterization of displacements along with the normalized displacement operator $\hat{d}_N$, we achieve a refinement of the field theory apparatus constructed in [2] (Sec. 2). The required changes reduce to a redefinition of the operator for a boson quasi-field. We define this operator by

$$\varphi(l) = (\mu^2/4 + l^2)^{-1/2}\{c(l) + c^+(-l)\}, \quad (5.1)$$

where $c(l)$ and $c^+(l)$ are the particle annihilation and creation operators, which are related by the commutators

$$[c(l), c^+(t)] = \delta(l, t). \quad (5.2)$$

(Commutators equal to zero are not written out.)

From (5.1) and (5.2) we obtain for the average of $\varphi(l)\varphi(t)$ over the vacuum

$$\langle\varphi(l)\,\varphi(t)\rangle_0 = (\mu^2/4 + l^2)^{-1}\,\delta(l, -t). \quad (5.3)$$

The boson propagation function in the right-hand side of (5.3) was selected so that for small transferred momenta k the contribution from the nonclassical solution l_2 would, in virtue of (2.5), be negligibly small compared with that of the classical solution l_1. This agrees with the refined correspondence principle formulated in Sec. 2.

The operator $\hat{\varphi}$ generalizing the field $\varphi(x)$ is defined by

$$\hat{\varphi} = \frac{1}{2\pi^2}\int d\Omega_l\,\varphi(l)\,\hat{d}_N(l). \quad (5.4)$$

The normalization coefficient of this integral was selected to satisfy the correspondence principle in making a transition to the ordinary theory. We use $\hat{\varphi}$ to define the interaction operator

$$\Lambda = g\,\langle\bar{\psi}\,|\,\gamma_5\hat{\varphi}\,|\,\psi\rangle, \quad (5.5)$$

which in virtue of (4.5) is Hermitian:

$$\Lambda = \Lambda^+. \quad (5.5a)$$

Beginning at this point, all further construction of the theory is exactly the same as in [2]. For a direct four-fermion interaction (5.5) is replaced by the operator

$$\Lambda = \int d\Omega_l\,J(l)\,J(-l), \quad (5.6)$$

where the current operator is

$$J(l) = \langle\bar{\psi}\,|\,\hat{O}\,\hat{d}_N(l)\,|\,\psi\rangle. \quad (5.6a)$$

6. DISPLACEMENT OPERATIONS IN THE PHYSICAL REGION

In this section we shall consider certain relations for displacements in a pseudo-elliptic space that generalizes the ordinary pseudo-Euclidean p-space and possesses a non-definite metric. All 4-momentum vectors mentioned in this section belong to the pseudo-elliptic space. The scalar product of two vectors in this space is written as $pq = p_0q_0 - \mathbf{p}\cdot\mathbf{q}$. The relationship between the elliptic and pseudo-elliptic spaces has been considered in [2]. A displacement in pseudo-elliptic space with l-parameterization is given by

$$q = d_0(l)\,p = l + \frac{1 - l^2}{1 + l^2 + 2lp}(l + p). \quad (6.1)$$

The vector l is related to the transferred momentum k by

$$k = 2l/(1 + l^2) \quad (6.2)$$

*ctg = cot.

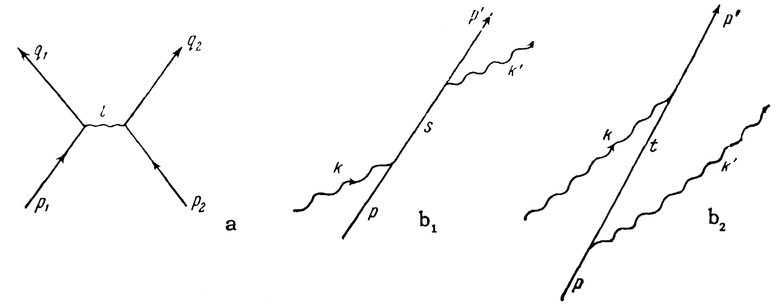

FIG. 2

As in Sec. 2, we obtain two solutions of (6.2):

$$l_1 = k/(1+\sqrt{1-k^2}), \quad l_2 = k/(1-\sqrt{1-k^2}). \quad (6.3)$$

For a real physical particle l must [according to (5.3)] satisfy $l^2 = \mu^2/4 \ll 1$.[3)] Therefore the solution l_2 drops out for real particles and we obtain a unique correspondence between the vectors k and l, with l always belonging to the classical region.

Using the simplest scattering processes as examples, we shall now consider to what extent energy and momentum are not conserved when ordinary p-space is replaced by a space of constant curvature.

1. Fermion-fermion scattering (Fig. 2a). The initial and final momenta are related by

$$q_1 = d_0(l)\,p_1, \quad q_2 = d_0(-l)\,p_2. \quad (6.4)$$

Since all particles participating in the reaction are real, their momenta satisfy the conditions

$$p_i^2 = m_i^2, \quad q_i^2 = M_i^2 \quad (i = 1, 2), \quad (6.5)$$

where all masses are different in general. We use a relation analogous to (2.20) for the case of the pseudo-elliptic space. We can limit ourselves to the classical solution l_{pq}, since the nonclassical solution l'_{pq} either leads to infinitely large values of l, which make a vanishingly small contribution, or, if l is finite, they must satisfy

$$q = -p. \quad (6.6)$$

Equation (6.6), representing the focusing property, can be satisfied only for processes involving the creation or annihilation of pairs of particles. The contribution of the focusing property to the annihilation diagrams is discussed in the following section. Here we shall limit ourselves to scattering processes not involving pairs. From (6.4) and (6.5) we have the following relation between momenta:

$$\frac{q_1/\sqrt{1-M_1^2} - p_1/\sqrt{1-m_1^2}}{1/\sqrt{1-M_1^2} + 1/\sqrt{1-m_1^2}} = \frac{p_2/\sqrt{1-m_2^2} - q_2/\sqrt{1-M_2^2}}{1/\sqrt{1-m_2^2} + 1/\sqrt{1-M_2^2}}. \quad (6.7)$$

Equation (6.7) shows that the energy-momentum conservation law is not of universal character in the theory under discussion. The relation between the momenta of the considered particles depends on their masses. In its usual form the energy-momentum conservation law is satisfied, as a general rule, up to m^2 (in dimensionless units). For elastic scattering ($m_1 = M_1 = m$, $m_2 = M_2 = m'$, since $m = m'$ is not obligatory) Eq. (6.7) is simplified and leads to the usual conservation law

$$p_1 + p_2 = q_1 + q_2. \quad (6.8)$$

Equation (6.7) was obtained by considering the simplest second-order diagram. It must be emphasized, however, that this relation is of purely kinematic character and is based on the definition (6.1) of displacements. For higher-order corrections the situation is complicated by the fact that different displacements do not commute. In virtue of certain considerations discussed in [2] effects associated with mass smearing arise; this in turn requires a fundamentally new approach to the description of a state of real particles. These questions require further investigation; however, we can assume that the final results will not be essentially different.

2. Boson-fermion scattering (Fig. 2b). For the sake of simplicity we shall consider the Compton effect: $k^2 = 0$. In this case we have, from (6.2), $l = k/2$. Momenta are related by

$$s = d_0(k/2)\,p = d_0(k'/2)\,p_1 \quad \text{(diagram } b_1\text{)}, \quad (6.9)$$
$$t = d_0(-k'/2)\,p = d_0(-k/2)\,p' \quad \text{(diagram } b_2\text{)}.$$

The momenta of the initial and final fermions satisfy

$$p^2 = p'^2 = m^2. \quad (6.10)$$

[3)]The relation $l^2 = \mu^2$ is not satisfied rigorously in the theory being discussed because of "mass smearing" resulting from the noncommutativity of displacements (Sec. 5 of [2]). However, we shall here neglect the small smearing effect and shall consider these relations as applying to real particles.

Eqs. (6.9) and (6.10) yield the following result. Let $\mathscr{P} = p + k$ be the energy-momentum vector of the system before scattering, and let $\mathscr{P}' = p' + k'$ be the corresponding vector after scattering. We conserve the relativistic invariant

$$E_0^2 = \mathscr{P}^2 = \mathscr{P}'^2. \tag{6.11}$$

E_0 is the total energy in the center-of-mass system. An essential difference of the present case from the usual theory consists in the fact that the center-of-mass system before scattering does not coincide with the center-of-mass system after scattering. The relative velocity of these two reference systems differs for the two diagrams b_1 and b_2 in Fig. 2 and is a second-order small quantity with regard to certain relativistic invariants formed from the vectors p, k, and k′.

7. ANOMALOUS PARTICLE-ANTIPARTICLE INTERACTIONS

A special case of $q = -p$ (6.6) which was excluded from the preceding section corresponds to the focusing property for nonclassical displacements (Sec. 2). Eq. (6.6) can be satisfied only for particle-antiparticle scattering where the two particles have equal momenta; by Feynman's rules the amplitude contains the antiparticle momentum with the opposite sign. It can therefore be expected that in the theory being developed here specific singularities will appear in expressions describing particle-antiparticle interactions in states with equal (or nearly equal) momenta.

On the other hand, the contribution of the focusing property to the interaction of two identical particles is extremely small and leads to no interaction anomaly. This sharp qualitative difference between particle-particle and particle-antiparticle interactions (which like the focusing property itself has no analogue in the ordinary theory) is characteristic of the field theory in a p-space of constant curvature.

We shall therefore consider briefly the formation of bound particle-antiparticle states, which play a fundamental role in the construction of composite elementary particle models. [4-7] In contrast to the kinematic analysis in Sec. 6 the present question is of dynamical character. The quantitative description of the pertinent effects is extremely sensitive to the selected type of particle interaction. Therefore for the purpose of an exact quantitative calculation we must, first of all, refine the form of the interaction operator Λ. This is not included among our present tasks, and we shall confine ourselves to a qualitative analysis, using the simplest four-fermion interaction type (5.6) as an example, and selecting the operator Λ in the form

$$\Lambda = g \int \langle \bar{\psi} | \gamma_5 \hat{d}_N(l) | \psi \rangle \langle \bar{\psi} | \gamma_5 \hat{d}_N(-l) | \psi \rangle \, d\Omega_l. \tag{7.1}$$

In contrast to Sec. 6 the particle momenta do not obey the "reality" condition $p^2 = m^2$; therefore our subsequent examination will be performed in the elliptic space. Using (4.3), (4.4), and (3.8), we easily obtain the matrix element of a normalized displacement:

$$\langle p | \hat{d}_{N0}(l) | q \rangle = \{\delta(l, l_{pq}) + \delta(l, l'_{pq})\}/16(\cos\lambda)^{3/2}. \tag{7.2}$$

The singularity of the matrix element at $\lambda = \pi/2$, which corresponds to the focusing property of displacements, enables us to simplify the determination of particle-antiparticle scattering amplitudes. We shall consider in first approximation the amplitude corresponding to a sum of annihilation diagrams (Fig. 3). It can be shown that in this approximation singularities of the two-particle Green's function are determined by the singularities of the matrix element $\langle l \mid \hat{\Pi} \mid t \rangle$, where the operator $\hat{\Pi}$ is related to the polarization operator $\hat{P}$ by

$$\hat{\Pi} = (1 + g\hat{P})^{-1}. \tag{7.3}$$

In (7.3), $\hat{P}$ denotes the polarization operator corresponding to the simplest fermion loop. In accordance with the general rules in [2], its matrix element is

$$\langle l | \hat{P} | t \rangle = \mathrm{Tr}\,\{(m + \hat{p})^{-1} \gamma_5 \hat{d}_N(l) (m + \hat{p})^{-1} \gamma_5 d_N(-t)\}. \tag{7.4}$$

This matrix element is based largely on (4.6), from which it follows that the operator contains a nondiagonal part [the second term in (4.6)]. The nondiagonal part of $\hat{P}$ leads to "smearing" of the meson mass, thus greatly complicating the analysis. Neglecting the smearing effect, we shall consider only the diagonal part of $\hat{P}$ associated with the first term of (4.6).

Omitting the details of the calculation,[4)] we

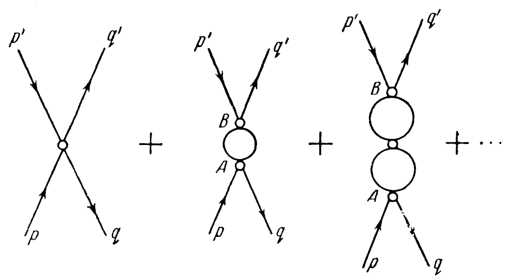

FIG. 3

[4)]The procedure for calculating (7.4), the generalization of Feynman parametrization, and a more consistent solution of the present problem will be treated in a separate paper.

present the final form of the diagonal matrix element:

$$\langle l\,|\,\hat{P}_D\,|\,t\rangle = \pi^2\delta(l,t)\,[\Phi(\cos^2\zeta)-\Phi(\sin^2\zeta)], \qquad (7.5)$$

where we have the function

$$\Phi(z) = z\Big\{-\frac{1}{3}+\frac{1}{2}\int_0^1 \frac{dx}{1-4x(1-x)z}\Big(\frac{1}{\sqrt{1-m^2}\sqrt{1-4x(1-x)z}}$$

$$\times\ \ln\frac{1+\sqrt{1-m^2}\sqrt{1-4x(1-x)z}}{1-\sqrt{1-m^2}\sqrt{1-4x(1-x)z}}-2\Big)\Big\}, \qquad (7.6)$$

and the quantity ζ is related to l by

$$l^2 = \mathrm{tg}^2\,\zeta. \qquad (7.7)^*$$

Because of the singularity of the matrix element (7.2) at the vertices A and B of the diagrams in Fig. 3, we can consider (7.5) only for the values $l = l'_{pq}$, with $q = -p$, given by (2.22). From the latter we obtain

$$\cos^2\zeta = \frac{1}{1+l^2} = \frac{(pn)^2}{(1+p^2)(1+p^2-(pn)^2)} = \frac{p^2\cos^2\alpha}{1+p^2}, \qquad (7.8)$$

where α is the angle between the vectors p and n that is measured at the point p in the metric of the elliptic p-space.

The total momentum of the system is

$$\mathcal{P} = p - q = 2p. \qquad (7.9)$$

Using μ to denote the mass of a bound state ($\mathcal{P}^2 = -\mu^2$) and neglecting μ^2 compared with unity, we obtain from (7.8) and (7.9)

$$\cos^2\zeta = -\tfrac{1}{4}\mu^2\cos^2\alpha. \qquad (7.10)$$

For the existence of a bound state it is required that for some value of μ the expression

$$\Big\langle\{1+g\pi^2[\Phi(-\tfrac{1}{4}\mu^2\cos^2\alpha)-\Phi(1+\tfrac{1}{4}\mu^2\cos^2\alpha)]\}^{-1}\Big\rangle \qquad (7.11)$$

should exhibit a singularity. The angular brackets in (7.11) denote an average over α [over the directions of the vector n in (2.22)]. The explicit form of this average can be determined only through a complete solution of the problem. For the purpose of an estimate we can use the fact that as μ^2 increases the denominator in (7.11) vanishes for the first time when $\cos^2\alpha = 1$. This gives the following equation for μ^2:

$$1+g\pi^2[\Phi(-\mu^2/4)-\Phi(1+\mu^2/4)] = 0. \qquad (7.12)$$

In virtue of $\mu^2 \ll 1$, it can be shown that (7.12) will have a solution for

$$g \approx (\pi^2[\Phi(1)-\Phi(0)])^{-1} \approx 6/\pi^2. \qquad (7.13)$$

The coupling constant g in (7.1) is related to the usual four-fermion interaction constant G by $G = \pi^4 g$. The estimate (7.13) in dimensionless units therefore gives $G \approx 6\pi^2$. By Kadyshevskiĭ's hypothesis [3] we have $G_F \approx 1$ for the coupling constant of weak interactions. Therefore the interaction (7.1) is only 60 times stronger than the universal Fermi interaction, and is thus very weak.

Several authors (in [8], for example) have attempted to calculate bound states of a particle-antiparticle system for a contact interaction within the framework of the ordinary theory. In our view the results of these calculations are accidental to a considerable extent, because they necessarily introduce an arbitrary cutoff of divergent integrals. It is even more significant that the ordinary theory includes no intrinsic requirement of a sharp difference between the characters of particle-particle and particle-antiparticle interactions at small distances.

The situation is entirely different for the field theory in the p-space of constant curvature. First, this theory leads to no divergent expressions, because there is no arbitrariness in the results of the calculation. Secondly, the anomaly in the particle-antiparticle interaction follows from the intrinsic properties of the constant-curvature p-space. In conclusion, it must again be emphasized that the present analysis is only tentative, and that the qualitative results must be confirmed by detailed calculations.

8. CONCLUDING REMARKS

1. It has been shown in Sec. 6 that in the theory developed here the law of energy-momentum conservation is only approximate. However, in the extremely important case of elastic scattering this law is satisfied exactly [Eq. (6.8)] or in any event in the first nonvanishing approximation. An important consequence of general character follows therefrom for the theory of macroscopic bodies. Since in the theory of such bodies the only elementary interactions between particles involve Coulomb and gravitational forces, it can be expected that this theory will not be essentially changed. This eliminates to a considerable degree the general objections raised against the field theory in constant-curvature p-space which are based on the difficulties in treating macroscopic bodies within this theory.

2. The conclusions in Sec. 7 regarding the possibility of forming bound states in a baryon-antibaryon (meson) system with a small coupling constant provide a basis for expecting that the newly developed theory will require no special mechanism of strong interactions. We thus have the attractive prospect of basing the theory of elementary particles solely on universal weak and

*tg = tan.

electromagnetic interactions. In this picture strong interactions will be "secondary" effects associated with the formation of mesons as composite particles.

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