THE PROBLEM OF PAIRINGS WITH NONZERO ANGULAR MOMENTUM IN A FERMI SYSTEM

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The properties of a Fermi system with an interaction in the case of pairs of nonzero angular momentum are investigated. A decoupling of the three-particle Green's function is proposed which allows one to obtain solutions both with an isotropic and an anisotropic gap. It is shown that the solution with an anisotropic gap is asymptotically exact. A discussion is given of the difficulties arising in obtaining a basis for solutions with an isotropic gap.

1. INTRODUCTION

THE problem of pairings with nonzero angular momentum in a Fermi system has been repeatedly discussed recently [1-10] in connection with the problem of the superfluidity of He^3. The author has shown [11] that production of Cooper pairs in the triplet state with a total component of spin equal to zero is possible in antiferromagnetic substances as a result of the interaction between conduction electrons induced by spin waves. Possibly such a mechanism is responsible for the occurrence of superconductivity in ruthenium and osmium, for which no isotopic effect has been found [12,13].

The theory of Fermi systems with an interaction in the case of pairs of nonzero angular momentum is developed on the basis of the reduced BCS Hamiltonian [14]:

$$H_{red} = \sum_{\mathbf{p},\sigma} \zeta_p a^+_{\mathbf{p}\sigma} a_{\mathbf{p}\sigma} + \frac{1}{2V} \sum_{\mathbf{p},\mathbf{p}',\sigma} V(\mathbf{p}, \mathbf{p}')\, a^+_{\mathbf{p}'\sigma} a^+_{-\mathbf{p}',-\sigma} a_{\mathbf{p}\sigma} a_{-\mathbf{p},-\sigma} \quad (1)$$

Here ζ_p is the kinetic energy of a fermion of momentum $\mathbf{p}$ referred to the Fermi energy, V is the volume of the system, $a_{\mathbf{p}\sigma}$, $a^+_{\mathbf{p}\sigma}$ are the creation and annihilation operators for a fermion of momentum $\mathbf{p}$ and spin component σ. The "potential" of the interaction $V(\mathbf{p},\mathbf{p}')$ in the isotropic model depends only on the scattering angle: $V(\mathbf{p},\mathbf{p}') = V(\mathbf{nn}')$, where $\mathbf{n} = \mathbf{p}/p$, and differs from zero in an energy shell of thickness 2ω near the Fermi surface.

For the investigation of the ground state of the system with the "reduced" interaction Gor'kov and Galitskiĭ [3] have generalized the technique of decoupling the chain of equations for the Green's function developed by Gor'kov [15] to the case of condensation of Cooper pairs with nonzero relative angular momentum. In this paper we propose a solution with an isotropic gap. Other authors [1,2,6-10] assume that in the ground state the gap is anisotropic in spite of the fact that the original Hamiltonian is isotropic. In particular, Anderson and Morel have obtained this result with the aid of a variational principle by minimizing the average value of the energy over states of the Bardeen type:

$$\Phi = \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}} a^+_{\mathbf{p}} a^+_{-\mathbf{p}})\, \Phi_0,$$

where Φ_0 is the "vacuum" amplitude.

In the next section we shall utilize a technique of calculation which differs inappreciably from the technique of Gor'kov and Galitskiĭ [3], and we shall obtain, by decoupling the chain of equations for the Green's functions, solutions both with an isotropic and with an anisotropic gap. In Sec. 3 it is shown that the anisotropic solution is asymptotically exact for $V \to \infty$. We shall also describe the difficulties arising in attempting to find a basis for the isotropic solutions.

2. STATES WITH ISOTROPIC AND ANISOTROPIC GAP

The energy spectrum of the system is determined by poles of the Fourier transform of the Green's function

$$G(\mathbf{p}, t-t') = -i\langle T a_{\mathbf{p}\sigma}(t)\, a^+_{\mathbf{p}\sigma}(t')\rangle \equiv \langle\langle a_{\mathbf{p}\sigma}(t);\, a^+_{\mathbf{p}\sigma}(t')\rangle\rangle, \quad (2)$$

where the angle brackets denote averaging over a Gibbs ensemble.

The equation of motion for the function $G(\mathbf{p}, t-t')$ has the form

$$i\frac{d}{dt}G(\mathbf{p},t-t') = \delta(t-t') + \zeta_p G(\mathbf{p},t-t') - \frac{1}{V}\sum_{\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\Gamma(\mathbf{p},\mathbf{p}',t-t'), \quad (3)$$

where Γ is the two-particle Green's function:

$$\Gamma(\mathbf{p},\mathbf{p}',t-t') = \langle\langle a^+_{-\mathbf{p},-\sigma}(t)\, a_{-\mathbf{p}',-\sigma}(t)\, a_{\mathbf{p}'\sigma}(t);\, a^+_{\mathbf{p}\sigma}(t')\rangle\rangle. \quad (4)$$

We set up the equation of motion for the function Γ:

$$i\frac{d}{dt}\Gamma(\mathbf{p},\mathbf{p}',t-t') = -\zeta_p\Gamma(\mathbf{p},\mathbf{p}',t-t') - \frac{1}{V}\sum_{\mathbf{p}''} V(\mathbf{p},\mathbf{p}'')\langle\langle a_{\mathbf{p}\sigma}a^+_{\mathbf{p}''\sigma}a^+_{-\mathbf{p}'',-\sigma}a_{-\mathbf{p}',-\sigma}a_{\mathbf{p}',\sigma};\, a^+_{\mathbf{p}\sigma}(t')\rangle\rangle + \langle\langle a^+_{-\mathbf{p},-\sigma}\, i\frac{d}{dt}(a_{-\mathbf{p}',-\sigma}a_{\mathbf{p}'\sigma});\, a^+_{\mathbf{p}\sigma}(t')\rangle\rangle. \quad (5)$$

We shall achieve the simplest interpolation decoupling of three-particle Green's functions appearing in (5) by pairing the operators $a_{\mathbf{p}\sigma}$ and $a^+_{\mathbf{p}\sigma}$. In order to simplify the resulting system of equations we impose the additional condition

$$\langle\langle a^+_{-\mathbf{p},-\sigma}\, i\frac{d}{dt}(a_{-\mathbf{p}'-\sigma}a_{\mathbf{p}'\sigma});\, a^+_{\mathbf{p}\sigma}(t')\rangle\rangle = 0. \quad (6)$$

The solutions which we shall obtain below satisfy this condition. Thus, we shall obtain a closed system of equations for $G(\mathbf{p}, t-t')$ and $\Gamma(\mathbf{p},\mathbf{p}', t-t')$ which on going over to Fourier components assume the form

$$(\omega-\zeta_p)\,G(\mathbf{p},\omega) + \frac{1}{(2\pi)^3}\int d^3\mathbf{p}'V(\mathbf{p},\mathbf{p}')\,\Gamma(\mathbf{p},\mathbf{p}',\omega) = 1, \quad (7)$$

$$(\omega+\zeta_p)\,\Gamma(\mathbf{p},\mathbf{p}',\omega) = \frac{i}{(2\pi)^4}G(\mathbf{p},\omega)\int d^3\mathbf{p}''d\omega V(\mathbf{p},\mathbf{p}'')\,\Gamma(\mathbf{p}'',\mathbf{p}',\omega). \quad (8)$$

A. States with isotropic gap. By taking into account the symmetry of the desired solution we shall assume $\Gamma(\mathbf{p},\mathbf{p}',\omega)$ to be a function of ζ, ζ', ω and of the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$. We expand $V(\mathbf{p},\mathbf{p}')$ and $\Gamma(\mathbf{p},\mathbf{p}',\omega)$ in a series of spherical harmonics:

$$V(\mathbf{p},\mathbf{p}') = \sum_l (2l+1)V_lP_l(\mathbf{nn}') = 4\pi\sum_{l,m}V_lY_{lm}(\mathbf{n})\,Y^*_{lm}(\mathbf{n}'), \quad (9)$$

$$\Gamma(\mathbf{p},\mathbf{p}',\omega) = \sum_l \Gamma_l(\zeta,\zeta',\omega)\,P_l(\mathbf{nn}') = 4\pi\sum_{l,m}\frac{1}{2l+1}\Gamma_l(\zeta,\zeta',\omega)\,Y_{lm}(\mathbf{n})\,Y^*_{lm}(\mathbf{n}'); \quad (10)$$

$P_l(\mathbf{nn}')$ are the Legendre polynomials. On substituting these expressions into (7) and (8), we shall obtain equations for $G(\mathbf{p},\omega)$ and for the coefficients in the expansion of the function $\Gamma(\mathbf{p},\mathbf{p}',\omega)$:

$$(\omega-\zeta)\,G(p,\omega) + \sum_l \rho_l\int d\zeta'\Gamma_l(\zeta,\zeta',\omega) = 1, \quad (11)$$

$$(\omega+\zeta)\,\Gamma_l(\zeta,\zeta',\omega) = \frac{i}{2\pi}G(p,\omega)\,\rho_l\int d\zeta''d\omega\Gamma_l(\zeta'',\zeta',\omega). \quad (12)$$

Here we have introduced the notation $\rho_l = V_lp_0^2/2\pi^2v_0$, where p_0, v_0 are the Fermi momentum and velocity.

The function $G(\mathbf{p},\omega)$ whose poles determine the spectrum of single particle excitations can be easily found:

$$G(p,\omega) = \frac{\omega+\zeta_p}{\omega^2-\varepsilon_p^2} + i\frac{\pi}{2}\,\mathrm{th}\,\frac{\beta\varepsilon_p}{2}\left\{\left(1-\frac{\zeta_p}{\varepsilon_p}\right)\delta(\omega+\varepsilon_p) - \left(1+\frac{\zeta_p}{\varepsilon_p}\right)\delta(\omega-\varepsilon_p)\right\}, \quad (13)^*$$

the imaginary part is determined in accordance with Landau's theorem [16], $\epsilon_p = (\zeta_p^2+\Delta^2)^{1/2}$ is the energy of an elementary excitation, β is the reciprocal temperature, and

$$\Delta^2 = -\frac{i}{2\pi}\sum_l\rho_l^2\int\Gamma_l(\zeta,\zeta',\omega)\,d\zeta d\zeta' d\omega. \quad (14)$$

On dividing Eq. (12) by $\omega+\zeta$ and on integrating over ζ, ζ', ω, we obtain

$$A_l\left(1-\rho_l\int_0^{\omega}d\zeta\frac{1}{\varepsilon}\,\mathrm{th}\,\frac{\beta\varepsilon}{2}\right) = 0, \quad (15)$$

$$A_l = \int\Gamma_l(\zeta,\zeta',\omega)\,d\zeta d\zeta' d\omega. \quad (16)$$

The system (15) admits the trivial solution $A_l = 0$ for all l corresponding to the normal state. There also exists a number of nontrivial solutions determined by equations

$$1 = \rho_{l_0}\int_0^{\omega}d\zeta\frac{1}{\varepsilon}\,\mathrm{th}\,\frac{\beta\varepsilon}{2}, \qquad \rho_{l_0}>0,$$
$$A_l = 0 \ \text{for}\ l\neq l_0. \quad (17)$$

In particular, at the temperature $T = 0$ we obtain from these equations

$$\Delta(0) = 2\omega\exp(-1/\rho_{l_0}), \quad (18)$$

which coincides exactly with the solution of Gor'kov and Galitskiĭ [3]. The maximum gap Δ, and consequently, the thermodynamically most advantageous state corresponds to maximum ρ_l. The ratio of the critical temperature T_c to $\Delta(0)$ is the same as in the case of a pairing with $l = 0$: $T_c = 1.75\times\Delta(0)$.

So far we have not used the additional condition (6) which can be written in the form

$$2\varepsilon'\Gamma_l(\zeta,\zeta',\omega) = \rho_l\,\mathrm{th}\,\frac{\beta\varepsilon'}{2}\int d\zeta''\Gamma_l(\zeta,\zeta'',\omega). \quad (19)$$

*th = tanh.

It can be easily shown that equations (11), (12), (19), can be satisfied simultaneously[1)] by setting all the Γ_l except Γ_{l_0} equal to zero, and

$$\Gamma_{l_0}(\zeta, \zeta', \omega) = \frac{i}{2\pi} F_{l_0}(\zeta, \omega) \int F^*_{l_0}(\zeta', \omega)\, d\omega$$

$$= iF_{l_0}(\zeta, \omega)\, F^*_{l_0}(\zeta'), \tag{20}$$

where

$$F_{l_0}(\zeta, \omega) = \frac{i\Delta}{\omega + \zeta} G(p, \omega). \tag{21}$$

A physical interpretation of the solution obtained was proposed by Gor'kov and Galitskiĭ[3]. The decoupling of the function $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ adopted by us means that the operator $\psi^+\psi^+$ can create a pair of relative angular momentum l, while the component of the angular momentum m can be arbitrary. Averages of four Fermi operators can be represented in the present scheme in the following manner:

$$\langle \psi\psi\psi^+\psi^+ \rangle$$

$$= \sum_m \langle N | \psi\psi | N+2, l, m \rangle \langle N+2, l, m | \psi^+\psi^+ | N \rangle, \tag{22}$$

where the state $|N+2, l, m\rangle$ is obtained from the state $|N\rangle$ of a system of N particles by the addition of a pair of relative angular momentum l and component m.

B. States with anisotropic gap. Anisotropic solutions can be obtained if we assume that the function $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ can be factorized:

$$\Gamma(\mathbf{p}, \mathbf{p}', \omega) = iF(\mathbf{p}, \omega)\, F^*(\mathbf{p}'), \tag{23}$$

where

$$F(\mathbf{p}) = \frac{1}{2\pi} \int d\omega\, F(\mathbf{p}, \omega). \tag{24}$$

From relation (8) one can easily show that

$$(\omega + \zeta_p)\, F(\mathbf{p}, \omega) = \frac{i}{(2\pi)^4} G(\mathbf{p}, \omega) \int d^3\mathbf{p}'\, d\omega V(\mathbf{p}, \mathbf{p}')\, F(\mathbf{p}', \omega). \tag{25}$$

The function $G(\mathbf{p}, \omega)$ is of the form (13) where, however, $\epsilon_p = (\zeta_p^2 + |\Delta(\mathbf{p})|^2)^{1/2}$, while the function $\Delta(\mathbf{p})$ can be expressed in terms of $F(\mathbf{p}, \omega)$ in the following manner:

$$\Delta(\mathbf{p}) = \frac{1}{(2\pi)^4} \int d^3\mathbf{p}'\, d\omega V(\mathbf{p}, \mathbf{p}')\, F(\mathbf{p}', \omega). \tag{26}$$

By eliminating the function $F(\mathbf{p}, \omega)$ we obtain the equation for the energy gap $\Delta(\mathbf{p})$:

$$\Delta(\mathbf{p}) = \frac{\pi}{(2\pi)^4} \int d^3\mathbf{p}' V(\mathbf{p}, \mathbf{p}') \frac{\Delta(\mathbf{p}')}{\varepsilon(\mathbf{p}')} \operatorname{th} \frac{\beta\varepsilon_{p'}}{2}, \tag{27}$$

which, gnerally speaking, is complex [a physical meaning can be ascribed to the absolute value of the function $\Delta(\mathbf{p})$]. The energy gap $\Delta(\mathbf{p})$ depends only on the direction of the vector $\mathbf{p}$: $\Delta(\mathbf{p}) = \Delta(\mathbf{n})$. On expanding $\Delta(\mathbf{n})$ in terms of spherical harmonics:

$$\Delta(\mathbf{n}) = \sum_{l, m} \Delta_{lm} Y_{lm}(\mathbf{n}), \tag{28}$$

we shall obtain for the expansion coefficients Δ_{lm} a system of transcendental equations:

$$\Delta_{lm} = \rho_l \sum_{l', m'} \Delta_{l'm'} \int_0^\omega d\zeta' \int d\mathbf{n}'$$

$$\times \frac{Y_{l'm'}(\mathbf{n}')\, Y^*_{lm}(\mathbf{n}')}{\sqrt{\zeta'^2 + |\Delta(\mathbf{n}')|^2}} \operatorname{th} \frac{\beta\sqrt{\zeta'^2 + |\Delta(\mathbf{n}')|^2}}{2}. \tag{29}$$

The additional condition (6) can be rewritten in the following manner:

$$2\zeta'\Gamma(\mathbf{p}, \mathbf{p}', \omega) + \frac{1}{(2\pi)^3} \int d^3\mathbf{p}'' V(\mathbf{p}', \mathbf{p}'')$$

$$\times \{n(\mathbf{p}')\, \Gamma(\mathbf{p}, -\mathbf{p}'', \omega)$$

$$- [1 - n(-\mathbf{p}')]\, \Gamma(\mathbf{p}, \mathbf{p}'', \omega)\} = 0, \tag{30}$$

where $n(\mathbf{p}) = \langle a^+_{\mathbf{p}\sigma} a_{\mathbf{p}\sigma} \rangle$.

The anisotropic solution obtained by us satisfies Eq. (30) only if the function $F(\mathbf{p}, \omega)$, and consequently also $\Delta(\mathbf{p})$, is even (or odd)[2)] with

$$F(\mathbf{p}, \omega) = \pm F(-\mathbf{p}, \omega) = \frac{i\Delta(\mathbf{p})}{\omega + \zeta_p} G(\mathbf{p}, \omega). \tag{31}$$

This means that among the solutions of the equation for $\Delta(\mathbf{p})$ only those are acceptable in which $\Delta(\mathbf{p})$ is expanded in terms of spherical harmonics with only even values of l (singlet state) or only with odd values of l (triplet state).

It is of interest to note that the critical temperature T_c, determined from the condition that the function $\Delta(\mathbf{p})$ vanishes identically, satisfies the same equation both in the case when the gap is isotropic and in the case when the gap is anisotropic. This equation is of the form

$$1 = \rho_{l_0} \int_0^\omega \frac{d\zeta}{\zeta} \operatorname{th} \frac{\zeta}{2T_c}, \tag{32}$$

where ρ_{l_0} is the maximum positive value of ρ_l. The critical temperature is equal to

$$T_c = 3.5\, \omega \exp(-1/\rho_{l_0}). \tag{33}$$

Thus, if in the interaction $V(\mathbf{p}, \mathbf{p}')$ at least one

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[1)]Generally speaking, there is no need to give such a proof, since the system of equations for a pairing of angular momentum $l = l_0$ is formally the same as in the case $l = 0$, when the solution is well known to exist.

[2)]Incidentally, such symmetry properties of the function $\Delta(\mathbf{p})$ are obvious.

of the quantities V_l is positive, then at sufficiently low temperatures the system will exhibit the property of superfluidity (or superconductivity). A positive sign of V_l denotes attraction of a pair of fermions in a state of angular momentum l.

3. ASYMPTOTIC EXACTNESS OF SOLUTIONS IN THE CASE OF THE REDUCED HAMILTONIAN

The problem of the symmetry properties of the state of thermodynamic equilibrium of a system with interaction in pairs of angular momentum different from zero is at the present moment controversial [1-3,6-10]. The solution with an isotropic gap proposed by Gor'kov and Galitskiĭ [3] cannot be obtained, for example, by the variational method, since the structure of the wave function of the isotropic ground state, if the latter exists, is very complicated. In the paper by Hone [7] it is shown that it is very difficult to construct a chain of many-particle isotropic Green's functions satisfying the equations of motion obtained with the aid of the reduced Hamiltonian. However, we note that such objections are not valid as proofs. At the same time it is asserted that if in the reduced Hamiltonian one neglects the kinetic energy, then the method of Anderson and Morel gives the correct value for the energy of the ground state [8]. We shall show in this section that solutions with an anisotropic gap are asymptotically exact, and we shall describe the difficulties arising in efforts of finding a basis for the isotropic solutions.[3)]

In the preceding section by decoupling the three-particle Green's function we obtained a closed system of equations for the one- and two-particle Green's functions. We now turn to the solution of an infinite chain of equations for the Green's functions.

Following Bogolyubov [17,18] we shall consider the temperature Green's functions

$$G_{\mathfrak{M}}(\mathbf{p}, t-t') = \left\langle\!\left\langle a_{\mathbf{p}\sigma}(t)\,\mathfrak{M}(t);\ a^{+}_{\mathbf{p}',\sigma}(t')\right\rangle\!\right\rangle, \tag{34}$$

$$\Gamma_{\mathfrak{N}}(\mathbf{p}, \mathbf{p}', t-t') = \left\langle\!\left\langle a^{+}_{-\mathbf{p},-\sigma}(t)\ a_{-\mathbf{p}',-\sigma}(t)\ a^{+}_{\mathbf{p}',\sigma}(t)\ \mathfrak{N}(t);\ a^{+}_{\mathbf{p}\sigma}(t')\right\rangle\!\right\rangle, \tag{35}$$

where $\mathfrak{M}$, $\mathfrak{N}$ are products of the form

$$\ldots a^{+}_{\mathbf{q}\sigma_1}a_{\mathbf{q}\sigma_1}\ldots a^{+}_{\mathbf{g}\sigma_2}a^{+}_{-\mathbf{g},-\sigma_2}a_{\mathbf{h}\sigma_3}a_{-\mathbf{h},-\sigma_3}\ldots,$$

where all the operators appearing in $G_{\mathfrak{M}}$ and $\Gamma_{\mathfrak{N}}$, are different.

We construct equations of motion for these functions:

$$i\frac{d}{dt}G_{\mathfrak{M}}(\mathbf{p}, t-t') = \delta(t-t')\langle\mathfrak{M}\rangle + \zeta_p G_{\mathfrak{M}}(\mathbf{p}, t-t') + \frac{1}{V}\sum_{\mathbf{p}'}V(\mathbf{p},\mathbf{p}')\left\langle\!\left\langle a^{+}_{-\mathbf{p},-\sigma}a_{\mathbf{p}'\sigma}a_{-\mathbf{p}',-\sigma}\mathfrak{M};\ a^{+}_{\mathbf{p}\sigma}\right\rangle\!\right\rangle + \left\langle\!\left\langle a_{\mathbf{p}\sigma}i\frac{d\mathfrak{M}}{dt};\ a^{+}_{\mathbf{p}\sigma}(t')\right\rangle\!\right\rangle, \tag{36}$$

$$\begin{aligned} i\frac{d}{dt}\Gamma_{\mathfrak{N}}(\mathbf{p},\mathbf{p}', t-t') = &-\zeta_p\Gamma_{\mathfrak{N}}(\mathbf{p},\mathbf{p}', t-t') - \frac{1}{V}\sum_{\mathbf{p}''}V(\mathbf{p},\mathbf{p}'')\left\langle\!\left\langle a_{\mathbf{p}\sigma}a^{+}_{\mathbf{p}''\sigma}a^{+}_{-\mathbf{p}'',-\sigma}a_{-\mathbf{p}',-\sigma}a_{\mathbf{p}'\sigma}\mathfrak{N};\ a^{+}_{\mathbf{p}\sigma}(t')\right\rangle\!\right\rangle \\ &+ \left\langle\!\left\langle a^{+}_{-\mathbf{p},-\sigma}i\frac{d}{dt}(a_{-\mathbf{p}',-\sigma}a_{\mathbf{p}'\sigma})\,\mathfrak{N};\ a^{+}_{\mathbf{p}\sigma}(t')\right\rangle\!\right\rangle \\ &+ \left\langle\!\left\langle a^{+}_{-\mathbf{p},-\sigma}a_{-\mathbf{p}',-\sigma}a_{\mathbf{p}'\sigma}i\frac{d\mathfrak{N}}{dt};\ a^{+}_{\mathbf{p}\sigma}(t')\right\rangle\!\right\rangle. \end{aligned} \tag{37}$$

In Eqs. (36) and (37) we can, preserving an accuracy up to terms of order $1/N$ (N is the number of particles in the system, $N/V = \text{const}$), neglect terms in which the same operator is repeated several times. Then the chain of equations (36) and (37) will be closed even though it is infinite.

We shall attempt to obtain an asymptotically exact solution of our chain of equations in the form

$$G_{\mathfrak{M}}(\mathbf{p}, t-t') = \langle\mathfrak{M}\rangle\, G(\mathbf{p}, t-t'), \tag{38}$$

$$\Gamma_{\mathfrak{N}}(\mathbf{p},\mathbf{p}', t-t') = \langle\mathfrak{N}\rangle\, \Gamma(\mathbf{p},\mathbf{p}', t-t'). \tag{39}$$

Since the averages

$$\langle id\mathfrak{M}/dt\rangle = \langle id\mathfrak{N}/dt\rangle = 0,$$

the chain of equations (36) and (37) reduces to the pair of equations for $G(\mathbf{p}, t-t')$ and $\Gamma(\mathbf{p},\mathbf{p}',t-t')$ which on going over to Fourier components assume the form (7) and (8) [taking into account the additional conditions (6)]. Thus, the solutions obtained in the preceding section satisfy the chain of asymptotically exact equations for the Green's function.

The decoupling of (38) and (39) is justified in the case of the solution with an anisotropic gap since in this case all the many-particle Green's functions under consideration can be factorized and, in particular, $\Gamma(\mathbf{p},\mathbf{p}',\omega) = iF(\mathbf{p},\omega)F^{*}(\mathbf{p}')$. This shows that the properties of a superconducting system with an anisotropic gap can be described by means of a quadratic approximating Hamiltonian of the type of Hamiltonian proposed by Bogolyubov [17,18]:

$$H = \sum_{\mathbf{p},\sigma}\{\zeta_p a^{+}_{\mathbf{p}\sigma}a_{\mathbf{p}\sigma} + \sigma|\Delta(\mathbf{p})|(a^{+}_{\mathbf{p}\sigma}a^{+}_{-\mathbf{p},-\sigma} + a_{-\mathbf{p},-\sigma}a_{\mathbf{p}\sigma})\}, \tag{40}$$

which can be diagonalized with the aid of a canonical transformation.[4)] The approximating Hamil-

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[3)]The asymptotic exactness of the BCS solution was first proved by Bogolyubov.[17,18] Recently the same result was also obtained by Klein [19] and by Haag.[20]

[4)]Solutions with an anisotropic gap can be obtained in the case of the Hamiltonian H_{red} with the aid of Bogolyubov's tonian, however, is not suitable for obtaining solutions with an isotropic gap ($l \neq 0$), since averages of the type $\langle a_{\mathbf{p}} a_{-\mathbf{p}} a^{+}_{-\mathbf{p}'} a^{+}_{\mathbf{p}'} \rangle$ on being averaged using the Hamiltonian (40) would have the structure of products $F(\mathbf{p}) F^{*}(\mathbf{p}')$ which in the scheme of Gor'kov and Galitskiĭ would denote pairing in an S-state.

However, if we seek such a system of Green's functions that

$$\Gamma(\mathbf{p}, \mathbf{p}', \omega) = \Gamma_l(\zeta, \zeta', \omega) P_l(\mathbf{nn}'),$$

then the decoupling of (38) and (39) is unacceptable, since the symmetry of many-particle Green's functions is violated. For example, in the case of such decoupling averages of the type

$$\langle a^{+}_{\mathbf{p}_1} a^{+}_{-\mathbf{p}_1} a_{\mathbf{p}'_1} a_{-\mathbf{p}'_1} a^{+}_{\mathbf{p}_2} a^{+}_{-\mathbf{p}_2} a_{\mathbf{p}'_2} a_{-\mathbf{p}'_2} \rangle$$

turn out to be proportional to $P_l(\mathbf{n}_1\mathbf{n}'_1) P_l(\mathbf{n}_2\mathbf{n}'_2)$ and are altered by the interchange of arguments $\mathbf{n}_1 \rightleftarrows \mathbf{n}_2$, i.e., the required symmetry is absent. Therefore, one should seek other methods of laying a basis for the isotropic solution.

In the preceding section we decoupled the three-particle Green's function. One can proceed somewhat further in this direction. If we restrict ourselves to a discussion of Green's functions in which products of the type $a^{+}_{\mathbf{g}} a^{+}_{-\mathbf{g}} a_{\mathbf{f}} a_{-\mathbf{f}}$ occur not more than twice, while the number of products of the type $a^{+}_{\mathbf{p}} a_{\mathbf{p}}$ can be arbitrary, then one can pair the operators $a^{+}_{\mathbf{p}\sigma}$ and $a_{\mathbf{p}\sigma}$, by assuming that the four-particle Green's function is of the following form:

$$\langle\langle a^{+}_{-\mathbf{p}, -\sigma} a_{-\mathbf{p}', -\sigma} a_{\mathbf{p}'\sigma} a^{+}_{-\mathbf{p}', -\sigma} a_{-\mathbf{p}'_1, -\sigma} a_{\mathbf{p}'_1\sigma} a^{+}_{\mathbf{p}_1\sigma};\ a^{+}_{\mathbf{p}\sigma}(t') \rangle\rangle$$
$$= -iF_l(\zeta, t - t') F^{*}_l(\zeta') F_l(\zeta_1) F^{*}_l(\zeta'_1) [P_l(\mathbf{nn}') P_l(\mathbf{n}_1\mathbf{n}'_1)$$
$$+ P_l(\mathbf{nn}'_1) P_l(\mathbf{n}_1\mathbf{n}') - P_l(\mathbf{nn}_1) P_l(\mathbf{n}_1\mathbf{n}'_1)]. \qquad (41)$$

The functions $F_l(\zeta, \omega)$ and $F_l(\zeta)$ are defined by relations (20) and (21) of the preceding section. The expression (41) possesses the necessary symmetry.

In the third term of the right hand side of Eq. (36) as a result of integrating over $\mathbf{p}'$ using the weighting function $V(\mathbf{p}, \mathbf{p}')$ the dependence on $\mathbf{n}$ disappears in virtue of the relation

$$\int dn' P_l(\mathbf{nn}') P_l(\mathbf{n}_1\mathbf{n}') = \frac{4\pi}{2l+1} P_l(\mathbf{nn}_1), \qquad (42)$$

canonical transformation. The present author[11] has obtained the Hamiltonian for the interaction of conduction electrons of an antiferromagnetic substance with spin waves. By applying the method of canonical transformation to this Hamiltonian one can also obtain the energy spectrum for a system with an anisotropic gap.

and the integral turns out to be proportional to $\langle \mathfrak{M} \rangle$, and this leads to the equation for $G(\mathbf{p}, \omega)$ discussed in the preceding section in the case of an isotropic gap. A similar situation holds also for Eq. (37).

Unfortunately, we cannot demonstrate a decoupling of the functions $G_{\mathfrak{M}}$ and $\Gamma_{\mathfrak{N}}$ containing a greater number of operators of the type $a^{+}_{\mathbf{g}} a^{+}_{-\mathbf{g}} a_{\mathbf{g}'} a_{-\mathbf{g}'}$ and the question of whether there exist isotropic solutions of the infinite chain of equations remains open. It is possible that the scheme of Gor'kov and Galitskiĭ is an approximate one even in the case of the reduced interaction.

Thus, at the present moment there have been proposed a number of solutions of the problem of pairings with nonzero angular momentum. For a given interaction $V(\mathbf{p}, \mathbf{p}')$ only one solution is acceptable. The problem of the choice of this solution is very complicated. The system (29) can be solved in the general case only by numerical methods. Vaks, Galitskiĭ, and Larkin[10] have succeeded in showing that certain particular anisotropic solutions are unstable. It is possible that an experimental investigation of superconducting ruthenium and osmium will clarify the situation.

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