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THE 3π *DECAY MODE OF THE* K^+ *MESON AND THE* $\pi\pi$ *INTERACTION*

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Experimental data for the 3π decay mode of the K^+ meson are compared with formulas obtained on the assumption that the $\Delta T = 1/2$ rule holds and that $a_0, a_2 \lesssim 1$ (a_0 and a_2 are the s-wave scattering lengths of pions in states with isotopic spin 0 and 2, in units of $\hbar/\mu c$). It is shown that these formulas agree with experiment if a_0 and a_2 satisfy the relations $|a_0 a_2| \lesssim 0.25$, $a_0^2 - a_0 a_2 + 0.5 a_2^2 \sim 0.7 \pm 1$. No more precise information about a_0 and a_2 can be obtained from the presently available experimental data. An integral equation for the decay amplitude is written without the assumption that a_0 and a_2 are small. This integral equation has a unique solution and can be solved numerically.

THE attention of many authors has been attracted to the $K \to 3\pi$ decay reactions in the hope of obtaining information on the $\pi\pi$ scattering amplitudes.

In the work of Thomas and Holladay[1] and Mitra[2] a strong interaction was assumed between the produced pions in the s state with isotopic spin T = 2. The effect of this interaction was evaluated by means of the Watson-Migdal formalism.[3] However it will be seen from the analysis below that the effect of a strong $\pi\pi$ interaction in the T = 2 state does not reduce to just the appearance of the multiplicative factor $(1 - ika_2)^{-1}$ (k being the relative momentum of the pair of particles).

Dispersion relations for the $K \to 3\pi$ reaction, in which the interaction of pions produced in s states was taken into account, were written by Khuri and Treiman[4] and Sawyer and Wali.[5] The solution of the resultant equations was expressed in the form of a power series in the momenta of the produced particles. This power series expansion, however, was obtained incorrectly—a number of terms which contribute substantially in the approximation considered was left out. We shall discuss the work of Khuri and Treiman[4] in more detail in Sec. 2 in connection with the integral equation for the K-decay amplitude.

In the papers[6-9] the amplitude for the $K^+ \to 3\pi$ decay was expanded in a power series in the momenta of the relative motion of the pions. The expansion was carried up to terms cubic in the momenta. The parameters in the expansion were the quantities $a_0 k_{il}$ and $a_2 k_{il}$. The expansion is valid if a_0 and a_2 are sufficiently small: $a_0, a_2 \lesssim 1$. The decay amplitude contains, in addition to the terms arising from expansion in the quantities $a_T k_{il}$, also quadratic terms αk_{il}^2 where the α are undetermined constants not expressible in terms of the a_T (the effect of only this type of terms was considered by Alles[10]). In the papers[7-9] the rule $\Delta T = 1/2$ is not invoked and consequently the produced mesons are in a superposition of states with isotopic spin 1, 2, and 3.

In the present paper a comparison is carried out of the formulas for the probabilities of the $K^+ \to 3\pi$ decays[7-9] with the experimental data[11] on the assumption that the rule $\Delta T = 1/2$ holds. The formulas for the $K^+ \to 3\pi$ decays derived on the assumption that a_0 and $a_2 \lesssim 1$ are qualitatively in agreement with the experimental data. However, owing to the large experimental errors it is not yet possible to draw conclusions about the magnitudes of a_0 and a_2. From experiment it follows only that

$$|a_0 a_2| \lesssim 0.25, \quad a_0^2 - a_0 a_2 + 0.5\, a_2^2 \sim 0.7 \pm 1.$$

In Sec. 2 we write a dispersion relation for the K-decay amplitude without assuming a_0 and a_2 to be small (this is necessary in view of the appearance of experiments indicating that a_0 is large[12]). This relation can be reduced to an integral equation for the decay amplitude of the Skornyakov-Ter-Martirosyan type.[13] Such equations were studied by Danilov.[14] The equation here obtained has a unique solution and can be solved numerically.

1. COMPARISON WITH EXPERIMENT OF THE FORMULAS FOR THE $K^+ \to 3\pi$ DECAY OBTAINED ON THE ASSUMPTION THAT a_0 AND a_2 ARE SMALL ($a_0, a_2 \lesssim 1$)

An expression for the probabilities of the $K^+ \to 3\pi$ decay accurate to terms quadratic in the momenta was obtained by Gribov.[8] If it is assumed that the K^+ decay satisfies the $\Delta T = 1/2$ rule then the produced mesons can be only in states of total isotopic spin equal to unity. In order to obtain expressions for cross sections in this case it is necessary to set the parameter ρ that appears in Gribov's work[8] equal to $-1/2$ (the minus sign arises from the definition of the charge exchange amplitude for the pions: $a_e = 2/3\,(a_2 - a_0)$, where $a_T = k^{-1}e^{i\delta_T}\sin\delta_T$ for $k \to 0$). In addition, the undetermined constants $\alpha_\pm$ appearing in the expressions for the probabilities will no longer be independent.

Let us rewrite for this case the expressions for the probabilities of the decays $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$:

$$dW/d\Gamma = 4\lambda^2\{1 + \beta_1[k_{12}(k_{13} + k_{23}) + 2J(k_{12}) + J(k_{13}) + J(k_{23})] + \beta_2[k_{13}k_{23} + J(k_{13}) + J(k_{23})] + \beta_3[J(k_{13}) + J(k_{23})] + 2\alpha k_{12}^2 + [\alpha + \delta - \tfrac{25}{36}(a_0 - a_2)^2](k_{13}^2 + k_{23}^2)\}, \quad (1a)$$

$$dW'/d\Gamma = \lambda^2\{1 + \gamma_1[k_{12}(k_{13} + k_{23}) + 2J(k_{12}) + J(k_{13}) + J(k_{23})] + \gamma_2[k_{13}k_{23} + J(k_{13}) + J(k_{23})] + 2\gamma_3 J(k_{12}) + 2\delta k_{12}^2 + 2\alpha(k_{13}^2 + k_{23}^2)\}. \quad (1b)$$

Here k_{il} are the momenta of the relative motion of the i-th and l-th pion. The subscripts 1 and 2 refer to the identical particles,

$$J(k) = -\frac{\sqrt{3}}{\pi}\varkappa^2\frac{x\arccos x}{(1-x^2)^{1/2}}\left(1 - \frac{8}{9}x^2\right) = \varkappa^2 I(x),$$
$$x = k/\varkappa, \qquad \varkappa^2 = M_K - 3 \approx 0.56, \quad (2)$$

M_K is the mass of the K meson, the mass of the pion is taken to be unity. λ, α, δ are real numbers to be determined from experiment. β_i and γ_i are numbers that can be expressed in terms of the pion scattering lengths a_0 and a_2:

$$\beta_1 = 1.67\,a_0a_2 + 0.33\,a_2^2, \qquad \gamma_1 = 3.33\,a_0a_2 - 1.33\,a_2^2,$$
$$\beta_2 = 1.39\,a_0^2 + 0.56\,a_2a_0 + 0.06\,a_2^2, \qquad \gamma_2 = 2a_2^2,$$
$$\beta_3 = -0.28\,(a_0 - a_2)^2, \qquad \gamma_3 = 1.11\,(a_0 - a_2)^2. \quad (3)$$

As already mentioned above, formulas (1a) and (1b) without the $\Delta T = 1/2$ rule taken into account were obtained by Gribov.[8]

A derivation of analogous formulas from dispersion relations is contained in [15]. The principle on the basis of which they are derived will also be explained in Sec. 2 in connection with the derivation of the integral equation for the $K \to 3\pi$ decay.

Corrections to the formulas (1a) and (1b), cubic in the momenta of the produced particles, were obtained in [9]. For $\rho = -1/2$ at the point $k_{12}^2 = k_{13}^2 = k_{23}^2 = 1/2\,\kappa^2$ these corrections are equal to zero. It is clear that also at other points in the physical region of the $K \to 3\pi$ decay these corrections will be small. Indeed, one can convince oneself that they contribute no more than 0.1 of the contributions from the quadratic terms.

Making use of the formulas (1a) and (1b) one can obtain expressions for the total probabilities for the processes $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$:

$$W = 4\lambda^2\Omega\{1 + \varkappa^2[2\alpha + \delta - 0.71(a_0 - a_2)^2 + 0.04\beta_1 + 0.02\beta_2 - 0.42\beta_3]\}, \quad (4a)$$

$$W' = \lambda^2\Omega'\{1 + \varkappa^2[2\alpha + \delta + 0.04\gamma_1 + 0.02\gamma_2 - 0.42\gamma_3]\}. \quad (4b)$$

where Ω and Ω' are the phase space volumes of the produced pions. Because $2\mu_{\pi^+} - 2\mu_{\pi^0} \approx 9.2$ MeV, Ω' is slightly larger than Ω.

Accurate to terms of order $0.1\,a_T^2$ the ratio of the decay probabilities is equal to

$$W'/W = 0.31. \quad (5)$$

In this formula the difference in the phase space volumes of the produced pions is taken into account in terms of order unity.[16,17] Terms of order $0.01\,a_T^2$ are not included in Eq. (5)—they contain contributions from terms cubic in the momenta and the mass difference of the pions affects them significantly.

From Eqs. (1a) and (1b) one may also obtain the energy distribution of the produced pions:

$$W_{\pi^-} = 1 + \varkappa^2[\beta_1F_1(x) + \beta_2F_2(x) + \beta_3F_3(x) + \tfrac{25}{36}(a_0 - a_2)^2x^2 + (\alpha - \delta)x^2], \quad (6a)$$

$$W_{\pi^+} = 1 + \varkappa^2[\beta_1(\tfrac{1}{2}F_1(x) + F_2(x)) + \tfrac{1}{2}\beta_2F_1(x) + \beta_3(I(x) + \tfrac{1}{2}F_3(x)) - \tfrac{25}{72}(a_0 - a_2)^2x^2 - \tfrac{1}{2}(\alpha - \delta)x^2], \quad (6b)$$

$$W'_{\pi^+} = 1 + \varkappa^2[\gamma_1F_1(x) + \gamma_2F_2(x) + 2\gamma_3I(x) - 2(\alpha - \delta)x^2]. \quad (6c)$$

x^2 is related to the energy E of the meson considered by $\kappa^2 = \kappa^2x^2 + 3E/2$.

In Eq. (6) all constant terms of order κ^2 may be omitted since after their inclusion into the normalization coefficient they will affect only terms of order κ^4. The functions $F_1(x)$, $F_2(x)$, and $F_3(x)$ were obtained in [8] [in the expression for $F_3(x)$ given in [8] the term $2x^2/3\sqrt{3}\pi$ was left out]. In the region $0 < x < 1$ the functions $F_1(x)$, $F_2(x)$, and $F_3(x)$ are well approximated by the following polynomials in x^2:

$$F_1(x) = -0.33 + 0.74\,x^2, \qquad F_2(x) = 0.42 - 0.74\,x^2,$$

$$F_3(x) = -0.33 - 0.16\,x^2. \tag{7}$$

If one introduces $\epsilon = E/E_{max}$ then Eq. (6) may be rewritten as follows:

$$W_{\pi^-} = 1 + (\varepsilon - {}^1/_2)\,[0.2\,a_0^2 + 0.4\,a_0a_2 - 0.5\,a_2^2 - 0.5\,(\alpha - \delta)], \tag{8a}$$

$$W_{\pi^+} = 1 + (\varepsilon - {}^1/_2)\,[-0.1\,a_0^2 - 0.2\,a_0a_2 + 0.2\,a_2^2 + 0.25\,(\alpha - \delta)] - 0.2\,(a_0 - a_2)^2\,[I\,(\sqrt{1-\varepsilon}) + 0.1], \tag{8b}$$

$$W'_{\pi^+} = 1 + (\varepsilon - {}^1/_2)\,[-1.4\,a_2a_0 + 1.4\,a_2^2 + 1.2\,(\alpha - \delta)] + 1.3\,(a_0 - a_2)^2\,[I\,(\sqrt{1-\varepsilon}) + 0.1]. \tag{8c}$$

The energy spectra in the reactions under consideration were measured by a number of authors.[11] The experiment gives

$$W_{\pi^-} = 1 + (\varepsilon - {}^1/_2)\,(0.53 \pm 0.07),$$

$$W_{\pi^+} = 1 - (\varepsilon - {}^1/_2)\,(0.26 \pm 0.09),$$

$$W'_{\pi^+} = 1 - (\varepsilon - {}^1/_2)\,(1.0 \pm 0.4). \tag{9}$$

One a priori unknown constant $\alpha - \delta$ enters Eq. (8) so that a study of the energy spectrum of one pion is not sufficient to yield information on a_0 and a_2. Since $|I(\sqrt{1-\epsilon}) + 0.1| \lesssim 0.1$ for $0 < \epsilon < 1$, the last term in Eq. (8b) is of the order of 10^{-2}. Therefore approximately $(W_{\pi^-} - 1)/(W_{\pi^+} - 1) \approx -2$, which is in agreement with experiment. In order to obtain information on a_0 and a_2 from energy distributions it is necessary to study both the reaction $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$.

The experimental data on the energy distribution of the π^+ from the reaction $K^+ \to 2\pi^0 + \pi^+$ are rather rough. They allow only to estimate a combination of the quantities a_0 and a_2. The last term in Eq. (8c) is of the order of 0.1. If it is ignored one can convince oneself that it follows from a comparison of Eqs. (8) and (9) that $a_0^2 - a_0a_2 + 0.5\,a_2^2 \approx 0.7 \pm 1$.

One may also obtain the distribution in z [the difference in the energies of the identical pions divided by its maximum value $z = \sqrt{3}\,\kappa^{-2}(E_1 - E_2)$]. For the reaction $K^+ \to 2\pi^+ + \pi^-$ this distribution is of the form [8]

$$W(z) = 1 + z^2\,(0.04\,a_0^2 + 0.4\,a_0a_2). \tag{10}$$

Experiment yields [11]

$$W(z) = 1 + z^2\,(0.0 \pm 0.1). \tag{11}$$

From here it follows that $|a_0^2 + 10\,a_0a_2| \le 2.5$. If a_0 is not very much larger than a_2 then $|a_0a_2| \lesssim 0.25$.

And so it follows from the above analysis that the experimental data on the $K^+ \to 3\pi$ decay are not in contradiction with the formulas derived under the assumption that a_0 and a_2 are not large. The accuracy of the experiment, however, does not as yet allow one to determine a_0 and a_2 from an analysis of the $K \to 3\pi$ decay.

There exist, however, experimental data that indicate that a_0 is large.[12] If that is so then $k_{il}a_0 \sim 1$, and Eqs. (1a) and (1b) are not valid. In that case the amplitude for $K \to 3\pi$ decay consists of a complicated function of k_{il}. The fact that Eqs. (1a) and (1b) are in qualitative agreement with experiment may not yet mean much, in view of the large experimental errors. In this connection it becomes necessary to find the amplitude for the $K \to 3\pi$ process without assuming a_0k_{il} to be small. In the next section we shall write an integral equation for such an amplitude.

2. DERIVATION OF THE INTEGRAL EQUATION FOR THE $K \to 3\pi$ DECAY WITHOUT ASSUMING a_0 AND a_2 TO BE SMALL

In this section we take exactly into account the interaction of the produced pions in s states with isotopic spins $T = 0$ and 2.

In the derivation of the integral equation, as in previous papers,[9,15] dispersion diagrams are utilized. Let us briefly recall what is involved. For example, the amplitude for the scattering of two particles on each other near threshold is simply given by the scattering length a_{12}; the corresponding diagram is shown in Fig. 1a. The correction of the order of the momentum of the scattering particles is given by $ia_{12}^2k_{12}$ as can be seen, for example, from the dispersion relation for the scattering amplitude (k_{12} stands for the relative momentum of the particles 1 and 2). This term may be represented by the diagram of Fig. 1b. The diagram of Fig. 1c describes the term $a_{12}^3(ik_{12})^2$, etc.

The sum of the diagrams, Fig. 1a, 1b, 1c, etc., gives the well known expression for the amplitude

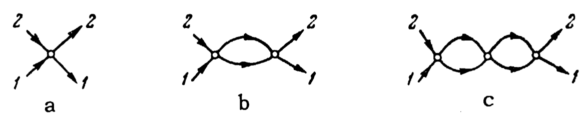

FIG. 1

for s-wave scattering near threshold when a_{12} is not small:

$$a_{12}(1 - ik_{12}a_{12})^{-1}.$$

This formula takes into account the singularities of the amplitude near threshold.

The regular part of the amplitude cannot be obtained in this way (it is determined, for example, by three-particle and four-particle intermediate states which have not been here taken into account). In this approach the regular terms of the amplitude for $1 + 2 \rightarrow 1 + 2$ near threshold can only be expressed as a power series in k^2 with undetermined coefficients. This manifests itself in the fact that the scattering length near threshold depends analytically on k^2 and may be expanded in a power series in k^2.

The amplitude for the decay of one particle into three, when the kinetic energy of the produced particles is not large, may also be expressed in terms of analogous diagrams. At that, as in the process $1 + 2 \rightarrow 1 + 2$, we shall obtain that part of the amplitude that is singular near the physical region for the decay (near $k_{il}^2 = 0$). The regular part can only be expanded in a power series in k_{il}^2 with undetermined coefficients. In the zeroth order approximation in the momenta the decay amplitude equals simply a constant λ. This term is represented by the diagram of Fig. 2a. Taking into account the interaction of only particles 1 and 2, say, gives rise to the appearance of diagrams of Fig. 2b, 2c, etc., which correspond to terms $\lambda ik_{12}a_{12}$, $\lambda(ik_{12}a_{12})^2$, etc. The sum of the diagrams, Fig. 2b, 2c, . . . , results in the quantity $\lambda ik_{12}a_{12}(1 - ik_{12}a_{12})^{-1}$.

In addition to the diagrams that take into account the interaction between some pair of particles there exist diagrams in which all three particles interact. The simplest diagram of this kind is shown in Fig. 2d. This diagram is expressed in the form of a dispersion integral

$$-\frac{1}{\pi} k_{23}^2 \int_0^\infty dk'^2_{23} a_{23} k'_{23} \frac{1}{k'^2_{23}(k'^2_{23} - k^2_{23} - i\varepsilon)} \int_{-1}^{1} \frac{dz}{2} ik'_{12} a_{12} \lambda. \quad (12)$$

One subtraction has been carried out in this integral in accordance with the fact that for $k_{il}^2 = 0$ the decay amplitude is given by the constant λ.

The absorptive part in the dispersion integral, Eq. (12), consists of a product of two amplitudes —the amplitude $ik'_{12}a_{12}\lambda$ corresponding to the diagram which in Fig. 2d lies to the left of the lines marked by crosses, and the amplitude a_{23} corresponding to the part of the diagram to the right of these lines. Integration over the phase space volume of particles 2 and 3 in the intermediate state gave rise to the appearance of the factor k'_{23} and of the integration over z (the cosine of the angle between the momentum of the relative motion of particles 2 and 3 in the intermediate state k'_{23} and the momentum of the first particle). It can be shown that if the masses of particles 1, 2, and 3 are equal to unity then

$$k'^2_{12} = \frac{1}{4} k'^2_{23} + \frac{3}{4}(\varkappa^2 - k'^2_{23}) + \frac{1}{2} z \sqrt{3}\, k'_{23} \sqrt{\varkappa^2 - k'^2_{23}}. \quad (13)$$

For $z = \pm 1$ (which corresponds to the limits of integration over z in the absorptive part) the expression (13) goes over into

$$k'_{12\pm} = \pm \frac{1}{2} k'_{23} + \frac{1}{2}\sqrt{3}\sqrt{\varkappa^2 - k'^2_{23}}. \quad (14)$$

A more detailed discussion of questions connected with these diagrams may be found in [8,9,15].

It is seen from the diagrams of Fig. 2 and other analogous ones, that the decay amplitude consists of a series of terms each depending on one invariant only. The decay amplitude has the form

$$A^{\pm}(k_{12}k_{13}k_{23}) = \lambda^{\pm} + A_1^{\pm}(k_{23}) + A_2^{\pm}(k_{13}) + A_3^{\pm}(k_{12}). \quad (15)$$

The plus sign refers to the reaction $K^+ \rightarrow 2\pi^0 + \pi^+$, the minus sign to the reaction $K^+ \rightarrow 2\pi^+ + \pi^-$.

If the decay satisfies the $\Delta T = 1/2$ rule then the produced mesons must be in a state of total isotopic spin $T = 1$. In that case it is easy to show (see, for example, [4,17]) that

$$\lambda^- = 2\lambda,\ A_3^-(k) = 2A(k),\ A_1^-(k) = A_2^-(k) = A(k) + D(k); \quad (16a)$$

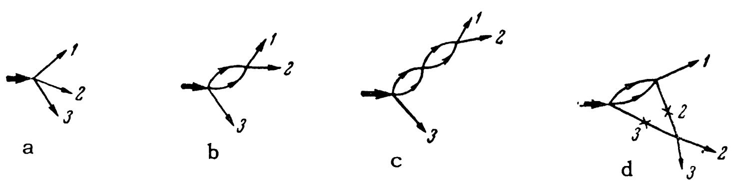

FIG. 2

$$\lambda^+ = -\lambda,\ A_3^+(k) = -D(k),\ A_1^+(k) = A_2^+(k) = -A(k). \tag{16b}$$

A(k) and D(k) vanish for k = 0. The minus sign in Eq. (16b), as was already mentioned in Sec. 1, is due to the choice of sign in the pion charge exchange amplitude.

If a_0 and a_2 are not small one must sum all diagrams analogous to those shown in Fig. 2. All diagrams may be divided into three kinds: 1) the produced particles in the final state do not interact, 2) only one pair of particles interacts in the final state, 3) all three particles interact in the final state. Only one diagram belongs to the first kind, that shown in Fig. 2a. This diagram describes the constant term (λ^+ or λ^-, depending on the reaction). Diagrams of the second kind are shown in Fig. 2b, 2c. The sum of all diagrams of this kind is represented, depending on the type of the reaction and the type of particles considered, by a certain superposition of expressions of the form $\lambda i k_{il} a_T (1 - i k_{il} a_T)^{-1}$. Diagrams of the third kind, depending on k_{12}, are shown in Fig. 3. The cross-hatched blob in Fig. 3a and 3b represents the totality of diagrams of the second and third kind in which particles 1 and 3 come out at one point.

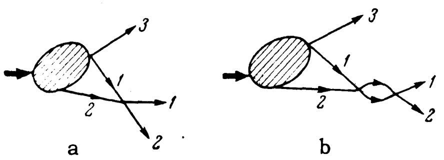

FIG. 3

In the diagrams of Fig. 3 particles 1 and 2 can, while interacting, form various chains of the type shown in Fig. 1. The summing of diagrams with such chains will result in the multiplication of diagram, Fig. 3a, by the factor $(1 - i k_{il} a_T)^{-1}$. In addition to the diagrams shown in Fig. 3 one also has, of course, diagrams in which particles 1 and 2 are interchanged in the intermediate state. From Eq. (15) it is seen that A(k) and D(k) are expressed in terms of diagrams of the second and third kind. It is easy to show that

$$A(k) = \lambda \frac{ika_2}{1 - ika_2} + \frac{1}{1 - ika_2} \frac{k^2}{\pi} \int_0^\infty dk'^2 \frac{k'a_2}{k'^2 (k'^2 - k^2 - i\varepsilon)} [\bar{A}(k') + \bar{D}(k')], \tag{17a}$$

$$D(k) = -\frac{2}{3} \lambda \frac{ika_2}{1 - ika_2} + \frac{5}{3} \lambda \frac{ika_0}{1 - ika_0} + \frac{1}{1 - ika_2} \frac{k^2}{\pi} \int_0^\infty dk'^2 \frac{k'a_2 [-2\bar{A}(k')/3 - 2\bar{D}(k')/3]}{k'^2 (k'^2 - k^2 - i\varepsilon)} + \frac{1}{1 - ika_0} \frac{k^2}{\pi} \int_0^\infty dk'^2 \frac{k'a_0 [8\bar{A}(k')/3 + 2\bar{D}(k')/3]}{k'^2 (k'^2 - k^2 - i\varepsilon)}. \tag{17b}$$

The functions $\bar{A}(k')$ and $\bar{D}(k')$ are obtained from A(k) and D(k) by averaging over z. For example

$$\bar{A}(k') = \frac{1}{2} \int_{-1}^{1} dz A(k''). \tag{18}$$

z is related to k′ and k″ by Eqs. (13) and (14) (it is necessary to replace in these equations k'_{12} by k″ and k'_{23} by k′).

In the derivation of Eq. (17) the following formulas for the amplitudes for scattering and charge exchange of pions were used

$$a_{++} = 2a_2/(1 - ika_2), \qquad a_{+0} = a_2/(1 - ika_2),$$
$$a_{+-} = a_2/3(1 - ika_2) + 2a_0/3\,(1 - ika_0),$$
$$a_{00} = 4a_2/3\,(1 - ika_2) + 2a_0/3\,(1 - ika_0),$$
$$a_{+-}^{00} = 2a_2/3\,(1 - ika_2) - 2a_0/3\,(1 - ika_0). \tag{19}$$

In summing the various diagrams of the type shown in Fig. 1 we have taken into account the singularities of the amplitude for s-wave scattering near threshold. In addition, a_T depends near threshold on the square of the relative momentum in a certain analytic way. The reason for such a dependence may, for example, be due to distant singularities of the s-wave scattering amplitude arising from the contributions of three-particle intermediate states, etc. For the same reasons λ depends analytically on k_{12}^2, k_{13}^2 and k_{23}^2 near $k_{il}^2 = 0$. In order to take this circumstance into account one must replace in Eq. (17) a_T by $a_T(k^2)$ and λ by $\lambda(k_{12}^2, k_{13}^2, k_{23}^2)$. The a_T appearing in the integrand should be replaced by $a_T(k'^2)$.

Equation (1) may be obtained by expanding Eq. (17) in a power series in k up to quadratic terms (more precisely up to terms of order κ^2). The terms with the undetermined constant α and δ arise from both the expansion of A(k) and D(k), as well as from the expansion of $\lambda_\pm(k_{12}^2, k_{13}^2, k_{23}^2)$. The latter terms, for example, enter with undetermined coefficients and therefore in such an approach α and δ cannot be expressed in terms of a_0 and a_2 or in terms of any other known quantities. Equation (17) is expressed as a series in k with the help of successive iterations—the terms of order k are obtained from the free term, the terms of order k^2 from the free term and from the dispersion integrals in which A and D are expressed as being linear in k, etc.

Khuri and Treiman[4] started out from dispersion relations in principle equivalent to Eq. (17). They expanded the $K \to 3\pi$ amplitude in a series in the momenta up to quadratic terms. The result of their expansion does not agree with Eq. (1). At the present time there exist a number of papers devoted to the determination of $a_0 - a_2$ from the experimental data on the basis of the Khuri-Treiman formulas, or to the discussion of the results of such a determination. For this reason we shall discuss this work in more detail.

The principal role in the determination of $a_0 - a_2$ in the work of Khuri and Treiman is played by terms of the type γk^2, where the coefficient γ is related to $a_0 - a_2$. As can be seen from the above analysis the terms obtained by Khuri and Treiman are not unique—quadratic terms are obtained, for example, from the expansion of $\lambda_\pm(k_{12}^2, k_{13}^2, k_{23}^2)$. There is no a priori basis for the belief that these terms are for some reason small. The fact that $a_0 - a_2$ cannot be determined from the Khuri-Treiman formulas follows already from the work of Alles, [10] who showed that there can be present in the decay amplitude terms quadratic in the momenta with undetermined coefficients.

Let us return to the integral equation for the $K \to 3\pi$ decay when a_0 is large and a_2 is small. In that case the amplitudes have in zeroth approximation ($a_2 = 0$) the form

$$A_0^-(k_{12}k_{13}k_{23}) = 2\lambda + D_0(k_{13}) + D_0(k_{23}), \tag{19a}$$

$$A_0^+(k_{12}k_{13}k_{23}) = -\lambda - D_0(k_{12}), \tag{19b}$$

where $D_0(k)$ is defined by the following integral equation:

$$D_0(k) = \frac{5}{3}\lambda\frac{ika_0}{1-ika_0} + \frac{1}{1-ika_0}\frac{2k^2}{3\pi}\int_0^\infty dk'^2\frac{k'a_0\bar{D}_0(k')}{k'^2(k'^2-k^2-i\varepsilon)}. \tag{20}$$

Keeping in Eq. (17) terms of order ka_2 and expanding $a_0(k^2)$ in a power series in k^2 one may obtain an equation for the decay amplitude to first order in κ, etc.

Upon averaging of the left and right hand sides of Eq. (2) one obtains an integral equation for $\bar{D}_0(k)$. After it has been solved numerically one can find $D_0(k)$. It is more convenient, however, to obtain directly an equation for $D_0(k)$. To that end we rewrite the integral in Eq. (20) as follows:

$$k^2\int_0^\infty dk'^2\frac{k'\bar{D}_0(k')}{k'^2(k'^2-k^2-i\varepsilon)}$$

$$= k^2\int_0^\Lambda dk'^2\,\frac{k'}{k'^2(k'^2-k^2-i\varepsilon)}\,\frac{1}{\sqrt{3}k'\sqrt{\varkappa^2-k'^2}}\int_{k_-'^2}^{k_+}dk''^2 D(k''). \tag{21}$$

We have made use here of Eqs. (13) and (14). The limit $\Lambda \to \infty$ will be taken in the final formulas.

Let us now vary $\kappa^2 + i\epsilon$. We make κ^2 negative. For $\kappa^2 < 0$ $k_\pm'^2$ varies in a certain complex region of k'^2, with $|k_\pm'^2| < \Lambda$. Equation (21) may be rewritten in the form

$$k^2\int_0^\Lambda dk'^2\frac{k'}{k'^2(k'^2-k^2-i\varepsilon)}\frac{1}{\sqrt{3}k'\sqrt{\varkappa^2-k'^2}}$$

$$\times\int_{k_-'^2}^{k_+'^2} dk''^2\frac{1}{2\pi i}\int_C ds\frac{1}{s-k''^2}D(s). \tag{22}$$

The contour of integration C is shown in Fig. 4. The integration over the large circle proceeds in such a way that $|s| > \Lambda$. The dashed line shows the cut drawn from the singularity of $D(s)$ at $s = 0$.

Having performed the integration over k'^2 and k''^2 we proceed to deform the contour C by making it narrower. Then after taking the limit $\Lambda \to \infty$ we obtain in place of Eq. (22):

$$\frac{1}{\sqrt{3}}\int_{\varkappa^2}^{-\infty} ds\, D(s)\left[\frac{1}{\sqrt{\varkappa^2-k^2}}\ln\frac{(-\frac{1}{2}\sqrt{\varkappa^2-s}+\sqrt{\varkappa^2-k^2})^2-\frac{3}{4}s}{(\frac{1}{2}\sqrt{\varkappa^2-s}+\sqrt{\varkappa^2-k^2})^2-\frac{3}{4}s}\right.$$

$$\left.-\frac{1}{\sqrt{\varkappa^2}}\ln\frac{(-\frac{1}{2}\sqrt{\varkappa^2-s}+\sqrt{\varkappa^2})^2-\frac{3}{4}s}{(\frac{1}{2}\sqrt{\varkappa^2-s}+\sqrt{\varkappa^2})^2-\frac{3}{4}s}\right]. \tag{23}$$

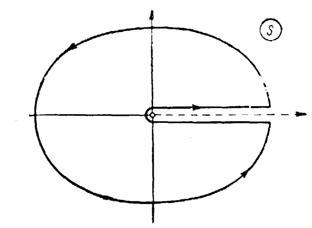

FIG. 4

In Eq. (23) one may, by varying $\kappa^2 + i\epsilon$, go over to $\kappa^2 > 0$. Introducing the notation

$$\varkappa^2 - s = \frac{3}{4}p'^2, \qquad \varkappa^2 - k^2 = \frac{3}{4}p^2,$$

$$D_0(k)(1 + k^2a_0^2) = \lambda d(p),$$

we obtain finally

$$(1 + i\sqrt{\varkappa^2 - 3p^2/4}\,a_0)^{-1}d(p) = \frac{5}{3}i\sqrt{\varkappa^2 - \frac{3}{4}p^2}\,a_0$$

$$+\frac{a_0}{3\pi}\int_0^\infty dp'^2\frac{d(p')}{1+\varkappa^2a_0^2-3p'^2a_0^2/4-i\varepsilon}$$

$$\times\left[\frac{1}{p}\ln\frac{p'^2+p^2+pp'-\varkappa^2-i\varepsilon}{p'^2+p^2-pp'-\varkappa^2-i\varepsilon}\right.$$

$$\left.-\frac{\sqrt{3}}{2\varkappa}\ln\frac{p'^2+\frac{1}{3}\varkappa^2+2p'\varkappa/\sqrt{3}}{p'^2+\frac{1}{3}\varkappa^2-2p'\varkappa/\sqrt{3}}\right]. \tag{24}$$

This is an integral equation of the Skornyakov-Ter-Martirosyan type [13] with one subtraction.

Such equations were studied by Danilov.[14] It has a unique solution. The method for the numerical solution of equations of this type is contained in the work of Danilov and Lebedev.[18]

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