ATTENUATION OF MAGNETIC SOUND WAVES AND THE WIDTH OF SHOCK WAVES IN ANISOTROPIC RELATIVISTIC MAGNETOHYDRODYNAMICS

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Submitted to JETP editor December 12, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 1640-1643 (May, 1963)

The basic equations of anisotropic relativistic magnetohydrodynamics are formulated. The problem of the attenuation of weak magnetic-sound waves is considered on the basis of these equations and the relation between the attenuation coefficients of weak magnetic-sound waves and the width of weak shock waves is derived.

WITH the decrease in density of conducting material and the increase in the intensity of the magnetic field, it can happen that the Alfvén velocity will be of the order of the light velocity. In this case, fast magnetic-sound waves which are propagated in a nonrelativistic medium become relativistic. Therefore, the development of the relativistic theory of magnetic-sound waves is of importance.

In the present work, the attenuation of magnetic-sound waves will be considered in relativistic magnetohydrodynamics; it is kept in mind here that the material located in the magnetic field is anisotropic. The width of weak shock waves will also be calculated.

1. First of all, let us write down the basic equations of relativistic magnetohydrodynamics with account of dissipative processes in the anisotropic medium. These equations were previously obtained for the isotropic case,[1] and now we need only generalize them. In the generalization, it is necessary to replace the anisotropy condition of [1] by the condition that the resultant equations will transform in the nonrelativistic limit to the corresponding equations of nonrelativistic anisotropic magnetohydrodynamics,[2] and, for symmetric form of the dissipation tensors, the equations will transform into the corresponding equations of relativistic isotropic magnetohydrodynamics. These conditions are satisfied by the equations

$$\partial T_{ik}/\partial x_k = 0, \tag{1.1}$$

$$\partial n_i/\partial x_i = 0, \tag{1.2}$$

$$\partial F_{ik}/\partial x_k = 4\pi c^{-1} j_i, \tag{1.3}$$

$$\partial F_{ik}/\partial x_l + \partial F_{kl}/\partial x_i + \partial F_{li}/\partial x_k = 0, \tag{1.4}$$

$$T_{ik} = (F_{il}F_{kl} - \tfrac{1}{4}F_{lm}^2\delta_{ik})/4\pi + p\delta_{ik} + wu_iu_k + \tau_{ik}, \tag{1.5}$$

$$n_i = nu_i + \nu_i, \tag{1.6}$$

$$F_i = (R_{ik} + u_iu_lR_{lk})(j_k - u_kc\rho_0) - T(A_{ik} + u_iu_lA_{lk} + u_ku_lA_{il} + u_iu_ku_su_pA_{sp})\frac{\partial}{\partial x_k}\frac{\mu}{T}, \tag{1.7}$$

$$\nu_i = -(A_{ik} + u_iu_lA_{lk})(j_k - u_kc\rho_0) - (K_{ik} + u_iu_lK_{lk} + u_ku_lK_{il} + u_iu_ku_su_pK_{sp})\frac{\partial}{\partial x_k}\frac{\mu}{T}, \tag{1.8}$$

$$\tau_{ik} = -(\eta_{ikml} + u_iu_p\eta_{pkml} + u_ku_p\eta_{piml} + u_lu_p\eta_{ikmp} + u_ku_su_lu_p\eta_{sipm} + u_iu_ku_su_p\eta_{spml} + u_iu_su_lu_p\eta_{skmp} + u_iu_ku_su_tu_lu_p\eta_{stmp})\frac{\partial u_m}{\partial x_l}, \tag{1.9}$$ [1)]

$$F_i = c^{-1}F_{ik}u_k. \tag{1.10}$$

In these expressions T_{ik} is the energy-momentum tensor, n_i is the 4-vector of matter flux density, F_{ik} is the 4-tensor of the electric field, j_i is the 4-vector of the electric current density, p is the pressure, w is the enthalpy of a unit of characteristic volume of the material, u_i is the 4-velocity, n is the number of particles per unit volume, c is the velocity of light, R_{ik}, A_{ik}, K_{ik} η_{iklm} are 4-tensors associated with the dissipative properties of the material, ρ_0 is the density of electric charge in the characteristic system, μ is the relativistic chemical potential, and T is the temperature.

The system (1)—(10) transforms into the corresponding system of equations of isotropic relativistic magnetohydrodynamics [1] for the case

$$R_{ik} = \frac{\delta_{ik}}{c\sigma_0}, \quad A_{ik} = \frac{n\varkappa_0\delta_{ik}}{cw}, \quad K_{ik} = \frac{\varkappa_0}{c}\left(\frac{nT}{w}\right)^2\delta_{ik},$$

$$\eta_{ikml} = c\mu_0(\delta_{im}\delta_{kl} + \delta_{il}\delta_{km}) + c\left(\zeta - \frac{2}{3}\mu_0\right)\delta_{ik}\delta_{lm}. \tag{1.11}$$

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1)It is also assumed here that η_{iklm} has the following symmetry property: $\eta_{iklm} = \eta_{kiml} = \eta_{kilm}$.

2. For an investigation of the attenuation of weak magnetic-sound waves, we shall work in a system of coordinates x, y, z attached to a fixed medium. The x axis is directed along the external magnetic field H_0, while the y and z axes are so chosen that the component k_z of the wave vector is equal to zero. The waves are assumed to be weak. Therefore the quantities u_α, E_α, $h_\alpha = H_\alpha - H_0\delta_{x\alpha}$ $(\alpha = x, y, z)$, $p' = p - p_0$, $w' = w - w_0$, etc., are small parameters (the zero index here refers to the unperturbed medium). The dissipation coefficients are also assumed to be small. Neglecting all terms above second order in these small quantities, we shall look for a solution of the system (1.1)—(1.10) in the form $f = f_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$. The homogeneous linearized system of equations obtained in this manner has a non-trivial solution when

$$- U^2\left(1 - \frac{\omega^2}{c^2k^2}\right)\frac{\omega^2}{c^2k^2} + \frac{\omega^4}{c^4k^4} - c_0^2\frac{\omega^2}{c^2k^2}$$
$$+ U^2\left(1 - \frac{\omega^2}{c^2k^2}\right)c_0^2\cos^2\theta$$
$$+ i\frac{\omega}{ck}\left\{\frac{H_0cc_0^2B}{4\pi n}\left[\left(1 - \frac{\omega^2}{c^2k^2}\right)A_z - \overset{*}{A_z}\right]\sin\theta + \left(1 - \frac{\omega^2}{c^2k^2}\right)\right.$$
$$\times\left(\frac{\omega^2}{c^2k^2} - c_0^2\right)k\beta_{zz}\frac{c_0^2}{kw}\left[N_{xx}\sin^2\theta - (N_{yx} + N_{xy})\right.$$
$$\times\sin\theta\cos\theta + N_{yy}\cos^2\theta\big] + \frac{\omega^2}{c^2wk^3}\,(N_{xx} + N_{yy})$$
$$- \frac{1}{kw}\left(1 - \frac{\omega^2}{c^2k^2}\right)U^2N_{xx} - \frac{c_0^2wB^2}{kn^2T}$$
$$\times\left.\left[\left(1 - \frac{\omega^2}{c^2k^2}\right)U^2 - \frac{\omega^2}{c^2k^2}\right]K\right\} = 0. \qquad (2.1)$$

Here

$$B = 1 - \frac{1}{c_0}\sqrt{\left(\frac{c}{C_V} - \frac{1}{C_p}\right)\frac{w}{nT}}, \quad \mathbf{U} = \frac{\mathbf{H}}{\sqrt{4\pi w}}, \quad c_0 = \frac{v_0}{c},$$
$$\beta_{zz} = \frac{c^3}{4\pi}R_{zz},$$

v_0 is the velocity of sound, θ is the angle between **k** and **H**,

$$N_{\alpha\beta} = k_x^2\eta_{x\alpha\beta x} + k_xk_y\,(\eta_{y\alpha\beta x} + \eta_{x\alpha\beta y}) + k_y^2\eta_{y\alpha\beta y}$$
$$(\alpha, \beta = x, y, z), \quad K = K_{xx}k_x^2 + K_{yy}k_y^2,$$
$$A_z = A_{zx}k_x + A_{zy}k_y, \qquad \overset{*}{A_z} = A_{xz}k_x + A_{yz}k_y.$$

On the basis of Eq. (2.1), we get the following for the attenuation coefficient, in linear approximation of the dissipation coefficients:

$$\gamma = -\operatorname{Im}\omega = -\left\{\beta_{zz}ck^2\,(1 - u_\pm^2)\,(u_\pm^2 - c_0^2) - \frac{KwB^2c_0^2c}{kn^2T}\right.$$
$$\times[(1 - u_\pm^2)\,U^2 - u_\pm^2]\,\frac{Hc^2c_0^2Bk}{4\pi n}\,[(1 - u_\pm^2)\,A_z - \overset{*}{A_z}]\sin\theta$$
$$+ u_\pm^2\,(N_{xx} + N_{yy})\,c - cc_0^2\,[N_{xx}\sin^2\theta - (N_{xy} + N_{yx})$$
$$\left.\times\sin\theta\cos\theta + N_{yy}\cos^2\theta]\,(1 - u_\pm^2)\,cU^2N_{xx}\right\}$$
$$\times\{2w\,[(1 + U_x^2)\,c_0^2 - (2u_\pm^2 + 2U^2u_\pm^2 - U^2)]\}^{-1}. \qquad (2.2)$$

Here $u_\pm = c^{-1}k^{-1}\operatorname{Re}\omega$ and, in accord with (2.1), satisfies the following well known[3-5] equation in the linear approximation in the dissipation coefficients:

$$u_\pm^4\,(1 + U^2) - (U^2 + c_0^2 + U_x^2c_0^2)\,u_\pm^2 + U_x^2c_0^2 = 0. \qquad (2.3)$$

On the basis of (1.11) we have from (2.2), for an isotropic medium,

$$\gamma = -\left\{\varkappa_0TB^2c_0^2\,[(1 + U^2)\,u_\pm^2 - U_x^2] + \beta_0\,(1 - u_\pm^2)\right.$$
$$\times(u_\pm^2 - c_0^2)\,w\,\mu_0c^2\,[u_\pm^2 - (1 - u_\pm^2)\,U_y^2 - c_0^2]$$
$$\left.+\left(\tfrac{4}{3}\mu_0 + \zeta\right)c^2\,[u_\pm^2\,(1 + U_x^2) - U_x^2]\right\}$$
$$\times\{2w\,[(1 + U_x^2)\,c_0^2 - (2u_\pm^2 - U^2 + 2U^2u_\pm^2)]\}^{-1}. \qquad (2.4)$$

Comparing Eq. (2.4) with the expression for the width of the weak shock wave in isotropic relativistic magnetohydrodynamics (which was calculated previously[1]), we can note that the behavior of the attenuation coefficient of weak magnetic-sound waves and the width of weak shock waves do not depend on the dissipative properties of the material:

$$\frac{l}{\gamma} = -\frac{8v_\pm\,[(1 + U_x^2)\,(1 + v_\pm^2)\,c_0^2 - (2v_\pm^2 + U^2v_\pm^2 - U^2)]\,w}{ck^2\left\{\frac{w^2c_0^4\,(U_x^2 - v_\pm^2)}{V}\left(\frac{\partial^2V}{\partial p^2}\right)_S - \frac{3}{U^2 - v_\pm^2}\,[c_0^2U_xU_y + c_0^2\,(U_x^2 - v_\pm^2)^2 - v_\pm^2U_y^2]\right\}\Delta p}, \qquad (2.5)$$

where $v_\pm^2 = u_\pm^2/(1 - u_\pm^2)$, $V = 1/n$, S is the entropy of the volume V, Δp is the pressure jump at the shock front.

The relation (2.5) is also valid in the case in which the medium is anisotropic. Therefore, on the basis of Eqs. (2.2) and (2.5), we can write the equation for the width when the medium is anisotropic. As a result we get

$$l_\pm = 4v_\pm\left\{\frac{\varkappa_{xx}w^2c_0^2B^2}{n^2T}(1 + v_\pm^2)\,(v_\pm^2 - U^2) + \beta_{zz}w\,\frac{U_y^2v_\pm^2}{v_\pm^2 - U_y^2}\right.$$
$$+ \frac{Hc_0^2B\,(1 + v_\pm^2)}{4\pi}\,[\alpha_{zx} - (1 + v_\pm^2)\,\alpha_{xz}]\sin\theta$$

$$+ c(\rho_{xx} + \lambda_{xy}) v_{\pm}^2 - (1 + v_{\pm}^2) c_0^2 c \lambda_{xy} - U^2$$
$$\times [\rho_{xx} \cos^2\theta - (\lambda_{xx} + \rho_{xy}) \sin\theta \cos\theta + \lambda_{xy} \sin^2\theta] \Big\}$$
$$\times \Big\{ \frac{w^2 c_0^4 (U_x^2 - v_{\pm}^2)}{V} \left(\frac{\partial^2 V}{\partial p^2}\right)_S$$
$$- \frac{3}{U^2 - v_{\pm}^2} [c_0^2 U_x^2 U_y^2 + c_0^2 (U_x^2 - v_{\pm}^2)^2 - v_{\pm}^2 U_y^2] \Big\}^{-1}$$
$$\times (1 + v_{\pm}^2)^{-1} \Delta p^{-1}. \quad (2.6)$$

Here the coefficients κ_{xx}, β_{zz}, $\alpha_{\alpha\beta}$, $\rho_{\alpha\beta}$, and $\lambda_{\alpha\beta}$ are chosen in correspondence to their nonrelativistic definitions.[2]

3. The principal interest lies in waves which are propagated in a nonrelativistic plasma for $U \gtrless 1$. In this case, $U \gg c_0$ and only the fast wave is relativistic. In accord with (2.6), the expression for the width differs appreciably from the nonrelativistic expression[2] only for $U^2 \gg 1$. In this case, in the region of isotropy,

$$l_+ = \frac{4cc_0^2}{3U\Delta p}\left[U^2\left(1 + \frac{\sin^2\theta}{3}\right)\mu_0 + \frac{\beta_0 w}{c^2} + \frac{4\varkappa_0 T U^4}{9c^2}\right]. \quad (3.1)$$

In the anisotropic region, (2.6) is simplified only for $U^2 \gg 10^{-2} c_0^{-2}$ and then

$$l_+ = \frac{16\varkappa_0 T U^3 c_0^2}{27 c \Delta p} \cos^2\theta \qquad (\theta \neq 90°). \quad (3.2)$$

Therefore, in very strong magnetic fields l_+ increases as H^3 with the strength of the magnetic field. Corresponding to (2.5), for $c_0^2 \ll 1 \lesssim U^2$,

$$\gamma_+ = 3Uck^2 l_+ \Delta p / 8c_0^2 (1 + U^2)^2 w \quad (3.3)$$

and therefore the attenuation coefficient decreases as H^{-2} with increase in H, and tends asymptotically to

$$\gamma_+ = \frac{2\varkappa_0 T \omega^2 \cos^2\theta}{9\rho c^2 U^2} \qquad (\theta \neq 90°). \quad (3.4)$$

In conclusion, the author expresses his gratitude to V. P. Silin, for direction of the present research and for a great deal of advice and discussion, and also to K. P. Stanyukovich, S. I. Syrovatskiĭ, and V. N. Tsytovich for useful discussions.

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Translated by R. T. Beyer
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