TWO-PHOTON ANNIHILATION OF POLARIZED ELECTRONS AND POSITRONS

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The differential cross section for two-photon annihilation of free polarized electrons and positrons is obtained in a relativistically invariant form for the case where the final state is resolved into linear polarizations of the γ quanta.

In recent years, in connection with the discovery of new effects in β decay, and also with the appearance of possibilities for experiments with colliding beams of positrons and electrons, interest has developed in more detailed study of the annihilation of electron-positron pairs, in particular the annihilation of polarized positrons.[1-9] This interest is also caused by the possibility of using the annihilation process for studying the behavior of charged particles in different materials and the determination of various properties of these materials.[5-7]

The differential cross section for two-photon annihilation "in flight" is found in this paper for polarized electrons and positrons, in an arbitrary reference frame, in the first nonvanishing approximation in e^2; this permits us easily to compare the cross sections which have been obtained previously.[1-3]

Page[1] calculated the differential cross sections for annihilation with longitudinal and transverse polarizations of the positrons and electrons summed over the polarization directions of the photons, and also the degree of circular polarization of one of the photons. In [4] formulas were given for the cross sections summed over the directions of polarization of the photons for arbitrary polarizations and momenta of the annihilating pair. And finally, in a paper of Theis[3] such cross sections were calculated in the system of the center of inertia (cms) for cases where the directions of linear polarization of the photons were taken to be perpendicular or parallel to the annihilation plane. In a paper of Frolov[9] general relativistically invariant formulas were obtained describing the polarization effects in the scattering of a spin $1/2$ particle, taking form factors into account, but the explicit form for the differential cross section for two-photon annihilation was not given or discussed there.

1. Using the polarization density matrix for positrons and electrons,[10] the differential cross section for two-photon annihilation of electrons and positrons in flight can be written in the form

$$d\sigma = r_0^2 \frac{\omega_1}{2^8 \omega_2} \left| \frac{\partial E_f}{\partial \omega_1} \right|^{-1} \frac{m^2}{[(pq)^2 - m^4]^{1/2}} \frac{\mathrm{Tr}\, X}{\varkappa_1^2 \varkappa_2^2} d\Omega_1, \tag{1}$$

where p_μ, q_μ, $k_{\mu i}$ are the four-momenta of the electron, positron and photon, and $a_{\mu\pm}$, $e_{\mu i}$ are their polarization vectors (here e_i give the linear polarizations of the photons)

$$X = \hat{\hat{Q}}_0 (1 + i\gamma_5 \hat{a}_+)(i\hat{q} + m)\, \hat{Q}_0 (1 + i\gamma_5 \hat{a}_-)(i\hat{p} - m)$$

$$\hat{Q}_0 = \varkappa_1 \hat{e}_1 [i(\hat{p} - \hat{k}_2) - m]\, \hat{e}_2 + \varkappa_2 \hat{e}_2 [i(\hat{p} - \hat{k}_1) - m]\, \hat{e}_1,$$

$$\hat{\hat{Q}}_0 = \gamma_4 \hat{Q}^+ \gamma_4, \quad r_0 = e^2/m, \quad E_f = \omega_1 + \omega_2,$$

$$\varkappa_i = (pk_i), \quad i = 1, 2, \tag{1a}$$

$d\Omega_1$ is the element of solid angle for the direction of the momentum of one of the photons.[1)]

After simple calculations one gets the following expressions for Tr X from (1):

$$\mathrm{Tr}\, X = F_{\mathrm{I}}(e_1 e_2) + F_{\mathrm{II}}(e_1 e_2 a_+) + F_{\mathrm{III}}(e_1 e_2 a_-) + F_{\mathrm{IV}}(e_1 e_2 a_+ a_-),$$

$$\tfrac{1}{8} F_{\mathrm{I}} = \varkappa_1 \varkappa_2 \varkappa^2 - 4s^2, \tag{2}$$

$$\tfrac{1}{8} F_{\mathrm{II}} = 2\varkappa_1 \varkappa_2 (e_1 e_2)\, m \left\{ \varkappa_1 (e_2 e_1 k_2 a_+)_\varepsilon + (qe_1)(k_1 k_2 a_+ e_2)_\varepsilon + \begin{pmatrix} e_1 \leftrightarrow e_2 \\ k_1 \leftrightarrow k_2 \end{pmatrix} \right\}, \tag{2a}$$

$$F_{\mathrm{III}} = F_{\mathrm{II}}|_{a_+ \to a_-,\ q \to -p}, \tag{2b}$$

$$\begin{aligned} \tfrac{1}{8} F_{\mathrm{IV}} = {} & (a_+ a_-)\left[\tfrac{1}{4} \varkappa_1 \varkappa_2 \varkappa^2 - s^2\right] \\ & + \varkappa_2 (k_1 a_+)(k_2 a_-)\left[\tfrac{1}{2}\varkappa^2 + 2(e_1 e_2)\, s\right] + 2s\,(e_2 a_+)\{\varkappa_2 \varkappa (e_1 a_-) \\ & + (k_2 a_-)[\varkappa_1 (qe_1) - \varkappa_2 (pe_1)] - \varkappa_2 (k_1 a_-)(k_2 e_1)\} \\ & + (e_2 a_+)\varkappa_2 \varkappa \{-\varkappa_1 \varkappa (e_2 a_-) + [\varkappa_1 (pe_2)(qa_-) \\ & - (qe_2)(\varkappa_2 (k_1 a_-) - \varkappa_1 (k_2 a_-))]\} \\ & - \varkappa^2 (pe_2)(qe_2)(k_1 a_+)(k_1 a_-) \\ & + 2(qa_-)(k_1 e_2)(pe_2)\varkappa_1 [\varkappa_1 (k_1 a_+) - \varkappa_2 (k_2 a_+)] \\ & + (e_1 \leftrightarrow e_2,\ k_1 \leftrightarrow k_2) + (a_+ \leftrightarrow a_-,\ q \leftrightarrow p), \end{aligned} \tag{2c}$$

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1)The notation and units used here are the same as in [10], but the sign of $\varkappa_i$ is reversed.

where

$$(abcd)_\varepsilon = \varepsilon_{\mu\nu\alpha\beta} a_\mu b_\nu c_\alpha d_\beta,$$

$$s = 2\varkappa_1\varkappa_2 (e_1e_2) - \varkappa_2 (pe_1)(qe_2) - \varkappa_1 (pe_2)(qe_1).$$

As they should be, the expressions obtained for F_J are symmetric under the interchange ($e_1 \longleftrightarrow e_2$, $k_1 \longleftrightarrow k_2$) and ($p \longleftrightarrow q$, $a_- \longleftrightarrow a_+$), which allows us to abbreviate the expressions for F_J. In the calculations of the F_J, only the following conditions imposed on k_i, e_i, p, q, $a_\pm$ were used,

$$k_1 + k_2 = p + q, \quad k_i^2 = 0, \quad p^2 = q^2 = -m^2, \tag{3a}$$

$$(k_ie_i) = 0, \quad (qa_+) = (pa_-) = 0, \quad i = 1, 2. \tag{3b}$$

Thus the expressions for F_J are valid for arbitrary p and q.

This allows us to carry out the summation over the directions of polarization of the photons very simply. Remembering that

$$\sum_{e_i} e_i^2 = 2, \quad \sum_{e_1e_2} (e_1e_2)^2 = 2, \quad \sum_{e_i} e_{i\mu}e_{i\nu} = \delta_{\mu\nu}, \tag{4}$$

we easily get

$$\tfrac{1}{32}\sum_{e_1e_2} F_{\rm I} \equiv F_0 = \varkappa_1\varkappa_2(\varkappa_1^2 + \varkappa_2^2) - 2m^2\varkappa_1\varkappa_2\varkappa - m^4\varkappa^2, \tag{5a}$$

$$\sum_{e_1e_2} F_{\rm II} = \sum_{e_1e_2} F_{\rm III} = 0, \tag{5b}$$

$$\tfrac{1}{32}\sum_{e_1e_2} F_{\rm IV} \equiv F_\zeta = -(a_+a_-)(m^4\varkappa^2 + 2\varkappa_1\varkappa_2\varkappa m^2 + 2\varkappa_1^2\varkappa_2^2) + (m^2\varkappa + 2\varkappa_1\varkappa_2)[\varkappa_1(k_1a_+)(k_2a_-) + \varkappa_2(k_2a_+)(k_1a_-)]. \tag{5c}$$

The expression for F_ζ coincides with the result obtained by Rogozinski,[2] and for F_0 agrees with formula (12)—(41) of [11]. In addition to the summation over polarizations of the photons, the arbitrariness of p and q allows us to use the "substitution law"[2)] to get the cross sections for Compton scattering and for formation of e^+-e^- pairs by two photons ([10], Sec. 32, par. 3).

Thus the differential cross section for annihilation of arbitrarily polarized electrons and positrons has the form

$$\frac{d\sigma}{d\Omega_1} = r_0^2 \frac{\omega_1}{2^8\omega_3} \left|\frac{\partial E_f}{\partial\omega_1}\right|^{-1} \frac{m^2}{\varkappa_1^2\varkappa_2^2[(pq)^2 - m^4]^{1/2}} \sum_{J=1}^{\rm IV} F_J(e_i, a_\pm), \tag{6}$$

where the F_J are given by formulas (2a)—(2c). From these formulas we see that the differential annihilation cross section has the required symmetry properties, and in addition is gauge invariant, i.e., $d\sigma/d\Omega\,|_{e_i \to k_i} = 0$.

If one of the incident particles is unpolarized, it is obvious that $F_{\rm IV}$ and the differential cross section $d\sigma/d\Omega_1$ are determined only by the functions $F_{\rm I} + F_{\rm II}$ (or $F_{\rm III}$). We note that for arbitrary p and q we have $F_{\rm II} = F_{\rm III} = 0$, if $e_1 = \pm e_2$ or $(e_1e_2) = 0$.

2. Now let us look at some special cases of formula (6). In the cms, where $\mathbf{p} + \mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2 = 0$ and $\epsilon_\pm = \omega_{1,2} = \omega$, we choose $\mathbf{q}/|\mathbf{q}| = \mathbf{n}$ as the Z axis. Then, writing $\beta = |\mathbf{q}|/\omega$, $\gamma = \omega/m$, $\mathbf{n}_1 = \mathbf{k}_1/\omega$, $\widehat{\mathbf{n}_1\mathbf{n}} = \theta$, we get

$$\varkappa_{1,2} = -\omega^2(1 \pm \beta\cos\theta), \quad \varkappa = \varkappa_1 + \varkappa_2 = -2\omega^2,$$

$$a_\pm = (\mathbf{a}_\pm, a_{0\pm}), \quad \mathbf{a}_\pm = \boldsymbol{\zeta}_\pm + (\gamma - 1)(\boldsymbol{\zeta}_\pm\mathbf{n})\,\mathbf{n},$$

$$a_{0\pm} = \pm\beta\gamma(\boldsymbol{\zeta}_\pm\mathbf{n}), \tag{7}$$

where $\zeta_+(\zeta_-)$ is the polarization of the positrons (electrons) in the rest frame.

We now choose the gauge so that $e_i = (\mathbf{e}_i, 0)$, $\mathbf{e}_i^2 = 1$. As basis vectors for the linear polarizations of the photons it is convenient to choose the following vectors: $\mathbf{e}_x \perp \mathbf{n}_1$, $\mathbf{e}_y \propto \mathbf{n}_1 \times \mathbf{n}$, for example,

$$\mathbf{e}_x = (1, \pi/2 + \theta, \Phi), \tag{8a}$$

$$\mathbf{e}_y \perp \mathbf{n} \text{ and } \mathbf{n}_1, \quad \mathbf{e}_y = (1, \pi/2, \Phi), \tag{8b}$$

and then we write $\mathbf{e}_i = \alpha_i\mathbf{e}_x + \beta_i\mathbf{e}_y$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, 2$.

It is not difficult to get the expression for the sum $\sum_{e_1e_2} d\sigma/d\Omega_1 \equiv d\sigma_\zeta/d\Omega_1$ in three-dimensional notation:

$$\frac{d\sigma_\zeta}{d\Omega_1} = \frac{r_0^2}{2\beta\gamma^2(1-\beta^2\cos^2\theta)^2}\Big\{\tfrac{1}{2}F_0 - (\gamma^{-4} + \beta^4\sin\theta)(\boldsymbol{\zeta}_+\boldsymbol{\zeta}_-) - 2\beta^2[\gamma^{-2} + \gamma^2\beta^2(1+\beta^2)\sin^4\theta](\mathbf{n}\boldsymbol{\zeta}_+)(\mathbf{n}\boldsymbol{\zeta}_-) + 2\beta^2\sin^2\theta(\mathbf{n}_1\boldsymbol{\zeta}_+)(\mathbf{n}_1\boldsymbol{\zeta}_-) + 2\beta^2(1-\gamma^{-1})[2\cos\theta(\mathbf{n}\boldsymbol{\zeta}_+)(\mathbf{n}\boldsymbol{\zeta}_-) - (\mathbf{n}\boldsymbol{\zeta}_-)(\mathbf{n}_1\boldsymbol{\zeta}_+) - (\mathbf{n}\boldsymbol{\zeta}_+)(\mathbf{n}_1\boldsymbol{\zeta}_-)]\Big\},$$

$$\tfrac{1}{2}F_0 = \tfrac{1}{2}\sum_{e_1e_2} F_{\rm I} = 1 - \beta^4 + 2\beta^2\sin^2\theta - \beta^4\sin^4\theta. \tag{9}$$

If we introduce spherical angles $\vartheta_\pm$, $\varphi_\pm$ to describe the polarization directions of the positron and electron, and use the notation

$$f = \cos\vartheta_+\cos\vartheta_-,$$

$$g = \cos\vartheta_-\sin\vartheta_+\cos(\varphi_+ - \Phi) + \cos\vartheta_+\sin\vartheta_-\cos(\varphi_- - \Phi),$$

$$h_\pm = \sin\vartheta_+\sin\vartheta_-\cos[\varphi_- - \Phi \pm (\varphi_+ - \Phi)],$$

$d\sigma_\zeta/d\Omega_1$ can be written as

[2)]Using this property of the cross sections, i.e., making the substitution $q \to -p'$, $p \to p$, $e_1 \to e$, $e_2 \to e'$, $k_2 \to k'$, $k_1 \to -k$ in F_J, it is not difficult to see that the expression for the cross section corresponding to X from formulas (11)–(13) in [11] contains several misprints.

$$\frac{d\sigma_\zeta}{d\Omega_1} = \frac{r_0^2}{2\beta\gamma^2(1-\beta^2\cos^2\theta)^2}\{\tfrac{1}{2}F_0 + \zeta_+\zeta_-[(-1+\beta^4+\beta^4\sin^4\theta + 2\beta^2\sin^2\theta\cos^2\theta)f + \gamma^{-2}(\beta^2\sin^4\theta-\gamma^{-2})h_- + \beta^2\sin^4\theta h_+ + 2\beta^2\sin^3\theta\cos\theta\, g/\gamma]\}, \qquad (10)$$

where $\zeta_\pm = |\boldsymbol{\zeta}_\pm|$ is the degree of polarization of the positron and electron.[3)] For $\vartheta_\pm = 0$ (longitudinal polarizations of the electron and positron) and for $\vartheta_\pm = \pi/2$, $\zeta_\pm = 1$ (transverse polarizations) we get the well known formulas of Page.[1,10]

The expressions for the functions F_J in (2a)—(2c) are easily transformed to three-dimensional notation using the relation (7). We give the complete expressions for F_I, F_{II}, and F_{III}, while we give only special cases for F_{IV} (the complete expression for F_{IV} in the cms is given in the Appendix)

$$\tfrac{1}{32}\omega^{-8}F_I = b - b^2(\mathbf{e}_1\mathbf{e}_2)^2 + 4b\beta^2(\mathbf{n}\mathbf{e}_1)(\mathbf{n}\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) - 4\beta^4(\mathbf{n}\mathbf{e}_1)^2(\mathbf{n}\mathbf{e}_2)^2, \qquad (11a)^*$$

$$\tfrac{1}{32}\omega^{-8}F_{II} = b\beta\gamma^{-1}(\mathbf{e}_1\mathbf{e}_2)\,|[\mathbf{e}_1\mathbf{e}_2]|\{-\gamma(\mathbf{n}\boldsymbol{\zeta}_+) + (\mathbf{n}_1\mathbf{a}_+)\cos\theta - (\mathbf{n}\mathbf{e}_1)(\mathbf{e}_1\mathbf{a}_+) - (\mathbf{n}[\mathbf{e}_1\mathbf{n}_1])(\mathbf{a}_+[\mathbf{e}_1\mathbf{n}_1])\}$$
$$= \tfrac{1}{2}b\beta\gamma^{-1}\sin 2\psi\{-2\gamma\sin^2\theta\cos\vartheta_+ + \sin\vartheta_+\cos(\varphi_+-\Phi)\}, \qquad (11b)$$

$$F_{III} = F_{II}|_{\zeta_+\to\zeta_-}, \qquad (11c)$$

where $b = 1-\beta^2\cos^2\theta$, $\psi = \widehat{\mathbf{e}_1\mathbf{e}_2}$, $\mathbf{e}_1 = \mathbf{e}_x$.

Since F_{II} and F_{III} are equal to zero when $\mathbf{e}_1 \parallel \mathbf{e}_2$ or $\mathbf{e}_1 \perp \mathbf{e}_2$, the polarization effects in the annihilation of polarized positrons by unpolarized electrons (or vice versa) can in principle be observed by measuring the degree of relative linear polarization of the γ quanta at an angle of $\pi/4$, or the corresponding circular polarizations (cf. [3,4]). In the analysis of the general case, it is convenient for simplicity to choose for $\mathbf{e}_1$ and $\mathbf{e}_2$ the basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ defined in (8); as an example we give $d\sigma/d\Omega_1$ for $\mathbf{e}_1, \mathbf{e}_2 = \mathbf{e}_x, \mathbf{e}_y$:

$$(d\sigma/d\Omega_1)_{y,y} \sim \beta^2 b\cos^2\theta + \zeta_+\zeta_-\beta^2\cos^2\theta\{(\sin^2\theta-\gamma^{-2}\cos^2\theta)f - \sin^2\theta h_+ + \gamma^{-2}\cos^2\theta h_- - 2\gamma^{-1}\sin\theta\cos\theta g\}, \qquad (12a)$$

$$(d\sigma/d\Omega_1)_{x,x} \sim \beta^2\{\sin^2\theta\cos^2\theta + \gamma^{-2}(1+\sin^2\theta)^2 + \zeta_+\zeta_-[(\cos^2\theta\sin^2\theta - \gamma^{-2}(1+\sin^2\theta)^2)f + \gamma^{-2}(1+\sin^2\theta)h_- - \sin^2\theta h_+ + 2\gamma^{-1}\sin\theta\cos\theta(1+\sin^2\theta)g]\}, \qquad (12b)$$

and also for $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, and $\mathbf{e}_1 = \mathbf{e}_y$, $\mathbf{e}_2 = \mathbf{e}_x$:

$$(d\sigma/d\Omega_1)_{\substack{x,y\\(y,x)}} \sim b + \zeta_+\zeta_-[(\beta^2\sin^2\theta-\gamma^{-2})f + \gamma^{-2}h_- + \beta^2\sin^2\theta h_+ \pm 2g_-\beta\gamma^{-1}\sin\theta], \qquad (12c)$$

where

$$g_- = \cos\vartheta_-\sin\vartheta_+\cos(\varphi_+-\Phi) - \cos\vartheta_+\sin\vartheta_-\cos(\varphi_--\Phi).$$

These formulas coincide with the corresponding formulas of Theis[3] if we take account of the note to formula (10) of this paper.

If in the experimental arrangement for studying two-photon annihilation no plane is distinguished (i.e., one does not observe the directions of **p**, **q**, $\mathbf{k}_1$), then $\sum_{\mathbf{e}_1} d\sigma/d\Omega_1 \sim \mathbf{e}_2$, i.e., the analysis of the linear polarization of the second photon, shows the natural polarization, in agreement with the general theoretical arguments in the nonrelativistic limit[12] (cf. the case of circular polarizations[4])—it is sufficient to assign the direction. But when there is such a plane, for example $\{\mathbf{n}, \mathbf{n}_1\}$, $\sum_{\mathbf{e}_1(\mathbf{e}_2)} d\sigma/d\Omega_1$ will be dependent on the state of polarization of the second photon relative to this plane.

Using formulas (12a)—(12c), it is easy to get the degree of linear polarization of one of the photons in the annihilation plane (for the definition of ξ_3, cf., for example, [10]):

$$\xi_3 = \Big[\sum_{e_2} d\sigma(\mathbf{e}_1=\mathbf{e}_x) - \sum_{e_2} d\sigma(\mathbf{e}_1=\mathbf{e}_y)\Big]\Big/\sum_{e_1e_2} d\sigma$$
$$= \frac{4\beta\sin\theta}{\gamma(F_0+F_\zeta)}\Big\{\frac{\beta}{\gamma}\sin\theta + \zeta_+\zeta_-\Big[g_- + \beta\cos\theta g_+ + \frac{\beta}{\gamma}\sin\theta(h_- - f)\Big]\Big\}. \qquad (13)$$

In conclusion we consider the nonrelativistic approximation for the differential annihilation cross section, which is gotten from formula (6) in the cms when $\beta \to 0$ (cf. also the Appendix):

$$(d\sigma/d\Omega_1)_{\text{nonrel}} = \tfrac{1}{4}r_0^2\beta^{-1}\{[1-(\mathbf{e}_1\mathbf{e}_2)^2][1+(\boldsymbol{\zeta}_+\boldsymbol{\zeta}_-) - 2(\mathbf{n}_1\boldsymbol{\zeta}_+)(\mathbf{n}_1\boldsymbol{\zeta}_-)] + 2[(\mathbf{e}_1\mathbf{e}_2)((\mathbf{e}_1\boldsymbol{\zeta}_+)(\mathbf{e}_2\boldsymbol{\zeta}_-) + (\mathbf{e}_2\boldsymbol{\zeta}_+)(\mathbf{e}_1\boldsymbol{\zeta}_-)) - (\mathbf{e}_1\boldsymbol{\zeta}_-)(\mathbf{e}_1\boldsymbol{\zeta}_+) - (\mathbf{e}_2\boldsymbol{\zeta}_+)(\mathbf{e}_2\boldsymbol{\zeta}_-)]\}. \qquad (14)$$

It is easy to see that for $\mathbf{e}_1 \parallel \mathbf{e}_2$

$$(d\sigma/d\Omega)_{\text{nonrel}} = 0,$$

while for $\mathbf{e}_1 \perp \mathbf{e}_2$

$$d\sigma/d\Omega_{1\,\text{nonrel}} = \tfrac{1}{4}r_0^2\beta^{-1}[1+(\boldsymbol{\xi}_+\boldsymbol{\xi}_-) - 2((\mathbf{n}_1\boldsymbol{\zeta}_+)(\mathbf{n}_1\boldsymbol{\zeta}_-) + (\mathbf{e}_1\boldsymbol{\zeta}_+)(\mathbf{e}_1\boldsymbol{\zeta}_-) + (\mathbf{e}_2\boldsymbol{\zeta}_+)(\mathbf{e}_2\boldsymbol{\zeta}_-))] = \tfrac{1}{4}r_0^2\beta^{-1}[1-(\boldsymbol{\zeta}_+\boldsymbol{\zeta}_-)],$$

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3)To compare these formulas with formula (5) in Theis' paper,[3] we must make the following substitutions: $\zeta_\pm \to \pm 1$, $\vartheta_- \to -\bar{\vartheta}$, $\varphi_- \to -\bar{\varphi}$. This corresponds to assigning the vector ζ relative to the axis $z = -\mathbf{n}$. We note that there is a misprint in the last term of formula (5) in [3].

*(**ne**) = **n**·**e**; [**ne**] = **n** × **e**.

where the last equality follows from the fact that $\mathbf{e}_1, \mathbf{e}_2 \perp \mathbf{n}_1$. Formula (14) directly verifies the general conclusions of Fano[12] regarding the nature of the polarization of the annihilation γ quanta (cf. also [10]).

The well known formula for the total annihilation cross section in this approximation is obtained from (14) after summing over the directions of polarization of the γ quanta and integrating over angles

$$\sigma_{\text{nonrel}} = (\pi r_0^2/\beta)\,[1 - (\zeta_+\zeta_-)]. \tag{15}$$

Formula (15) is also of course obtained from (9) for $\beta \rightarrow 0$.

In the case where $\boldsymbol{\zeta}_+$ and $\boldsymbol{\zeta}_-$ are perpendicular to the plane $\{\mathbf{e}_2, \mathbf{n}_1\}$, the differential cross section $(d\sigma/d\Omega_1)_{\text{nonrel}}$ has the form

$$(d\sigma/d\Omega_1)_{\text{nonrel}} = (r_0^2/4\beta)\,(1 - \varepsilon)\sin^2\psi,$$

where $\boldsymbol{\zeta}_+ = \pm\boldsymbol{\zeta}_- = \epsilon\boldsymbol{\zeta}_-$, $\psi = \widehat{\mathbf{e}_1\mathbf{e}_2}$.

It is not difficult to get the expressions for $d\sigma/d\Omega_1$ in the ultrarelativistic limit, when $\beta \rightarrow 1$, $\gamma \gg 1$; no essential simplifications of the cross section formula occur.

3. The transition to the laboratory coordinate system (l.s.): $\mathbf{p} = 0$ or $\mathbf{q} = 0$. The expression for the cross section $d\sigma/d\Omega_1$ in this system can be gotten from the formulas in the cms by a simple Lorentz transformation. But it is much simpler for this purpose to use formulas (2), (5) and (6), which are valid for any $\mathbf{p}$ and $\mathbf{q}$, and the relations

$$E = m(1 + \gamma_+) = \omega_1 + \omega_2, \qquad \mathbf{n}' = \mathbf{n}_1' + \mathbf{n}_2',$$

where $\gamma_+ = \epsilon_+'/m$, $\epsilon_-' = m$, $\mathbf{n}' = \mathbf{q}'/|\mathbf{q}'|$, $\mathbf{n}_i' = \mathbf{k}_i'/\omega_i$.

From the equation $(pk_2) = (qk_1)$ (which follows from (3)) we have

$$\omega_2 = \omega_1\gamma_+(1 - \beta_+\cos\theta_1) \equiv \omega_1\gamma_+ z,$$

where $\theta_1 = \widehat{\mathbf{k}_1'\mathbf{q}'}$, $\beta_+ = |\mathbf{q}'|/\epsilon_+'$; in addition we have $\partial E_f/\partial\omega_1 = m(\omega_1^{-1} + \omega_2^{-1}) = 1 - \cos\Theta = \lambda$, $\Theta = \widehat{\mathbf{n}_1'\mathbf{n}_2'}$.

The differential cross section for annihilation, summed over the polarization directions of the photons, in the l.s. has the form

$$d\sigma_\zeta/d\Omega_1 = (F_0' + F_\zeta')\,r_0^2/2\beta\gamma_+^2 z, \quad F_0' = (1 + \gamma_+) + (2 - \lambda - 2/\lambda)$$
$$F_\zeta' = (2 - \lambda - 2/\lambda)\,[\mathbf{n}'\zeta_-]\,[\mathbf{n}'\zeta_+] + [\gamma_+(1 - \lambda) + (\gamma_+ z - \gamma_+ + 1)(1 - 2/\lambda)/\gamma_+ z]\,(\zeta_-\mathbf{n}')\,(\zeta_+\mathbf{n}')$$
$$+ (2/\lambda - 1)\,[(1 + \gamma_+)\,(\mathbf{n}_1'\zeta_-)\,(\mathbf{n}_1'\zeta_+) - \beta\gamma\,((\mathbf{n}_1'\zeta_-)\,(\mathbf{n}'\zeta_+) + (\mathbf{n}_1'\zeta_+)\,(\mathbf{n}'\zeta_-))]/\gamma_+ z. \tag{16}$$

As an example, let us consider the summation over the polarization directions of the photons in the l.s.:

$$\sum_{e_2} \frac{d\sigma}{d\Omega_1}\bigg|_{\zeta_\pm = 0} \sim \frac{(1 + \gamma_+ z)^2}{\gamma_+ z} - 2\,[(\mathbf{e}_1\mathbf{e}_x')^2\cos^2\Theta + (\mathbf{e}_1\mathbf{e}_y')^2], \tag{17}$$

where the basis vectors for the polarizations of the photons are chosen as follows:

$$\mathbf{e}_y' \perp \mathbf{n}', \mathbf{n}_1', \qquad \mathbf{e}_x' \perp \mathbf{n}_1', \qquad \mathbf{e}_x' \in \{\mathbf{n}', \mathbf{n}_1'\}, \tag{17a}$$

$$\mathbf{e}_y'' = \mathbf{e}_y', \qquad \mathbf{e}_x'' = \mathbf{n}_1'\sin\Theta - \mathbf{e}_x'\cos\Theta. \tag{17b}$$

From formula (17) it is easy to find the degree of linear polarization of the photon relative to the plane $\{\mathbf{n}', \mathbf{n}_1'\}$.

In conclusion I want to thank A. A. Ansel'm and Yu. N. Demkov for useful criticism and comments.

APPENDIX

In the cms, F_{IV} has the form

$$\tfrac{1}{32}\omega^{-8}F_{IV} = [b - s^2]\,[(\mathbf{a}_+\mathbf{a}_-) + 2(\gamma^2 - 1)\,(\mathbf{n}\zeta_+)\,(\mathbf{n}\zeta_-)]$$
$$+ [(\mathbf{e}_1\mathbf{e}_2)\,s - 1]\,[(nn_2)\,(n_1a_+)\,(n_2a_-) + (nn_1)\,(n_2a_+)\,(n_1a_-)]$$
$$+ 2\beta^2\,[(\mathbf{n}\mathbf{e}_1)^2\,(n_2a_+)\,(n_2a_-) + (\mathbf{n}\mathbf{e}_2)^2\,(n_1a_-)\,n_1a_+)]$$
$$- 2s\,[(nn_2)\,(\mathbf{e}_1\mathbf{a}_+)\,(\mathbf{e}_2\mathbf{a}_-) + (nn_1)\,(\mathbf{e}_2\mathbf{a}_+)\,(\mathbf{e}_1\mathbf{a}_-) + \beta\,(\mathbf{n}\mathbf{e}_1)\,t_1$$
$$+ \beta\,(\mathbf{n}\mathbf{e}_2)\,t_2] + 2\beta\,[(\mathbf{n}\mathbf{e}_1)\,r_1 + (\mathbf{n}\mathbf{e}_2)\,r_2]$$
$$- 2b\,[(\mathbf{e}_1\mathbf{a}_-)\,(\mathbf{e}_1\mathbf{a}_+) + (\mathbf{e}_2\mathbf{a}_+)\,(\mathbf{e}_2\mathbf{a}_-)],$$

where

$$s = b\,(\mathbf{e}_1\mathbf{e}_2) - 2\beta^2\,(\mathbf{n}\mathbf{e}_1)\,(\mathbf{n}\mathbf{e}_2),$$
$$r_1 = (nn_1)\,(\mathbf{e}_1\mathbf{a}_-)\,(n_2a_+) - (nn_2)\,(\mathbf{e}_1\mathbf{a}_+)\,(n_2a_-),$$
$$t_1 = (\mathbf{e}_1\mathbf{a}_+)\,(n_1a_-) - (\mathbf{e}_1\mathbf{a}_-)\,(n_1a_+),$$

r_2 and t_2 are gotten, respectively, from r_1 and t_1 by the replacement $\mathbf{n}_1 \leftrightarrow \mathbf{n}_2$, $\mathbf{e}_1 \leftrightarrow \mathbf{e}_2$; n, n_1, n_2 are four-vectors with components $n_1, n_2 = (\pm\mathbf{n}, 1)$, $n = (\beta\mathbf{n}, 1)$.

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