APPLICATION OF THE METHOD OF MAJORIZATION OF FEYNMAN DIAGRAMS FOR PROVING THE DISPERSION RELATIONS

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The method of majorization of Feynman diagrams is used for the investigation of elastic scattering of pseudoscalar π and K mesons by K mesons. Primitive diagrams for these processes have been found and their analytical properties have been studied.

INTRODUCTION

THE method of majorizing diagrams arising from a perturbation theory series has been proposed by Nambu[1], and was then developed in the papers of Symanzik[2], Logunov, Tavkhelidze, Todorov, and Chernikov[3,4], and Wu[5]. The method consists of finding strongly connected diagrams with the smallest domain of analyticity in the case of Euclidean external momenta, the so-called majorizing diagrams. A study of the analytic properties of these diagrams has made it possible to prove the dispersion relations for the vertex parts and for the NN-scattering amplitude[1,2,6]. It was not possible to achieve this earlier on the basis only of general principles of field theory.

In this paper the method of majorization of Feynman diagrams is applied to the study of elastic πK and KK scattering, with both mesons being pseudoscalar. The primitive (majorizing) diagrams for these processes have been found, and one-dimensional dispersion relations have been proved.

MAJORIZING DIAGRAMS FOR ELASTIC SCATTERING OF π AND K MESONS BY K MESONS

We denote by R the set of all strongly connected diagrams for the processes

$$K + K \rightarrow K + K, \qquad (1)$$
$$K + \pi \rightarrow K + \pi. \qquad (2)$$

We shall assume that the diagrams of class R do not contain any closed baryon paths (cycles). This is not a restriction, since diagrams with an odd number of baryon lines in a cycle are excluded from consideration in accordance with Furry's theorem, while diagrams with an even number of baryon lines in a cycle can be majorized by means of diagrams with meson cycles[7] which are already contained in the class R.

By arbitrarily choosing the K-meson polygon (a certain broken line path consisting of lines characterized by strangeness and connecting external[1)] K-meson vertices) it is possible to assert that the remaining lines characterized by strangeness are closed, and that they can be replaced by π-meson cycles. The diagram obtained in this way belongs to R and majorizes the original diagram, since a decrease in the masses characterizing the internal lines diminishes the domain of analyticity of the diagrams[4]. This enables us to restrict our discussion to the class R′, each diagram of which can contain K-meson lines only as components of a strangeness polygon. We note that in the case of KK scattering the polygons can intersect one another.

By means of methods described in [1-9] it can be shown that all the diagrams of the class R′ are majorized by a finite set of diagrams defined by the following theorems I and II the proofs of which are omitted since they are quite cumbersome.

<u>Theorem I</u>. Any diagram belonging to the class R′ for the process (1) is majorized by at least one of the following diagrams:

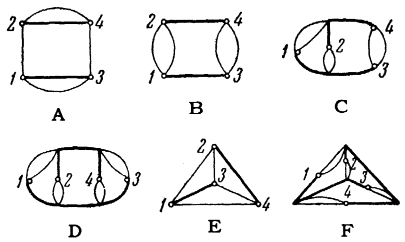

FIG. 1

[1)]We denote by external vertices those vertices which involve external lines. On the diagrams they are shown by circles.

Theorem II. Any diagram of the class R′ for the process (2) is majorized by at least one of the following diagrams:

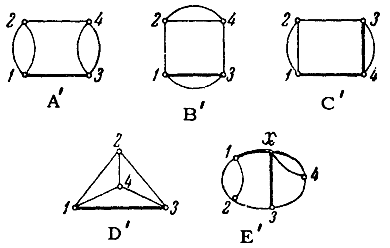

FIG. 2

Further, by utilizing Symanzik's theorem [2] and its generalization [4] it can be shown that for the domains of analyticity of the diagrams of Fig. 1 the following relations hold 2)

$$\Delta_{\rm D}^{KK(K\bar K)}(1a, bc) \supset \Delta_{\rm B}^{KK(K\bar K)}(1a, bc),$$
$$\Delta_{\rm F}^{KK(K\bar K)}(1a, bc) \supset \Delta_{\rm E}^{KK(K\bar K)}(1a, bc),$$
$$\Delta_{\rm C}^{KK}(13,24) \supset \Delta_{\rm B}^{KK}(13,24) \cap \Delta_{\rm B}^{KK}(14,23),$$
$$\Delta_{\rm E}^{KK}(13,24) \supset \Delta_{A}^{KK}(13,24) \cap \Delta_{A}^{KK}(14,23),$$
$$\Delta_{\rm C}^{K\bar K}(13,42) \supset \Delta_{\rm B}^{K\bar K}(13,42) \cap \Delta_{\rm B}^{K\bar K}(12,43),$$
$$\Delta_{\rm E}^{K\bar K}(13,42) \supset \Delta_{\rm B}^{K\bar K}(13,42) \cap \Delta_{\rm B}^{K\bar K}(13,24).$$

It can be verified that for any arbitrary permutation of the external junctions in the diagrams C and E diagrams A or B can be found which belong to the same class as C and E and whose intersection majorizes the latter diagrams.

For the domains of analyticity of the diagrams of Fig. 2 we have

$$\Delta_{\rm D'} \supset \Delta_{A'}(13,24) \cap \Delta_{A'}(13,42).$$

We note that the diagram C′ is obtained from E′ by the contraction of the meson lines (4, x) at the point x and that therefore

$$\Delta_{\rm C'} \supset \Delta_{\rm D'}.$$

Corollaries: 1. The intersection of the domains of analyticity of the diagrams A and B gives the domain of analyticity of the amplitudes of elastic KK and $K\bar K$ scattering

$$\Delta^{KK} = \Delta_{A}^{KK} \cap \Delta_{\rm B}^{KK},$$
$$\Delta^{K\bar K} = \Delta_{A}^{K\bar K} \cap \Delta_{\rm B}^{K\bar K}. \qquad (3)$$

2. The intersection of the domains of analyticity of the diagrams A′, B′, E′ determines the domain of analyticity for the amplitude of elastic πK scattering

$$\Delta^{\pi K} = \Delta_{A'}^{\pi K} \cap \Delta_{\rm B'}^{\pi K} \cap \Delta_{\rm E'}^{\pi K}, \qquad (4)$$

where $\Delta^{\rm KK}$, $\Delta^{\rm K\bar K}$, and $\Delta^{\pi \rm K}$ are domains in which any arbitrary strongly connected diagrams for the corresponding processes are analytic functions.

DISPERSION RELATIONS FOR THE ELASTIC SCATTERING OF K MESONS BY K AND $\bar{\rm K}$ MESONS

In order to establish a certain real domain of analyticity of the KK and $K\bar K$ scattering amplitudes in the space of the scalar products of external momenta lying on the mass surface we shall utilize relation (3). In this case the problem reduces to the determination of the domains $\Delta_{\rm A}^{\rm KK}$, $\Delta_{\rm B}^{\rm KK}$ and $\Delta_{\rm A}^{\rm K\bar K}$, $\Delta_{\rm B}^{\rm K\bar K}$.

The domain $\Delta_{\rm i}^{\rm KK}$ (i = A, B) is the intersection of the domains of analyticity of four topologically equivalent diagrams with the following notation for the external junctions: i (13, 24), i (14, 23), i (13, 42) and i (14, 32). Similarly $\Delta_{\rm i}^{\rm K\bar K}$ is also determined by the intersection of the domains of analyticity of the diagrams: i (13, 24); i (13, 42); i (12, 34); i (12, 43).

In order to obtain the domain $\Delta_{\rm A}^{\rm KK}$ it is sufficient to consider the domain of analyticity of only a single diagram, for example $\Delta_{\rm A}^{\rm KK}(13, 24)$, since the other domains are obtained from this one by a permutation of the external momenta.

Following the paper of Logunov et al [6] we consider the case when the squares of the external momenta are equal

$$P_1^2 = P_2^2 = P_3^2 = P_4^2 = \tfrac{1}{4}(S + T + U),$$

where $P_{\rm i}$ are Euclidean vectors, while S, T, U are invariants corresponding to Mandelstam's s, t, u (the connection between $P_{\rm i}$, S, T, U and s, t, u is established in [6]).

Moreover, since in virtue of Symanzik's theorem the self-energy part in the diagrams can be replaced by a single line characterized by total mass without altering the domain of analyticity, 3) we consider instead of the diagram A (B) the square diagram with masses μ and M + μ (2μ and M). Omitting all computations, since a diagram of this type has been studied in [6], we shall state the results.

2) $\Delta_{\rm i}$(1a, bc) denotes the domain of analyticity of the diagrams i = A, B, . . . , F; A′, . . . E′. a, b, c is a permutation of the numbers (2, 3, 4) corresponding to the positions of the external vertices in the diagrams. The expressions $\Delta_{\rm i} \cap \Delta_{\rm g}$ and $\Delta_{\rm i} \supset \Delta_{\rm k}$ denote respectively the intersection of the regions $\Delta_{\rm i}$ and $\Delta_{\rm k}$ and the inclusion of the region $\Delta_{\rm k}$ in the region $\Delta_{\rm i}$.

3) In the paper by Drabkin and Yappa [8] this assertion is proved without utilizing Symanzik's theorem.

In the real (s, t) plane Δ_A^{KK} and Δ_B^{KK} are triangular regions defined respectively by the inequalities

$$s < 4(M+\mu)^2, \quad t < 4\mu^2, \quad u < 4\mu^2,$$
$$s < 4M^2, \quad t < 16\mu^2, \quad u < 16\mu^2;$$
$$s + t + u = 4M^2.$$

We can now formulate the following theorem: any diagram for the elastic scattering of a pseudoscalar K meson by another K meson is an analytic function of s and t in the domain

$$\Delta^{KK}: \quad s < 4M^2, \quad t < 4\mu^2, \quad u < 4\mu^2 \tag{5}$$

in the plane $s+t+u = 4M^2$.

Having established the domain of analyticity of (5) we can write the dispersion relation for the scattering amplitude $T_{KK}(s,t)$ with respect to one of the variables while the second variable is held fixed. For example, the dispersion relation with respect to s for fixed t in the domain

$$-4\mu^2 < t < 4\mu^2 \tag{6}$$

has the form

$$\mathrm{Re}\, T_{KK}(s, t) = \frac{1}{\pi} P \left\{ \int\limits_{-\infty}^{4(M^2-\mu^2)-t} + \int\limits_{4M^2}^{\infty} \right\} \frac{\mathrm{Im}\, T_{KK}(s'k)}{s'-s} ds'. \tag{7}$$

Similarly, for a fixed s in the interval

$$4(M^2 - 2\mu^2) < s < 4M^2$$

the dispersion relation with respect to t has the same form (7), but with the limits of integration respectively given by $(-\infty, 4(M^2-\mu^2)-s)$ and $(4\mu^2, \infty)$.

From the inequality (6) it can be seen that the dispersion relations for a fixed t can be proved in perturbation theory if Δ_{max}^2 does not exceed μ^2, where Δ is the transferred momentum in Breit's system and $\Delta^2 = -t/4$.

By the same method we find that the domain of analyticity of $\Delta_{K\bar{K}}$ in the plane $s+t+u = 4M^2$ has the form of a triangle and is defined by the inequalities

$$s < 4\mu^2, \quad t < 4\mu^2, \quad u < 4M^2.$$

DISPERSION RELATIONS FOR THE ELASTIC SCATTERING OF π MESONS BY K MESONS

We have considered the case when $P_1^2 = P_3^2$, $P_2^2 = P_4^2$. In accordance with formula (4) in order to establish the analyticity properties for the πK scattering amplitude it is necessary to investigate the diagrams A′, B′, and E′ (Fig. 2). It can be easily noted that the domain $\Delta_i^{K\pi}$ (i = A′, B′, E′) is the intersection of the domains $\Delta_i^{K\pi}(13, 24)$ and $\Delta_i^{K\pi}(13, 42)$.

We now investigate diagrams A′ and B′. The improper domain of analyticity of the reduced diagrams in the plane $s+t+u = 2(M^2+\mu^2)$ is defined by the inequalities

$$s < (M+\mu)^2, \quad u < (M+\mu)^2, \quad t < 4(2\mu)^2$$

and

$$s < (M+3\mu)^2, \quad u < (M+3\mu)^2, \quad t < 4\mu^2,$$

respectively for the diagrams A′ and B′, while the region in which they intersect is defined by

$$s < (M+\mu)^2, \quad u < (M+\mu)^2, \quad t < 4\mu^2. \tag{8}$$

In order to obtain the proper singularities of the diagram A′ it is necessary to investigate the maximum with respect to α_i of the quadratic form of the Euclidean vectors

$$Q(\alpha_i P_i) = \left(\sum_{i=1}^{4} \alpha_i\right)^{-1} \{\alpha_2(\alpha_1+\alpha_3) P_2^2 + \alpha_4(\alpha_1+\alpha_3) P_1^2 + \alpha_2\alpha_4 S + \alpha_1\alpha_3 T\} - (\alpha_1+\alpha_3)(2\mu)^2 - \alpha_2\mu^2 - \alpha_4 M^2,$$

with all $\alpha_i > 0$. It can be easily verified that the equations

$$\partial Q(\alpha_i, P_i)/\partial\alpha_i = 0 \qquad (i = 1, 2, 3, 4) \tag{9}$$

with respect to α_i have positive roots only provided

$$P_1^2 > M^2 + 4\mu^2, \quad P_2^2 > 5\mu^2. \tag{10}$$

The curve of singularities obtained from the condition of vanishing of the determinant of the system (9), taking (10) into account, nowhere intersects the region (8). As an example we consider the region $t < 0$. As has been shown in [6], for the Euclidean momenta in this domain we can take $P_1^2 = M^2 - t/4$ and $P_2^2 = \mu^2 - t/4$ and then for the compatibility of Eqs. (9) with $\alpha_i > 0$ it is necessary to have $-t/4 > 4\mu^2$. Thus, only those parts of the singularities curve have a physical meaning which lie below $t = -16\mu^2$. On the other hand, it follows from (8) that the minimum transferred momentum is defined by the value $t = -4M\mu$, and since $M < 4\mu$, then the triangular domain of analyticity (8) lies above the line $t = -16\mu^2$.

All the properties of the diagram B′ are obtained from the results given above by replacing M by $M+\mu$ and $\mu \rightleftarrows 2\mu$. The condition that the roots of the equations of the type (9) should be positive assumes for the diagram B′ the form

$$P_1^2 > (M+\mu)^2 + \mu^2, \quad P_2^2 > 5\mu^2,$$

and in this case the curve of the proper singularities does not intersect the region (8).

We now consider the diagram E′. It can be easily shown that Landau's equations [10]

$$\alpha_i q_i = 0, \qquad q_i^2 = m_i^2, \qquad \alpha_i > 0$$

for the triangular part (Fig. 2) are satisfied for $P_2^2 > 5\mu^2$, and this is sufficient in order that the curve of the proper singularities of the diagram E′ should not intersect the region (8).

Thus, we have shown that the region (8) is the minimum common domain of analyticity of all the strongly connected diagrams for πK scattering and, consequently, we can write the dispersion relation for the πK-scattering amplitude.

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303