INSTABILITY THEORY FOR A LOW-PRESSURE INHOMOGENEOUS PLASMA IN A STRONG MAGNETIC FIELD

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The stability of an inhomogeneous plasma is considered. A method for constructing finite solutions that correspond to localized perturbations is presented. It is shown that a correspondence exists between these results and the results obtained by using the approximate quasiclassical theory in the same problem.[1-5] The shearing effect of nonparallel magnetic lines of force is taken into account; it is shown that shear effects lead to stabilization of the so-called universal instability of a low-pressure inhomogeneous plasma when particle collisions are neglected.

1. INTRODUCTION

Going beyond the framework of magnetohydrodynamics in the theory of plasma stability one finds new instabilities associated with the excitation of so-called drift waves.[1-5] In this case, in solving the problem of stability for a given plasma configuration, one makes use of an approximate quasiclassical theory and the dependence of the perturbed quantities on coordinates is written in the form

$$\psi(x, y, z, t) = \psi_0 \exp\left\{ i\int^x k_x(x)\,dx + ik_y y + ik_z z + i\omega t\right\}, \tag{1.1}$$

where the x axis is taken in the direction in which the plasma is inhomogeneous. In the final expressions for the growth rates of the kinetic instabilities and the frequencies of the oscillations that develop one introduces a total wave number $k_\perp(x) = \sqrt{k_x^2(x) + k_y^2}$ at a given point of the x axis [cf. (1.6)]. The exact value of the latter is determined by solving the equations for the perturbed quantities although, even if it is unknown, one can make certain qualitative statements about the stability of the plasma.[1-5]

Although the solution of the stability problem (but not that of determining the exact values of the growth rates) does not require the explicit dependence of the perturbed quantity $\psi(x)$ on the coordinate x one always requires that the solution be finite [$\psi(x) \to 0$ when $x \to \pm\infty$]. The finiteness of the solution itself must be obtained rigorously from the appropriate differential equation for $\psi(x)$ (we note that in the quasiclassical approximation (1.1) the latter can easily be obtained from the dispersion equations[2-4] by making the substitution $k_x \to -i\partial/\partial x$).

In the general case we obtain a differential equation of "infinite" order[1)] for $\psi(x)$; however, in two limiting cases: 1) the drift approximation $k_\perp r_i \ll 1$ ($r_i = \sqrt{T/m_i}\, m_i c/eH$ is the ion Larmor radius, and 2) in the approximation $k_\perp r_i \gg 1$, this equation reduces to a second-order equation

$$r_i^2 d^2\psi/dx^2 - [U(x, \omega, k) + iV(x, \omega, k)]\psi = 0, \tag{1.2}$$

which is in the form of the Schrödinger equation with a complex potential energy $U + iV$.[2)] Multiplying this by $\psi^*(x)$ and integrating between infinite limits we obtain for the finite solution the integral conditions

$$r_i^2 \int_{-\infty}^{+\infty} \left|\frac{d\psi}{dx}\right|^2 dx + \int_{-\infty}^{+\infty} U(x)|\psi|^2 dx = 0,$$
$$\int_{-\infty}^{+\infty} V(x)|\psi|^2 dx = 0. \tag{1.3}$$

In the usual quantum mechanical case we deal with (1.2) with $V(x) \equiv 0$. In the quasiclassical approximation for this case finite solutions always exist for a potential energy $U(x)$ in the form of a well. The condition for finiteness of the solution is found by joining the solutions that decay at both infinities with the solutions inside the well at the turning points x_1, x_2 [$U(x_1) = U(x_2) = 0$] and making use of the quasiclassical Bohr quantization rule

1) More precisely a second-order integro-differential equation (this is due to the fact that the particle density flux in Maxwell's equations is a functional of the electric and magnetic fields).

2) In which we also include the well known instabilities of the magnetohydrodynamic type.[6]

$$\int_{x_1}^{x_2} \sqrt{-U(x, \omega^{(p)}, k)}\, dx = \pi r_i \left(p + \frac{1}{2}\right). \quad (1.4)$$

In stability problems in which an equation of the form in (1.2) is obtained for the perturbed quantity with a real potential energy (cf., for example, [6]) it is not necessary to determine precisely the frequency spectrum $\omega^{(p)}$ from (1.4). It is sufficient to note that at the turning point $x_1^{(p)}$, which corresponds to a characteristic value of the frequency $\omega^{(p)}$, we have (cf. [8])

$$U(x^{(p)}, \omega^{(p)}, k) = 0, \quad (1.5)$$

from which it is possible to determine the sign of the imaginary part of the frequency and thus to solve the stability problem for a given plasma configuration.

If $V(x) \neq 0$, then the points at which the real and imaginary parts of the potential energy vanish are not the same in the general case; hence we do not have vanishing points for the total complex potential energy $U(x) + iV(x)$ on the real axis. However, if we regard $U(z) + iV(z)$ as a function of the complex variable $z = x + iy$ such points can exist in the complex plane z and by joining the quasiclassical solutions at these points we obtain the existence condition for finite solutions. The latter is the Bohr quantization condition over a length L of the real phase of the quasiclassical wave function $\operatorname{Im} \int_{z_2}^{z} \sqrt{-(U + iV)}\, dz = 0$ which connects the complex turning points.

After verifying the existence of finite solutions one can draw qualitative conclusions as to the stability of the inhomogeneous plasma from the following considerations. According to the integral condition (1.3), for each characteristic value of the frequency $\omega^{(p)}$ there is a point $x^{(p)}$ in the localization region of $\psi(x)$ such that[3)]

$$V(x^{(p)}, \omega^{(p)}, k) = 0, \quad \operatorname{Re} k_x^2(x^{(p)}) + U(x^{(p)}, \omega^{(p)}, k) = 0, \quad (1.6)$$

where the second condition is obtained from (1.2). We note that (1.6), which plays the same role as (1.5) in the stability problem with $V(x) \neq 0$ is written now not at the turning point, but at the point at which the imaginary part of the potential energy vanishes. In order to find the exact boundaries of the instability and the growth rates we must solve the problem starting from the analytic dependence of $U(z)$ and $V(z)$ and the Bohr quantization condition.

We now consider the case in which the real part of the potential energy is in the shape of a hill. The nontrivial question now arises of whether finite solutions are possible if there is an imaginary part $V(x)$ (it is known from quantum mechanics that if $V(x)$ vanishes there are no such solutions in this case). We consider this problem using the simple example of the equation of an inverted oscillator:

$$d^2\psi/dx^2 + 2\{E + {}^1/_2\, \Omega^2 x^2 e^{i\delta}\}\psi = 0. \quad (1.7)$$

Assume first that $\delta = 0$. Then, on the lines $z^4/|z^4| = -1$ the relative sign of the second derivative $d^2\psi/dz^2$ (with respect to the potential energy $\Omega^2 z^2/2$) changes and we obtain a potential well at these lines. The complex turning points lie on these lines ($\arg z = \pm \pi/4$) and the eigenvalues of the energy corresponding to solutions that are finite on these lines are $E = \mp i\Omega(p + 1/2)$. The position of the Stokes lines M and the lines L of the real phase for $E = i\Omega(p + 1/2)$ are shown in Fig. 1.

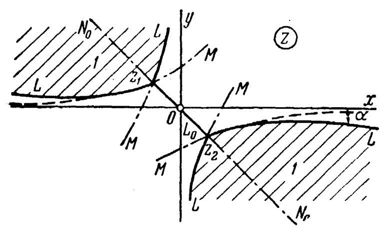

FIG. 1

Since the asymptotes of the L lines are in this case the semi-axes of x, it is sufficient to take an arbitrarily small negative imaginary part $V(x) = -|\delta|\Omega^2 x^2/2$ in order that the lines L rotate and that the real semi-axes of x lie in the region 1 where all solutions decay as $|z| \to \infty$.

In the stability problem the approximation for the potential energy $U_1 + iV_1$ by an oscillator

$$U_1(x) + iV_1(x) = {}^1/_2\Omega^2 x^2 e^{i\delta} \quad (1.8)$$

holds only in the finite region $|x| < R$ while the potential goes to the constant value $U_1(\infty) + iV_1(\infty)$ when $|x| \to \infty$. In this case, for the lowest levels the turning points z_1 and z_2 do not go outside the region of validity of the approximation in (1.8) and the characteristic energy values are again $E = \mp i\Omega(p + 1/2)$ (for $V_1(x) \equiv 0$). But the asymptotes

[3)]Conditions like (1.6) derived earlier[2-4] without analysis of the spatial behavior of the function $\varphi(x)$ remain formally valid when $k_y^2 \to \operatorname{Re} k_\perp^2 = \operatorname{Re} k_x^2 + k_y^2$. However, the problem of forming a finite solution for the integro-differential equation obtained in this particular case can be solved by means of (1.1).

of the lines of real phase L now make some angle α; hence when $V_1(x) \equiv 0$ there are no finite solutions and we require a finite imaginary part $V_1(x)$ of definite sign so that rotation of the line L allows the real semiaxes $x \to \pm\infty$ to lie inside the region of finite solutions.

To determine which of the solutions is finite for a given sign of $V_1(x)$ we make use of the quasiclassical solutions continued analytically from region 1 to the x axis:

$$\psi \approx \frac{\text{const}}{(E-U-iV)^{1/4}} \exp\left\{\pm i \int^x \sqrt{2[E-U_1(x)-iV_1(x)]}\, dx\right\} \xrightarrow[x\to\infty]{} \text{const} \exp\{\pm ix\sqrt{2[E-U_1(\infty)-iV_1(\infty)]}\},$$

which correspond to the characteristic values $E = \mp i\Omega(p + 1/2)$ [if the imaginary part $V_1(x) = 0$ these solutions must grow with x along the real axis, and, as follows from this expression, the angle α of the asymptote to the line L is of order arc tan $(|E|/|U_1(\infty)|)$].

We see that when $|V_1(\infty)| > |E|$ and is negative [positive] the solution that corresponds to $E = -i\Omega(p + 1/2)$ [$E = i\Omega(p + 1/2)$] is finite. In instability problems we find $|V| \ll |U|$ so that local solutions can only exist at the hill itself (i.e., for small values of the quantum number p).

2. DERIVATION OF THE BASIC EQUATIONS

The equations we require can be obtained by the substitution $k_x \to -id/dx$ in the dispersion relations that have been obtained earlier;[2-5] for complete clarity, however, we shall derive them in the drift approximation[1] (the wavelength of a perturbation λ is much greater than the ion Larmor radius). The analysis is carried out as in[1-6] using the example of a plane low-pressure plasma layer $1 \gg \beta \equiv 8\pi nT/H^2 \gg m_e/m_i$ in a gravitational field g along the x axis.

The unperturbed distribution functions for the ions and electrons are taken in the form

$$f_{i,e}^{(0)}(x, v) = n(x)\left(\frac{m_{i,e}}{2\pi T(x)}\right)^{3/2} \times \exp\left\{-\frac{m_{i,e}(v_\parallel - v_{i,e}^{(0)})^2}{2T(x)} - \frac{m_{i,e}(v_\perp^2 + v_x^2)}{2T(x)}\right\}. \quad (2.1)$$

The lines of force of the magnetic field lie in the y-z plane and are straight lines, whose inclination angle $\theta(x)$ (with respect to the z axis) depends on x. This rotation of the lines of force is due to the current of electrons along the lines of force so that the inclination angle θ, in accordance with the Maxwell equation $\nabla \times \mathbf{H} = 4\pi\mathbf{j}/c$, is related to $v_e^{(0)}$ by

$$d\theta/dx = -(4\pi e n_0/cH_0)\, v_e^{(0)}(x); \qquad v_i^{(0)} \equiv 0.$$

Following the conventional approach in the linear theory of stability we superimpose small perturbations on the stationary background. We assume that the plasma density is so low that particle collisions can be neglected and write the kinetic equations for $f_j^{(1)}$, the correction to the ion and electron distribution functions in the drift approximation, neglecting the collision integral:

$$i(\omega + k_\parallel v_\parallel) f_j^{(1)} - \frac{ik_\perp g}{\omega_{Hj}} f_j^{(1)} + v_\parallel \frac{H_x}{H^{(0)}}\frac{df_j^{(0)}}{dx} + c\frac{E_\perp}{H^{(0)}}\frac{df_j^{(0)}}{dx} + \frac{e_j}{m_j}E_\parallel \frac{\partial f_j^{(0)}}{\partial v_\parallel} = 0;$$

$$k_\parallel = k_z + k_y\int \frac{d\theta}{dx}\,dx, \qquad \omega_{Hj} = \frac{e_j H^{(0)}}{m_j c}, \qquad j = \{i, e\}. \quad (2.2)$$

Here, we have used a local coordinate system $\{x, s_\perp, s_\parallel\}$ ($s_\parallel$ is a unit vector in the direction of the magnetic line of force, $s_\perp/|s_\perp| = [s_\parallel x]/|s_\parallel||x|$).

We consider the self-consistent problem so that the kinetic equations are supplemented by the Maxwell equations for the electric and magnetic fields associated with the perturbations **E** and **H**:

$$\frac{dH_\perp}{dx} - ik_\perp H_x = \frac{4\pi}{c} j_\parallel^{(1)}, \qquad j_\parallel^{(1)} = \sum_j e_j \int v_\parallel f_j^{(1)}\, d\mathbf{v}, \quad (2.3)$$

$$ik_\perp H_\perp + dH_x/dx = 0, \quad (2.4)$$

$$ik_\perp E_\parallel - ik_\parallel E_\perp = -i\omega H_x/c, \quad (2.5)$$

$$dE_\perp/dx - ik_\perp E_x = 0. \quad (2.6)$$

In these equations we have used the quasiclassical condition in going from the Cartesian variables to the local triads of coordinates. Furthermore, we have neglected the perturbation $H_\parallel$, as is valid when $\beta \ll 1$.

Finally, invoking the quasi-neutrality condition ($\int(f_i^{(1)} - f_e^{(1)})\,dv = 0$) the complete system of equations describing the plasma in the present approximations is written

$$ik_\parallel j_\parallel^{(1)} + i\frac{\omega}{\omega_{Hi}}\frac{cn_0}{H^{(0)}}\left(ik_\perp E_\perp + \frac{dE_x}{dx}\right) - \frac{i\omega}{\omega_{Hi}}\frac{c}{H}\left(\frac{d^2}{dx^2} - k_\perp^2\right)p^{(1)} - \frac{ik_\perp ge}{\omega_{Hi}}\int f_i^{(1)}\, d\mathbf{v} = 0,$$

$$p^{(1)} = \int \frac{m_i(v_\perp^2 + v_x^2)}{2} f_i^{(1)}\, d\mathbf{v}, \quad (2.7)$$

where the second and third terms come from the so-called second-order drift (inertia drifts), while the fourth term represents the drift due to the gravitational force.

Under the assumption that the derivatives with respect to x of the perturbed quantities are appreciably greater than the derivatives of the unperturbed quantities [quasiclassical approximation, which applies when $\lambda_x \ll R$; R is the distance in which there are appreciable changes in n(x), T(x), and $\theta(x)$], (2.3)—(2.8) can be reduced to the single fourth-order equation for frequencies $\omega \gg k_{\|}u_i$:

$$\left(1-\frac{\omega^2+k_\perp v_n(1+\eta)\omega}{k_\|^2V_A^2}\right)$$
$$\times\left[\frac{d^2}{dx^2}-k_\perp^2\left(1+\frac{gn'/n}{k_\|^2V_A^2-\omega^2-k_\perp v_n(1+\eta)\omega}\right)\right]$$
$$\times\left\{\frac{\omega-k_\perp v_n}{\omega}\frac{n}{T}-\left[\frac{\omega+k_\|v_e^{(0)}}{T}-\frac{k_\perp}{m_i\omega_{Hi}}\frac{d}{dx}\right]\int\limits_{-\infty}^{+\infty}\frac{f_e^{(0)}dv_\|}{\omega+k_\|v_\|}\right\}\psi$$
$$=\left(1+\frac{k_\perp v_n(1+\eta)}{\omega}\right)\left[\frac{d^2}{dx^2}-k_\perp^2\left(1-\frac{gn'/n}{\omega^2+k_\perp v_n(1+\eta)\omega}\right)\right]$$
$$\left(r_i^2\frac{d^2}{dx^2}-k_\perp^2r_i^2\right)\psi,$$

$$v_n(x)=k_\perp cTn'/eHn,\qquad n'=dn(x)/dx,\qquad \eta\equiv d\ln T/d\ln n,$$
$$V_A^2=H^2/4\pi nm_i,\qquad u_{i,e}=\sqrt{T/m_{i,e}},\qquad \psi(x)\equiv E_\perp(x). \tag{2.8}$$

This equation holds only for wavelengths $\lambda_x(x)$ that are appreciably greater than the ion Larmor radius. However, if the turning points of U(x) are far from each other this condition is violated in the region of the minimum of the potential well of U(x). Hence, we limit our analysis to the lowest levels, those almost at the bottom of the well. For these levels, using the quasiclassical Bohr quantization condition we can approximate U(x) in the entire region of integration by two terms of the expansion. The fact that (2.8) does not generally hold in the region of small gradients $n'/n \sim 0$ does not affect the validity of the results since the solution is negligibly small in this region.

Furthermore, we assume that the variation of density and temperature with the x coordinate is similar, that is to say $\eta \equiv d\ln T/d\ln n = \text{const}$ and that it is so small ($\Delta n \ll n$, $\Delta T \ll T$) that all the quantities that depend on it can be regarded as constant so long as they are not functions of their gradients.

3. INSTABILITY OF AN INHOMOGENEOUS PLASMA IN A STRONG MAGNETIC FIELD WITH PARALLEL LINES OF FORCE

In the preceding section we have obtained a differential equation for the perturbed quantities which, in the most interesting cases, reduces to a second-order equation (Schrödinger equation). In the theory of stability we are only interested in finite solutions that vanish at infinity. Hence, the entire stability problem for an inhomogeneous plasma reduces to an investigation of the characteristic values of the equations that are obtained.

Equation (2.8) includes the well-known magnetohydrodynamic instability of the flute type.[6] However, we shall not be interested in instabilities connected with the effect of the gravitational force which will therefore be neglected (with the exception of the last section of the paper, where we consider the effect of a weak gravitational force on the stabilization of drift instabilities).

In the absence of the gravitational field (g = 0) the instability is of oscillatory nature. Everywhere in what follows, in solving the problem of the characteristic values, we will use the laboratory coordinate system so that the frequencies in (2.8) are to be replaced, in accordance with the relation $\omega \to \omega_* = \omega - (1+\eta)\times k_\perp v_n(x)$ [(2.8) is written in the coordinate system in which $E_0 \equiv 0$]. Then, for perturbations in the intermediate frequency region ($k_z u_i < \omega_* < k_z u_e$), in terms of the approximations made above, we have from (2.8)

$$r_i^2\frac{d^2\psi}{dx^2}-\left\{k_y^2r_i^2+\left(1-\frac{\omega^2-k_yv_n(x)(1+\eta)\omega}{k_z^2V_A^2}\right)\right.$$
$$\times\left(1-\frac{(2+\eta)k_yv_n(x)}{\omega}\right.$$
$$-i\sqrt{\frac{\pi}{2}}\frac{\omega-k_yv_n(1+\eta)}{|k_z|u_e\omega}$$
$$\left.\times\left[\omega-2k_yv_n(1+\eta/4)\right]\right\}\psi=0. \tag{3.1}$$

A. We first consider the case of potential perturbations ($\nabla\times\mathbf{E}=0$, which applies when $\omega^* \ll k_zV_A$). In this case the real part of the potential energy U(x) is a well[4)] while the imaginary part V(x) is small so that the required finite solutions always exist. Because of the interaction of the wave with resonance electrons ($v_\| = [\omega - k_yv_n(x)(1+\eta)]/k_\|$) the amplitude can grow.

Mathematically this interaction is described by the residue in the integrals containing the electron distribution function. The growth rate of the instability can be determined from the integral condition (1.3):

[4)]Here, and everywhere below in the analysis of drift waves, we assume that $k_\perp v_n > 0$. As is evident from (3.1) and (3.8), the case $k_\perp v_n < 0$ coincides with the preceding if we make the formal substitution

$$\mathrm{Re}\,\omega\to-\mathrm{Re}\,\omega,\qquad \mathrm{Im}\,\omega\to\mathrm{Im}\,\omega,\qquad k_\perp v_n\to-k_\perp v_n.$$

$$\int_{-\infty}^{+\infty} |\psi|^2 \Big\{k_y v_n(x)(2+\eta)\frac{\mathrm{Im}\,\omega}{\mathrm{Re}\,\omega^2} - \sqrt{\frac{\pi}{2}}\frac{\mathrm{Re}\,\omega - k_y v_n(1+\eta)}{|k_z|u_e}\Big[1-\frac{2k_y v_n(1+\eta/4)}{\mathrm{Re}\,\omega}\Big]\Big\}dx = 0, \qquad (3.2)$$

which expresses the conservation of energy for the system consisting of the wave and the plasma particles. When $X \ll r_i\sqrt{m_i/m_e}\,|k_z|u_i/\omega_*$ (X is the distance between the zeroes of the function $U(x)$) we can apply perturbation theory and in this condition we can substitute the solution $\psi_0(x)$ and the frequency ω found from (3.1) neglecting the imaginary part $V(x)$.

However, as indicated in the introduction, we use an expansion of the function $k_y v_n(x)$ in a series about the point $x = x_0$ ($[n'(x_0)/n]' = 0$). Then our equation reduces to the equation for a harmonic oscillator with energy E and frequency Ω [cf. (1.7)]:

$$E = \frac{1}{2r_i^2}\Big[\frac{\omega_0(2+\eta)-\omega}{\omega}(1-i\gamma) + i\gamma\frac{\omega_0\eta}{2\omega}\Big] - \frac{k_y^2}{2},$$

$$\gamma = \sqrt{\frac{\pi}{2}}\frac{\omega-(1+\eta)\omega_0}{|k_z|u_e} \ll 1,$$

$$\Omega^2 = \frac{1}{r_i^2R^2}\Big\{\frac{(2+\eta)\omega_0}{\omega} - i\gamma\Big[\frac{(2+\eta)\omega_0}{\omega} + \frac{\omega_0^2(1+\eta)\eta}{2\omega(\omega-\omega_0(1+\eta))} - \frac{\omega_0\eta}{2\omega}\Big]\Big\},$$

$$k_y v_n(x) = \omega_0\Big(1-\frac{(x-x_0)^2}{R^2}\Big).$$

The finite solutions in this case are expressed in terms of the Hermite polynomials $H_p(x)$:

$$\psi_p(x) = (\Omega/\pi)^{1/4} H_p(x\sqrt{\Omega})\,2^{-p/2}(p!)^{-1/2}\exp\{-\Omega x^2/2\} \qquad (3.3)$$

(far from the turning point where the series expansion is not valid, the solution of (3.3) is replaced by

$$\psi_p(x) = \frac{\mathrm{const}}{|U+iV|^{1/4}} \times \exp\Big\{-\int_{z_2}^{x}\sqrt{U(z,\omega^{(p)},k) + iV(z,\omega^{(p)},k)}\,dz\Big\}\Big)$$

corresponding to characteristic frequencies

$$\omega^{(p)} = \omega_0(2+\eta) + {}^1/_2\, i\gamma\eta\omega_0 - k_y^2r_i^2\omega_0(2+\eta)(1+i\gamma) - 2r_iR^{-1}(p+{}^1/_2)(2+\eta) \times \omega_0\{1 - i\gamma\,[1+\eta^2/(4+2\eta)]\}^{1/2}(1+i\gamma). \qquad (3.4)$$

We see from the last expression that an instability occurs when

$$\eta < 4k_y^2r_i^2 + 4r_iR^{-1}(p+{}^1/_2), \qquad (3.5)$$

that is to say, it is of universal nature.

B. For perturbations that have a strong distorting effect on the lines of force of the magnetic field ($\omega^2 \gg k_z^2V_A^2$) the real part of the potential energy has the form of a hill when $\eta > -2$. Again expanding the function $k_y v_n(x)$ in a series, from (3.1) and the results of Sec. 1, we find that solutions with characteristic frequency

$$\omega^{(p)} = \omega_0(2+\eta) + {}^1/_2\, i\gamma\omega_0\eta + k_y^2r_i^2k_z^2V_A^2\omega_0^{-1}(1+i\gamma) + i2r_iR^{-1}(p+{}^1/_2)\sqrt{2+\eta}\{1-i\gamma\,[1+(2\eta^2+\eta)/(4+2\eta)]\}^{1/2} \times (1+i\gamma)|k_z|V_A \qquad (3.6)$$

increase with time and are finite when

$$\eta < -2k_y^2r_i^2k_z^2V_A^2/\omega_0^2 - 8r_iR^{-1}(\beta m_i/\pi m_e)^{1/2}(p+{}^1/_2)k_z^2V_A^2/\omega_0^2 < 0. \qquad (3.7)$$

If, however, the relative temperature gradient is twice as big as the relative density gradient, and in the opposite direction ($\eta < -2$) the real part of the potential energy is again a well; finite solutions exist and are unstable while the corresponding characteristic values of the frequency are obtained from (3.6) by the substitution $\sqrt{\eta+2} \to i\sqrt{-\eta-2}$.

C. In the case of high frequency perturbations ($\omega_* \gg k_z u_e$) for a plasma in which the density is not too low ($\beta \gg m_e/m_i$), such as that being considered here, (2.8) becomes

$$r_i^2\frac{d^2\psi}{dx^2} - \Big\{k_y^2r_i^2 + \frac{m_i}{m_e}\beta\Big[\frac{\omega-2k_yv_n(x)(1+\eta)}{\omega-k_yv_n(x)(1+\eta)} + i\sqrt{\frac{\pi}{2}}\frac{\omega_*^2}{(|k_z|u_e)^3}\Big[\omega-2k_yv_n\Big(1+\frac{\eta}{4}+\frac{\eta\omega_*^2}{4k_z^2u_e^2}\Big)\Big] \times \exp\Big(-\frac{\omega_*^2}{2k_z^2u_e^2}\Big)\Big]\Big\}\psi = 0 \qquad (3.8)$$

and has finite solutions (the situation is completely analogous to that in A) that are unstable in the range (cf. [5])

$$-1 < \eta < -2\frac{m_e}{m_i\beta}\Big(\frac{k_zu_e}{\omega_0}\Big)^2k_y^2r_i^2 + 2\frac{r_i}{R}\Big(p+\frac{1}{2}\Big)\sqrt{\frac{2m_e}{m_i\beta}}\Big(\frac{k_zu_e}{\omega_0}\Big)^2. \qquad (3.9)$$

D. Up to this point we have only considered the excitation of drift waves with phase velocity ω/k_z greatly different from the Alfvén velocity V_A so that the boundaries of the potential well are determined by the vanishing points of the function $\omega-(2+\eta)k_yv_n(x)$ for approximately constant values of the factor $[k_\parallel^2V_A^2 - \omega^2 + k_yv_n(1+\eta)\omega]$. In the general case ($\omega \sim k_zV_A$), to make a qualitative investigation of stability we use (1.6), which yields

$$(\omega-\omega_1(x))(\omega-\omega_2(x))(\omega-\omega_d(x))=\mathrm{Re}\,k_\perp^2 r_i^2 k_z^2 V_A^2\omega,$$

$$-\mathrm{Im}\,\omega\equiv\nu=-\sqrt{\frac{\pi}{2}}\frac{(\omega-k_y v_n(1+\eta))(\omega-\omega_d(x)+{}^1/_2 k_y v_n\eta)}{|k_z|u_e}\times\left(1+\frac{(\omega-\omega_d(x))(\omega^2+k_z^2V_A^2)}{\omega(\omega-\omega_1(x))(\omega-\omega_2(x))}\right)^{-1},$$

$$\omega_{1,2}(x)=\frac{1+\eta}{2}k_y v_n(x)\left\{1\pm\sqrt{1+\frac{4k_z^2V_A^2}{k_y^2v_n^2(1+\eta)^2}}\right\},$$

$$\omega_d(x)\equiv k_y v_n(x)(2+\eta). \tag{3.10}$$

As we have already noted, a qualitative difference from the preceding cases arises only if the frequency can become equal to the frequency $\omega_1(x)$, $\omega_2(x)$ at some point x, causing the factor $(\omega-\omega_1)\times(\omega-\omega_2)$ to change sign at this point.

1. Suppose that $\omega=\omega_1(x)$ at some point. Then, as is evident from (3.1), a potential well is produced in the region where $\omega<\omega_d(x)$, $\omega<\omega_1(x)$, $\omega(\omega-\omega_2(x))>0$. In the absence of a temperature gradient we find from (3.10) that an instability must arise when $\omega>k_y v_n(x_0)$ (this inequality is satisfied, for example, for zeroes of the function $\omega-\omega_1(x)$ that lie close together).

The preceding case, in which the frequency is close to $\omega_d(x_0)$, corresponds to the usual drift wave; hence the growth rates, to an accuracy of order unity, coincide with the growth rates given in A.

Here, we shall only consider the excitation of a wave with frequency $\omega\approx\omega_1(x_0)$. Then, expanding the function $n'(x)/n$ we obtain the equation

$$\frac{d^2\psi}{dx^2}+\left\{-k_y^2-\frac{2}{k_z^2V_A^2r_i^2}\left[\omega-\omega_1(x_0)+(1+\eta)\omega_0\frac{2+\alpha^2+2\sqrt{1+\alpha^2}}{4\sqrt{1+\alpha^2}}\frac{x^2}{R^2}\right]\times\frac{\sqrt{1+\alpha^2}}{1+\sqrt{1+\alpha^2}}(\omega_d(x_0)-\omega_1(x_0))\times\left[1-i\sqrt{\frac{\pi}{2}}\frac{\omega_1(x_0)-\omega_0(1+\eta)}{|k_z|u_e}\times\left(1-\frac{\omega_0\eta}{2(\omega_d(x_0)-\omega_1(x_0))}\right)\right]\right\}\psi=0,$$

$$\alpha=2k_zV_A/(1+\eta)k_yv_n(x_0). \tag{3.11}$$

If $\eta>-1$ the finite solutions can be obtained only when $|k_z|V_A<\sqrt{2+\eta}\,\omega_0$ (then the real part of the potential energy U(x) is a well).

From the expression for the characteristic frequency corresponding to the finite solution[5]

$$\omega=\omega_1(x_0)-k_y^2r_i^2\frac{1+\sqrt{1+\alpha^2}}{2\sqrt{1+\alpha^2}}\frac{k_z^2V_A^2}{\omega_d(x_0)-\omega_1(x_0)}(1+i\gamma_1)-\frac{r_i}{2R}\sqrt{2(1+\eta)}\frac{\sqrt{2+\alpha^2+2\sqrt{1+\alpha^2}}}{\sqrt{1+\alpha^2}}\times\frac{|k_z|V_A\sqrt{1+\sqrt{1+\alpha^2}}}{\sqrt{\omega_d(x_0)-\omega_1(x_0)}}\left(1+\frac{i\gamma_1}{3}\right),$$

$$\gamma_1=\sqrt{\frac{\pi}{2}}\frac{\omega_1(x_0)-\omega_0(1+\eta)}{|k_z|u_e}\left[1-\frac{\eta\omega_0}{2(\omega_d(x_0)-\omega_1(x_0))}\right], \tag{3.12}$$

we obtain the instability limits

$$-1<\eta<2[\omega_d(x_0)-\omega_1(x_0)]/\omega_0.$$

We now note that the smallness of $k^2r_i^2$ does not enter in the expression for the instability limit. This is so because the new "turning points" [the zeroes of the function $\omega-\omega_1(x)$] limit the region of localization of the wave only to the region of strong electron excitation. The maximum value of the temperature gradient for which it is unstable is reached when $|k_z|V_A\ll\omega_0$ and is given by $\eta=2$ [the growth rates for this case go as $(|k_z|V_A\omega_0^{-1})^3$].

In cases in which $|k_z|V_A>\sqrt{2+\eta}\,\omega_0$ or $\eta<-1$ the potential energy U(x) is in the form of a hill and the finite solutions decay in time [for growing finite solutions, we require the opposite sign of the imaginary part V (cf. the introduction)].

2. We now consider further the case $\omega=\omega_2(x)$. When $\eta>-1$ there is now a potential well in the region where $\omega<\omega_2(x)$, $\omega<0$ and, as follows from (3.10), this wave attenuates in time. If $-2<\eta<-1$, then $\omega>0$ and a well is produced in the region between the roots of the equation

$$\omega-\omega_2(x_1)=0,\qquad\omega-\omega_d(x_2)=0.$$

It is difficult to approximate this well by an oscillator and we analyze the simplest case, in which the frequencies $\omega_2(x)$, $\omega_d(x)$ coincide [$k_zV_A=\sqrt{2+\eta}\,k_yv_n$].

From (3.10) we obtain the characteristic frequencies

$$\omega=\omega_d(x)\pm\mathrm{Re}\,k_\perp r_i\sqrt{\frac{2+\eta}{3+\eta}}|k_z|V_A+i\sqrt{\frac{\pi}{8}}\frac{k_y^2v_n^2}{|k_z|u_e}\left(\frac{\eta}{2}\pm\mathrm{Re}k_\perp r_i\frac{|k_z|V_A}{k_yv_n}\right), \tag{3.13}$$

[5]A growth rate somewhat larger than that in (3.4) and (3.12) $(\nu=\sqrt{\pi/12}k_yv_nk_\perp r_iV_A/u_e)$ is obtained from (3.10) when $\omega_1(x)=\omega_d(x)$ and $T(x)\equiv\mathrm{const.}$[4]

from which it is evident that the plasma is unstable in this region of η. If $\eta < -2$ the region of localization of the wave is bounded by the roots of the equation $\omega = \omega_2(x)$ and, as is evident from (3.10), an instability arises when

$$-1-\sqrt{9-4k_z^2V_A^2/k_y^2v_n^2} < \eta < -2.$$

The instability boundary is farthest of all in the region $\eta < 0$ where $k_zV_A \ll k_yv_n$ ($\eta_{\lim} = -4$). The solution of the equation

$$\frac{d^2\psi}{dx^2} + \left\{-k_y^2 + \frac{2}{(k_zV_A)^4} \times \left[\omega + \frac{k_z^2V_A^2}{\omega_0(1+\eta)}\left(1+\frac{x^2}{R^2}\right)\right]\omega_0^4(1+\eta)^2\left(1+\frac{\eta}{2}\right) \times \left[1 + i\sqrt{\frac{\pi}{2}}\frac{\omega_0}{|k_z|u_e}\frac{(1+\eta)(4+\eta)}{2(2+\eta)}\right]\right\}\psi = 0, \quad (3.14)$$

which applies in this case verifies this conclusion. The total region of instability in the parameter η for the last case ($\omega_* \sim k_zV_A$) is given by the expression $-4 < \eta < 2$.[6)]

Thus, when the lines of forces of the magnetic field are parallel $|d\theta/dx \equiv 0|$, the plasma indeed exhibits a universal instability. In this case perturbations are excited with wavelengths of order $\lambda_x \gtrsim r_i$ and phase velocities $u_i < \omega_*/k_z \lesssim V_A$. The growth rates of the instability are given by (3.5) and (3.12). Perturbations with phase velocities $\omega_*/k_z > V_A$ can also be excited in an inhomogeneous plasma but only in the presence of a temperature gradient (i.e., the instability associated with these perturbations is not a universal one).

In this section we have studied the stability with respect to perturbations which are highly extended along the lines of force ($k_\parallel \ll k_\perp$); it is then desirable to analyze the effect of small departures from parallelism (of the lines of force of the magnetic field) that violate the condition required for the existence of the instability.[7)]

4. STABILITY OF AN INHOMOGENEOUS LOW-DENSITY PLASMA IN A MAGNETIC FIELD IN WHICH THE LINES OF FORCE ARE NOT PARALLEL

The effect due to departure from parallelism of the lines of force of the magnetic field, which we will call shear below, can be described in terms of two mechanisms that tend to stabilize the instability.

1. In the presence of shear there are still finite solutions corresponding to local perturbations but the condition that the interaction be small between the perturbation waves and ions:

$$\frac{\omega_*}{k_\parallel(x)} > u_i, \qquad k_\parallel(x) = k_z + k_y\int\frac{d\theta}{dx}dx$$

can no longer be satisfied in the entire region of localization of the perturbation. This means that the instability with respect to the excitation of drift waves with phase velocities $u_i < \omega_*/k_\parallel < V_A$ is no longer "universal;" it does not even exist in the absence of a temperature gradient if[8)]

$$\int\frac{d\theta}{dx}dx > \frac{\omega_*}{k_yu_i} \sim \frac{r_i}{R}. \quad (4.1)$$

Here, the integration is carried out over the region of localization of the perturbation.

2. The region of localization of the perturbations with shear is changed in such a way that we can always neglect the interaction with the ions in this region. The stabilization of these perturbations arises from the fact that shear causes the potential well $U(x)$ to be contracted and when

$$\lambda_x \gtrsim \Delta X, \quad (4.2)$$

where λ_x is the wavelength of the perturbation and ΔX is the width of the well, it becomes impossible to localize perturbations in the well (mathematically this means the impossibility of forming finite solutions).

In analyzing the effect of shear on the excitation of drift waves we must keep in mind that the region of non-potential perturbations ($\omega_* > k_\parallel V_A$) occupies a narrow range ΔX_n of the entire localization region $\Delta X_n \sim (\omega_*/k_\parallel V_A)dx/d\theta$ even when $Rd\theta/dx > \sqrt{\beta}\, r_i/R$.

Outside of this range the perturbations can be assumed to be potential so that the well $U(x)$ is determined primarily by (3.1). The excitation of drift waves with frequencies in the range $k_\parallel u_i < \omega_* < k_\parallel u_e$ is possible only if the following inequality holds in the localization region:

$$k_\perp v_n(1+\eta) < \omega < k_\perp v_n(2+\eta/2)$$

[cf. (3.10)] and we can neglect the interaction of the waves with ions in this region.

The effect of ions on the excitation of the wave can be introduced by means of the integral condition (1.3) and (2.8) in the frequency range $k_\parallel u_i < \omega_* k_\parallel u_e$ taking account of the residues in the integrals

[6)]In the paper of Mikhaĭlovskiĭ and Rudakov[4] this region is given incorrectly: $0 < \eta < 2$.

[7)]This condition is also violated in short systems [length $L < (10-15)R$], where there is no instability.

[8)]Stabilization of the drift instability by rotation of the lines of force has been proposed by Rosenbluth.[7]

$$\int \frac{f_i^{(0)}}{\omega_* + k_\parallel v_\parallel} dv_\parallel$$

containing the ion distribution function, which describe the interaction of resonance ions ($v_\parallel = -\omega_*/k_\parallel$) with the drift wave

$$\int_{-\infty}^{+\infty} \Big\{ \frac{(2+\eta)\, k_\perp v_n}{\operatorname{Re}\omega} \operatorname{Im}\omega - \sqrt{\frac{\pi}{2}} \operatorname{Re} \frac{\omega - (1+\eta)\, k_\perp v_n}{|k_\parallel| u_e}$$

$$\times \Big[\omega - \Big(2 + \frac{\eta}{2}\Big) k_\perp v_n$$

$$+ k_\parallel v_e^{(0)} + \sqrt{\frac{m_i}{m_e}} \Big(\omega - \frac{k_\perp v_n \eta}{2} \Big(3 - \frac{\omega_*^2}{k_\parallel^2 u_i^2}\Big)\Big)$$

$$\times \exp\Big(-\frac{\omega_*^2}{2k_\parallel^2 u_i^2}\Big)\Big]\Big\} |\psi|^2 dx = 0. \qquad (4.3)$$

The last expression expresses the fact that the balance of energy of the wave is determined by the work of the electric field of the wave in its entire localization region. It is evident from this expression that in the absence of temperature gradients the interaction with ions leads to a damping of the wave if the phase velocity ω/k_z is comparable with u_i over a distance of the order of the dimensions of localization. However, if the phase velocity is much larger $\omega_*/k_\parallel > \sqrt{3}\, u_i$ so that only the ions in the tail of the Maxwell distribution are in phase with the wave, this interaction excites the wave for sufficiently high temperature gradients directed against the density gradients. It follows from the stabilization criterion (4.1) that it is most difficult to stabilize perturbations with a small localization region Δx. This is the case when the range of non-potential perturbations [$k_\parallel(x) < \omega_*/V_A$] which bounds the region of localization on one side (Fig. 2) lies close to the turning point x_1 (or x_2).

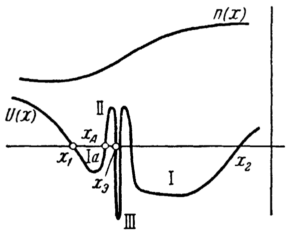

FIG. 2

Taking account of terms of order $k_\parallel^2 u_i^2/\omega^2$ in the expansion the well U(x) assumes the form [cf. (3.1)]

$$U(x) \approx k_\perp^2 r_i^2 + \Big[\Big(1 - \frac{k_\parallel^2 u_i^2}{\omega_*^2}\Big) - \frac{(2+\eta)\, k_\perp v_n}{\omega}\Big]$$

$$\times \Big[1 + \frac{k_\parallel^2 u_i^2}{\omega_*^2}\Big(1 + \frac{\eta k_\perp v_n}{\omega}\Big)\Big]^{-1}$$

and expands to dimensions $\Delta x \sim R$ in the presence of shear:

$$R d\theta/dx > {}^1\!/_2\, r_i R^{-1} \sqrt{R/\Delta x_1},$$

$$\Delta x_1/R = [(2+\eta)\, k_\perp v_n(x_A) - \omega(1 + k_\perp^2 r_i^2)]/\omega \qquad (4.4)$$

[this occurs because of the rapid increase of $k_\parallel(x)$]. Under these conditions it follows from (4.1) that the instability is stabilized.

Substituting everywhere from the condition (4.2) for the absence of finite solutions $\Delta x_1 > \lambda_x = r_i/\sqrt{U}$ the minimum Δx_1, given by $\Delta x_1 \sim r_i^{2/3} R^{1/3}$, we obtain the critical shear:

$$R\, d\theta/dx > (r_i/R)^{2/3} \qquad (4.5)$$

(if the shear is of order (4.5) a U(x) well with $\Delta x_1 < r_i^{2/3} R^{1/3}$ cannot support one wavelength and finite solutions do not exist).

At low values of β stabilization of these perturbations is afforded by a less stringent condition than that in (4.5). This is due to the fact that the potential barrier II (Fig. 2) is contracted and becomes transparent. Then the localization region for the perturbations is not limited to the narrow well Ia close to the turning point x_1 but also encompasses the well I, where the stabilizing effect of ion damping is large. Thus, the stabilization condition is essentially that the barrier II be transparent.

From the transparency condition (Fig. 2)

$$\frac{2}{r_i} \int_{x_3}^{x_A} \sqrt{U(x)}\, dx < 1$$

we find, taking account of the fact that

$$U(\kappa) = \frac{\omega - (2+\eta)\, k_\perp v_n}{\omega} \Big(1 - \frac{\omega^2 - k_\perp v_n (1+\eta)\, \omega}{k_\parallel^2(x)\, V_A^2}\Big),$$

where $x_{A,e}$ represents the boundaries of the barrier and is of order

$$x_A \sim (r_i \sqrt{\beta}/R)\, dx/d\theta, \quad x_e \sim \sqrt{m_e/m_i \beta}\, x_A),$$

that a sufficient condition for stabilization is

$$R d\theta/dx \gtrsim \sqrt{\Delta x_1/R} \sqrt{3\beta}\, (\ln(2x_A/x_e) - 1). \qquad (4.6)$$

The maximum possible width Δx_1 of the well Ia for a given shear, for which the ion damping is smallest, is determined by the inverse condition to (4.4), i.e., when $\sqrt{\Delta x_1/R} < r_i R^{-2} dx/d\theta$ and $(r_i/R)^2 < \beta < (r_i/R)^{2/3}$ stabilization occurs earlier because of the transparency of the barrier when

$$R d\theta/dx > (r_i/R)^{1/2} (3\beta)^{1/4} [\ln 2\sqrt{m_i\beta/m_e} - 1]^{1/2}. \qquad (4.7)$$

On the other hand, for large $\beta \sim 1$ the wall II is not very steep and the well Ia is smaller so that for a large well width $\Delta x < r_i/\sqrt{U} \sim r_i R/\Delta x$ there are no finite solutions and the stabilization criterion is given in order-of-magnitude terms by

$$R d\theta/dx > (r_i/R)^{3/4}. \tag{4.8}$$

Thus, the magnitude of the shear that stabilizes the potential perturbations is essentially a function of $\beta = 8\pi n_0 T/H^2$. This dependence is shown in Fig. 3.

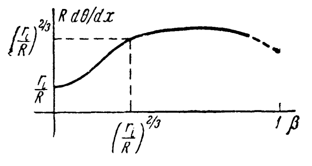

FIG. 3

In the presence of shear the phase velocity encompasses essentially the entire range from u_i to ∞ and this leads to the formation of additional potential wells for $\omega_*/u_e < k_{\|}(x) < \omega_*/V_A$ and $k_{\|}(x) < \omega_*/u_e$. If the non-potential range ($k_{\|} < \omega_*/V_A$) lies in the range $\omega > (2+\eta)k_\perp v_n$ then a well $U(x)$ is produced for $\omega_*/u_e < k_{\|} < \omega_*/V_A$ and, in the general case, is described by an integral differential equation; when $kr_i \ll 1$ this equation becomes a second-order differential equation (3.1) and when $kr_i \gg 1$ it becomes the equation

$$r_i^2 \frac{d^2\psi}{dx^2} - \left\{ k_\perp^2 r_i^2 - \frac{[\omega - k_\perp v_n(1+\eta)]^2 - k_\perp^2 v_n^2}{2k_\perp^2 V_A^2 (d\theta/dx)^2 x^2} \times \left[1 - i\sqrt{\frac{\pi}{8}} \frac{(\omega - \eta k_\perp v_n)^2 [\omega - (2+\eta/2) k_\perp v_n]}{(\omega_*^2 - k_\perp^2 v_n^2)\,|k_{\|}|\,u_e} \right] \right\} \psi = 0, \tag{4.9}$$

where we have assumed a linear dependence $k_{\|} = x k_\perp d\theta/dx$.

From the integral condition (1.3) we obtain the growth rate ν in the form (3.10) for (3.1) and

$$\nu = \sqrt{\frac{\pi}{32}} \frac{(\omega - \eta k_\perp v_n)^2}{|k_{\|}|\,u_e} \frac{\omega - (2+\eta/2) k_\perp v_n}{\omega - (1+\eta) k_\perp v_n}$$

for (4.9). In both cases the instability boundary is given by

$$\eta < -2\,[\omega - k_\perp v_n (2+\eta) + k_{\|} v_e^{(0)}]/k_\perp v_n < 0, \tag{4.10}$$

since the point $x^{(p)}$ in (1.6) lies in the region of localization of $\psi(x)$ where $\omega > (2+\eta)k_\perp v_n$ [cf. Eq. (4.9)]; we note that in accordance with footnote 4 $k_\perp v_n > 0$.

We can always neglect ion Landau damping in the region of localization of the perturbation so that stabilization of the instability occurs only by virtue of the "squeezing out" of the levels caused by strong compression of the well. The expressions in (3.1) and (4.9) assume the form

$$d^2\psi/dx^2 + (-k_y^2 + \varkappa^2/x^2)\psi = 0, \tag{4.11}$$

where

$$\varkappa^2 = (\omega_*^2 - k_\perp^2 v_n^2)/2k_\perp^2 r_i^2 V_A^2 (d\theta/dx)^2$$

can be assumed constant for the large shear we have assumed.

The expression in (4.11) has the formal solution:

$$\psi = \psi^{(0)} \sqrt{x}\, K_{i\nu}(k_\perp x), \quad \nu = \sqrt{\varkappa^2 - 1/4},$$

where $K_{i\nu}(k_\perp x)$ is the Macdonald function of imaginary order. When $\nu^2 > 0$ the number of zeroes of this solution close to $x = 0$ is infinite and a solution that vanishes at infinity can be joined to any solution close to $x = 0$. When $\nu^2 < 0$ this situation does not hold. The condition for the absence of finite solutions $\kappa^2 < 1/4$ and the condition for the instability (4.10) give us the magnitude of the shear that stabilizes the instability for a given temperature gradient η:

$$R d\theta/dx > \sqrt{\beta\,|\eta|}, \quad \eta < -(2r_i/R)^{2/3}, \tag{4.12}$$

where the boundary of the instability region is determined by the minimum value of the difference $[\omega - (2+\eta)k_\perp v_n/\omega]$ in the region of the well; this is reached when the point x_2 at which this difference vanishes is the turning point (in this case the point x_2 enters region II in Fig. 4).

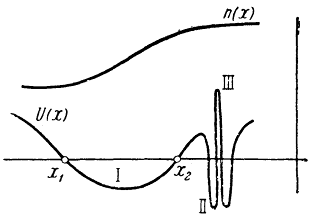

FIG. 4

Finally, we consider briefly high-frequency perturbations in the well III of Fig. 2 which arise close to the point x_0 [$k_{\|}(x_0) = 0$] when $2k_\perp v_n(x_0) \times (1+\eta) > \omega$. These perturbations are described by (3.8) both when $kr_i \ll 1$ and when $kr_i \gg 1$ (if $\beta \gg m_e/m_i$). From the latter, using (1.3) we obtain the growth rate

$$\nu = -\sqrt{\frac{\pi}{2}} \frac{\omega_*^4}{|k_{\|}|^3 u_e^3} \times \frac{\omega - 2(1+\eta)k_\perp v_n - 1/2\,\eta k_\perp v_n (3 - \omega_*^2/k_{\|}^2 u_e^2)}{k_\perp v_n (1+\eta)} \exp\left\{ -\frac{\omega_*^2}{2k_{\|}^2 u_e^2} \right\}.$$

Using the instability condition that follows from this and the condition for the absence of finite solutions ($\Delta x < \lambda_x$), and proceeding completely analogously with the preceding case, we obtain the value of the stabilizing shear:

$$Rd\theta/dx > \sqrt{\beta|\eta|},$$
$$-1 < \eta < -(m_e/\beta m_i)^{1/2}(r_i/R)^{2/3}. \qquad (4.13)$$

In concluding this section we wish to consider the effect of gravitational forces on stabilization of the instability [the role of this effective gravitational force in magnetic traps is, as is well known, played by the centrifugal inertial force arising by virtue of the motion of the particles along the curved line of force of the magnetic field, that is to say, $g_{eff} = v_{\parallel}^2/r$ (r is the radius of curvature of the lines of force)]. We limit ourselves to the most interesting case, in which the particle drift due to the inhomogeneity in density and temperature is much greater than the drift in the gravitational field ($g_{eff} \ll u_i^2/R$).

The gravitational field has no effect on the non-potential perturbations ($\omega_* \gg k_z V_A$) because for these (2.8) separates into equations for the flute and drift instabilities. For potential perturbations, with frequency ω of order $k_\perp v_n$ we can neglect the term

$$(k_\perp^2 r_i^2 g/k_\parallel^2 V_A^2)\, n'/n$$

compared with

$$(d^2/dx^2 - k_\perp^2)\, r_i^2 \psi,$$

and the equation for the perturbations in the frequency region $k_\parallel u_i < \omega_* < k_\parallel V_A$ assumes the form

$$r_i^2 \frac{d^2\psi}{dx^2} - \left\{k_\perp^2 r_i^2 \left(1 - \frac{gn'/n}{\omega^2 - k_\perp v_n (1+\eta)\omega}\right) + 1 - \frac{(2+\eta) k_\perp v_n}{\omega} - i\sqrt{\frac{\pi}{2}} \frac{\omega_*}{|k_\parallel| u_e}\left[1 - \frac{(2+\eta/2) k_\perp v_n}{\omega}\right]\right\}\psi = 0.$$

This equation contains only one small term $\sim k_\perp^2 r_i g n'/\omega^2 n \ll 1$ as compared with (3.2). A qualitative idea of the effect of this term can be obtained by expanding $n'(x)/n$. Then the problem proceeds in exactly the same way as that considered in section 3A; in particular, when $d\theta/dx \equiv 0$ the frequency spectrum of the perturbations is of the form

$$\omega^{(p)} = (2+\eta)\omega_0 + \frac{i\gamma\eta\omega_0}{2} - (2+\eta)\omega_0 k_\perp^2 r_i^2 (1+i\gamma) + \frac{k_\perp^2 r_i^2 g n'(x_0)/n}{\omega - \omega_0(1+\eta)}(1+i\gamma) - \frac{2r_i}{R}\left(p + \frac{1}{2}\right)(2+\eta) \times \omega_0\left\{1 + \frac{k_\perp^2 r_i^2 g n'(x_0)/n}{[\omega - \omega_0(1+\eta)]^2} - i\gamma\left[1 + \frac{\eta^2}{4+2\eta}\right]\right\}^{1/2}(1+i\gamma).$$

It is thus evident that a weak gravitational field can broaden the instability boundary

$$\eta < 4k_y^2 r_i^2 + 4r_i R^{-1}(p + 1/2) - \omega_0^{-2} k_\perp^2 r_i^2 g n'(x_0)/n,$$

if $gn'/n < 0$; however, this effect on the stability due to the gravitational field is expressed through a change in the characteristic frequencies of the plasma oscillations whereas the nature of the growth rate is, as before, determined by the interaction of the wave with electrons and ions. Thus it is clear that the increase in ion damping associated with the reduction in the phase velocity along the lines of magnetic force of the field $\omega/k_\parallel$ again leads to a stabilization of this instability for the condition in (4.5), that is to say, introduction of a weak gravitational force does not affect the stabilization provided by the curved lines of force.

Thus, the universal instability with respect to potential perturbations $k_z u_i < \omega_* < k_z V_A$ is stabilized when $R d\theta/dx > (r_i/R)^{2/3}$ while the instability with respect to non-potential perturbations, which arise only in the presence of a temperature gradient [$d \ln T/d \ln n < 0$] is stabilized when $R d\theta/dx > \sqrt{\beta}$.

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