COVARIANT CONSERVATION LAWS IN THE GENERAL THEORY OF RELATIVITY

D. E. BURLANKOV

Submitted to JETP editor November 31 [sic!], 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 1941-1949 (June, 1963)

For the investigation of problems dealing with energy in the general theory of relativity a method of covariant comparison of the space-time under investigation with an auxiliary comparison space (flat space is the most convenient) is utilized—i.e., comparative differential geometry is employed. Conservative tensors are derived from the invariance of action with respect to the motion of the comparison space. The coordinate conditions are replaced by completely covariant correspondence conditions. The relations thus derived are used to investigate the energy of closed systems and of the Einstein-Rosen waves.

FOR a covariant definition of conservation laws in the general theory of relativity we use in addition to physical space-time a certain flat Minkowski space—a "comparison space"—the concept of which, as well as the idea of the method of comparative analysis of two Riemann spaces—"comparative differential geometry," originates in the papers of Gutman [1] and of Pugachev [2]. Conservative tensors are derived from the invariance of action with respect to motions of the comparison space. This method can be utilized if physical space is topologically isomorphic with Minkowski space.

1. THE COMPARISON SPACE

We postulate a one-to-one correspondence between points of two topologically equivalent Riemann spaces, and a coordinate system in one of them. Then the objects belonging to the second space can be described in the same coordinate system with the corresponding points having the same coordinates. In the problems considered by us one of these spaces is given, while the second is defined by comparison with the first, and therefore we shall refer to the first space as the comparison space.

We denote the metric tensor of the comparison space in the given system of coordinates by $\overset{0}{g}_{ab}(x)$. Then the metric tensor of the second space in the same system of coordinates can be regarded as simply a tensor of the second rank defined in the comparison space independently of the tensor $\overset{0}{g}_{ab}(x)$. We denote it by $g_{ab}(x)$.

A parallel translation in each of the two spaces defines covariant differentiation with respect to the given space: the basic space

$$\nabla_k A^i \equiv A^i_{;k} = \partial A^i/\partial x^k + \Gamma^i_{jk} A^j, \tag{1}$$

and the comparison space

$$\overset{0}{\nabla}_k A^i \equiv A^i_{|k} = \partial A^i/\partial x^k = \overset{0}{\Gamma}{}^i_{jk} A^j, \tag{2}$$

where Γ^i_{jk} and $\overset{0}{\Gamma}{}^i_{jk}$ are the Christoffel symbols expressed respectively in terms of $g_{ab}(x)$ and $\overset{0}{g}_{ab}(x)$. Both quantities $A^i_{;k}$ and $A^i_{|k}$ are tensors of the same nature and, consequently, their difference is also a tensor of the same nature, and therefore the quantity $\Pi^i_{jk} = \Gamma^i_{jk} - \overset{0}{\Gamma}{}^i_{jk}$ is also a tensor [2].

Since $\nabla_k g_{ij} = 0$ and $\overset{0}{\nabla}_k \overset{0}{g}_{ij} = 0$, it follows that

$$\overset{0}{\nabla}_k g_{ij} = \Pi^s_{ik} g_{sj} = \Pi^s_{jk} g_{is}, \quad \nabla_k \overset{0}{g}_{ij} = -\Pi^s_{ik}\overset{0}{g}_{sj} - \Pi^s_{jk}\overset{0}{g}_{is}, \tag{3}$$

from which it follows that

$$\Pi^i_{jk} = \tfrac{1}{2} g^{is}(g_{sk|j} + g_{sj|k} - g_{jk|s}) = -\tfrac{1}{2}\overset{0}{g}{}^{is}(\overset{0}{g}_{sk;j} + \overset{0}{g}_{sj;k} - \overset{0}{g}_{jk;s}). \tag{4}$$

The relative curvature of the space at corresponding points is characterized by the tensor:

$$S^j_{kil} = \Pi^j_{kl|i} - \Pi^j_{ki|l} + \Pi^s_{kl}\Pi^j_{si} - \Pi^s_{ki}\Pi^j_{sl} = R^j_{kil} - \overset{0}{R}{}^j_{kil}. \tag{5}$$

The determinant $|g_{ij}| \equiv g$ is a relative invariant of weight 2, and so is the determinant $|\overset{0}{g}_{ij}| = \overset{0}{g}$, and therefore the quantity $\Pi = (g/\overset{0}{g})^{1/2}$ is an absolute invariant.

The following formulas also hold:

$$\overset{0}{\nabla}_i (\Pi g^{ij}) = -g^{kl}\Pi^i_{kl}, \tag{6}$$

$$\overset{0}{\nabla}_i \Pi = \Pi\Pi^s_{is}. \tag{7}$$

$$\nabla_i A^k = \overset{0}{\nabla}_i A^k + \Pi^k_{si} A^s, \tag{8}$$

which differ from the corresponding formulas of tensor analysis by the replacement of $\partial/\partial x^i$ by

$\overset{0}{\nabla}_i$, $\sqrt{g}$ by Π, Γ^i_{jk} by Π^i_{jk}. We shall say that these formulas refer to the comparative differential geometry of Riemann spaces.

2. THE COMPARATIVE SPACE IN THE GENERAL THEORY OF RELATIVITY

For the discussion of conserved quantities in the general theory of relativity it is convenient to choose a flat comparison space. In this case we have

$$S^i_{jkl} = R^i_{jkl},\ S \equiv g^{jl} S^i_{jil} = R, \tag{9}$$

while the covariant derivatives with respect to the comparison space commute.

3. COORDINATE CONDITIONS AND CORRESPONDENCE CONDITIONS

As is well known, the Christoffel symbols are not components of a tensor, and by a suitable transformation of coordinates can all be simultaneously made to vanish at an arbitrary (in the general case a single) point. As a tensor Π^i_{jk} can be equal to zero either in all coordinate systems or in none of them. "Pseudotensors" in the general theory of relativity are formed with the aid of the "pseudotensor" Γ^i_{jk}, and, therefore, can also be made to vanish at any arbitrarily prescribed point by a suitable transformation of coordinates. Therefore, in the discussion of problems of energy the coordinate conditions turn out to be essential, and they are usually chosen in such a way that all the pseudoquantities would have the most natural meaning.

The use of comparison geometry does not remove this ambiguity, but the auxiliary conditions take on a completely covariant form and meaning. This ambiguity manifests itself in the following: we can establish the correspondence between the points of two spaces in an infinite number of ways; but this is done independently of the coordinate system, and, therefore, having chosen a correspondence (and having replaced Γ^i_{jk} by Π^i_{jk}, etc.) we are then dealing only with tensors. Thus, the noncovariant coordinate conditions are replaced by completely covariant correspondence conditions.

As an example, we refer to the harmonic correspondence which agrees with the deDonder-Fock harmonicity condition if we select Cartesian coordinates in the comparison space

$$\overset{0}{\nabla}_i \Pi (g^{ij}) = - g^{sk} \Pi^l_{sk} = 0\,. \tag{10}$$

It is covariant in form.

4. CONSERVATION LAWS

Action for a system consisting of the gravitational field and other matter can be written in the following form[3]:

$$\begin{aligned} S &= \frac{c^3}{16\pi k} \int R \sqrt{-g}\, d\omega + \int \mathcal{L} \sqrt{-g}\, d\omega \\ &= \int \left[\frac{c^3}{16\pi k} R + \mathcal{L} \right] \Pi \sqrt{-\overset{0}{g}}\, d\omega, \end{aligned} \tag{11}$$

where $\mathcal{L}$ is the Lagrangian for nongravitating matter, while R is the scalar space curvature:

$$R = g^{kl} (\Pi^i_{kl|i} - \Pi^i_{ki|l} + \Pi^s_{kl}\Pi^i_{si} - \Pi^s_{ki}\Pi^i_{ls}). \tag{12}$$

The terms in R containing second derivatives can be put in the form of a divergence[3]:

$$\begin{aligned} \Pi R &= \overset{0}{\nabla}_l (\Pi g^{ik}\Pi^l_{ik} - \Pi g^{il}\Pi^k_{ik}) + \Pi G \\ &= \overset{0}{\nabla}_l (\Pi \overset{0}{\nabla}_i g^{il}) + \Pi g^{ik} (\Pi^m_{il}\Pi^l_{km} - \Pi^l_{ik}\Pi^m_{lm}). \end{aligned} \tag{13}$$

Since in discarding terms of the form of a divergence the equations of the fields obtained from a variational principle are not altered, the conserved quantities are altered only by terms which are identically conserved, and since the Lagrangian of the nongravitating fields depends on derivatives of order not higher than the first, then this identically conserved term is expressed only in terms of quantities describing the gravitational field, and, consequently, does not describe the conversion of energy-momentum, and therefore we can neglect the first term in (13). Thus, we have

$$S = \int \left[\frac{c^3}{16\pi k} G + \mathcal{L} \right] \Pi \sqrt{-\overset{0}{g}}\, d\omega = S_g + S_{\mathcal{L}}. \tag{14}$$

We note that no terms in (11) depend on $\overset{0}{g}_{ik}$, while in (14) ΠG does depend on this quantity, since it differs from ΠR, which does not depend on $\overset{0}{g}_{ik}$, by a divergence term which contains this tensor ($\overset{0}{g}_{ik}$).

We now consider infinitesimal transformations of coordinates[4]. Since $\mathcal{L}$ is a scalar, G involves only completely contracted tensor expressions, and $(-\overset{0}{g})^{1/2} d\omega$ is an invariant element of volume of the comparison space, the increments of each of the terms in (14) resulting from a variation of coordinates must be equal to zero.

The transformations of coordinates can be written in the form

$$\bar{x}^i = x^i + p^i(x), \tag{15}$$

where p^i is an infinitesimal vector. In this case the geometrical quantities at points having the original coordinates acquire so-called Lie-increments (cf. [5]). Thus, for example,

$$\delta_L g_{ij} = -(g_{is}p^s_{|j} + g_{sj}p^s_{|i} + g_{ij|s}p^s),$$
$$\delta_L \overset{0}{g}_{ij} = -(\overset{0}{g}_{is}p^s_{|j} + \overset{0}{g}_{sj}p^s_{|i}). \tag{16}$$

The variation of action consists of increments of the integrands and of a change in the region of integration:

$$\frac{16\pi k}{c^3}\delta S_g = \int \delta_L [G\sqrt{-g}]\, d\omega + \oint G\Pi \sqrt{-\overset{0}{g}}\, p^s d\sigma_s, \tag{17}$$

$$\delta_L [G\sqrt{-g}] = \frac{\partial (G\sqrt{-g})}{\partial g_{ik}} \delta_L g_{ik} + \frac{\partial (G\sqrt{-g})}{\partial g_{ik,s}} \delta_L g_{ik,s}$$
$$+ \frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik}} \delta_L \overset{0}{g}_{ik} + \frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik,s}} \delta_L \overset{0}{g}_{ik,s}$$
$$= \left[\frac{\partial (G\sqrt{-g})}{\partial g_{ik}} - \frac{\partial}{\partial x^s} \frac{\partial (G\sqrt{-g})}{\partial g_{ik,s}}\right] \delta_L g_{ik}$$
$$+ \left[\frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik}} - \frac{\partial}{\partial x^s} \frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik,s}}\right] \delta_L \overset{0}{g}_{ik}$$
$$+ \frac{\partial}{\partial x^s}\left[\frac{\partial (G\sqrt{-g})}{\partial g_{ik,s}} \delta_L g_{ik} + \frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik,s}} \delta_L \overset{0}{g}_{ik}\right]. \tag{18}$$

Here we have utilized the fact that the ordinary (not the covariant!) differentiation commutes with the Lie derivative. The quantity

$$\frac{\partial}{\partial x^s} \frac{\partial (G\sqrt{-g})}{\partial g_{ik,s}} - \frac{\partial (G\sqrt{-g})}{\partial g_{ik}} \equiv \sqrt{-g}\, G^{ik} \tag{19}$$

is the Einstein tensor which appears in the field equations [3]:

$$G^{ik} = 8\pi k T^{ik}/c^4, \tag{20}$$

where T^{ik} is the symmetric energy-momentum tensor of nongravitating matter.

We introduce the notation

$$\frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik}} - \frac{\partial}{\partial x^s} \frac{\partial (G\sqrt{-g})}{\partial \overset{0}{g}_{ik,s}} \equiv \frac{1}{2}\sqrt{-\overset{0}{g}}\, \vartheta^{ik} \tag{21}$$

and we shall call $c^4\vartheta^{ik}/16\pi k$ the totally symmetric energy-momentum tensor ($\vartheta^{ik} = \vartheta^{ki}$).

We now substitute (18), taking (19) and (21) into account, into (17):

$$\frac{16\pi k}{c^3}\delta S_g = \int \left[\frac{1}{2}\vartheta^{ik}\delta_L \overset{0}{g}_{ik} - \Pi G^{ik}\delta_L g_{ik}\right]\sqrt{-\overset{0}{g}}\, d\omega$$
$$+ \int \overset{0}{\nabla}_s \left(\frac{\partial G\Pi}{\partial g_{ik,s}} \delta_L g_{ik} + \frac{\partial G\Pi}{\partial \overset{0}{g}_{ik,s}} \delta_L \overset{0}{g}_{ik} + G\Pi p^s\right)\sqrt{-\overset{0}{g}}\, d\omega. \tag{22}$$

We have taken into account the fact that

$$\frac{\partial}{\partial x^s}(\sqrt{-\overset{0}{g}}A^s) = \sqrt{-\overset{0}{g}}\,\overset{0}{\nabla}_s A^s. \tag{23}$$

Now on substituting the expressions (16) for $\delta_L g_{ik}$ and $\delta_L \overset{0}{g}_{ik}$ and on integrating by parts we obtain

$$\frac{16\pi k}{c^3}\delta S_g = \int [\Pi G^{ik} g_{ik|s} - \overset{0}{\nabla}_l (\Pi G^l_s + \Pi G^l_s)$$
$$+ \overset{0}{\nabla} (\overset{0}{g}_{ks}\vartheta^{lk})]\, p^s \sqrt{-\overset{0}{g}}\, d\omega$$
$$- \int \overset{0}{\nabla}_s \left\{\left[\frac{\partial G\Pi}{\partial g_{ik,s}}(g_{il}\delta^t_k + g_{lk}\delta^t_i) + \frac{\partial G\Pi}{\partial \overset{0}{g}_{ik,s}}(\overset{0}{g}_{il}\delta^t_k + \overset{0}{g}_{lk}\delta^t_i)\right] p^l_{|t}\right.$$
$$\left. + \left[\frac{\partial G\Pi}{\partial g_{ik,s}} g_{ik|l} - G\Pi\delta^s_l - 2\Pi G^s_l + \vartheta^s_l\right] p^l\right\}\sqrt{-\overset{0}{g}}\, d\omega = 0, \tag{24}$$

$$\vartheta^s_l = \overset{0}{g}_{kl}\vartheta^{ks}. \tag{25}$$

Since we have taken p^l to be arbitrary and linearly independent of $p^l_{|i}$ and $p^l_{|ij}$, then in (24) the term which goes over into a surface integral (containing the divergence under the integral) and the remaining term must vanish independently of each other, since we can select an infinitely large number of vectors p^l which are equal to zero on the boundary of the region of integration. We obtain

$$\Pi G^{ik} g_{ik|s} - \overset{0}{\nabla}_l (2\Pi G^l_s) + \overset{0}{\nabla}_l \vartheta^l_s = 0. \tag{26}$$

On utilizing (7) and on differentiating the second term we obtain

$$\Pi (G^{ik} g_{il}\Pi^l_{ks} + G^{ik} g_{kl}\Pi^l_{is} - 2\Pi^r_{lr}G^l_s - 2\overset{0}{\nabla}_l G^l_s) + \overset{0}{\nabla}_l D^l_s$$
$$= -2\Pi (\overset{0}{\nabla}_l G^l_s + \Pi^r_{lr}G^l_s + \Pi^l_{ks}G^k_l) + \overset{0}{\nabla}_l \vartheta^l_s$$
$$= -2\Pi\nabla_l G^l_s + \overset{0}{\nabla}_l \vartheta^l_s = 0. \tag{27}$$

Since the Einstein tensor satisfies the relation

$$\nabla_l G^l_s = 0, \tag{28}$$

then the following relations must hold

$$\overset{0}{\nabla}_l \vartheta^l_s = 0, \quad \overset{0}{\nabla}_l \vartheta^{ls} = 0. \tag{29}$$

For the convenience of manipulating the second term in (24) we introduce the notation

$$G\Pi\delta^s_l - \frac{\partial G\Pi}{\partial g_{ik,s}} g_{ik|l} \equiv t^s_l, \tag{30}$$

$$\frac{\partial G\Pi}{\partial g_{ik,s}}(g_{il}\delta^t_k + g_{lk}\delta^t_i) + \frac{\partial G\Pi}{\partial \overset{0}{g}_{ik,s}}(\overset{0}{g}_{il}\delta^t_k + \overset{0}{g}_{lk}\delta^t_i) \equiv S^{st}_l. \tag{31}$$

Here $c^4S^{st}_l/16\pi k$ and $c^4t^s_l/16\pi k$ are respectively the spin tensor and the canonical energy-momentum tensor of the gravitational field. Then we have

$$\overset{0}{\nabla}_s [S^{st}_l p^l_{|t} + (t^s_l + 2\Pi G^s_l - \vartheta^s_l)\, p^l] = S^{st}_l p^l_{|ts} + (S^{st}_{l|s} + t^t_l$$
$$+ 2\Pi G^t_l - \vartheta^t_l)\, p^l_{|t} + [\overset{0}{\nabla}_s[t^s_l + 2\Pi G^s_l - \vartheta^s_l)]\, p^l = 0, \tag{32}$$

from which as a result of the arbitrary nature of p^l and of the commutativity of the covariant derivatives in the flat comparison space we obtain

$$S_l^{st} + S_l^{ts} = 0, \quad S_l^{st} = S_l^{[st]}, \tag{33}$$

$$\vartheta_l^t = 2\Pi G_l^t + t_l^t + S_{l\,|\,s}^{st}, \tag{34}$$

$$\frac{c^4}{16\pi k}\,\vartheta_l^t = \Pi T_l^t + \frac{c^4}{16\pi k}\,(t_l^t + S_{l\,|\,s}^{st}), \tag{35}$$

$$\overset{0}{\nabla}_s\,(t_l^s + 2\Pi G_l^s - \vartheta_l^s) = \overset{0}{\nabla}\,(t_l^s + 2\Pi G_l^s) = 0. \tag{36}$$

[The relations (35) and (36) are obtained by taking into account (20) and (29) respectively.]

If we introduce the notation

$$(c^4/16\pi k)\,(t_l^t + S_{l\,|\,s}^{st}) = \tau_l^t, \tag{37}$$

then (35) yields

$$(c^4/16\pi k)\,\vartheta_l^t = \tau_l^t + \Pi T_l^t, \tag{38}$$

i.e., we achieve a division of the total energy into the energy of "matter" and purely gravitational energy. On carrying out analogous calculations for the action of the nongravitational matter we shall obtain the well-known relations between the symmetrical and the canonical energy-momentum tensors.

5. THE EXPLICIT FORM OF TOTAL ENERGY-MOMENTUM TENSOR

We calculate ϑ^{st} by means of formula (21). We obtain

$$\sqrt{-\overset{0}{g}}\,\vartheta^{st} = 2\left(\frac{\partial G\sqrt{-g}}{\partial \overset{0}{g}_{st}} - \frac{\partial}{\partial x^r}\,\frac{\partial G\sqrt{-g}}{\partial \overset{0}{g}_{st,\,r}}\right). \tag{39}$$

$\overset{0}{g}_{ik}$ appears in $G(-g)^{1/2}$ only as a component part of Π_{jk}^i, and therefore

$$\vartheta^{st} = 2\left[\Pi\,\frac{\partial G}{\partial \Pi_{jk}^i}\,\frac{\partial \Pi_{jk}^i}{\partial \overset{0}{g}_{st}} - \frac{1}{\sqrt{-\overset{0}{g}}}\,\frac{\partial}{\partial x^r}\left(\frac{\partial G}{\partial \Pi_{jk}^i}\,\frac{\partial \Pi_{jk}^i}{\partial \overset{0}{g}_{st,\,r}}\,\sqrt{-g}\right)\right]. \tag{40}$$

We introduce the notation

$$\Pi g^{ij} \equiv b^{ij}, \quad \Pi\partial G/\partial \Pi_{jk}^i = \tfrac{1}{2}A_i^{jk}. \tag{41}$$

Calculations then yield

$$\tfrac{1}{2}A_i^{jk} = -b_{|\,i}^{jk} + \tfrac{1}{2}\,[\delta_i^k b_{|\,m}^{mj} + \delta_i^j b_{|\,m}^{mk}], \tag{42}$$

$$\partial \Pi_{ib}^a/\partial \overset{0}{g}_{st} = \overset{0}{g}{}^{as}\overset{0}{\Gamma}{}_{ib}^t, \tag{43}$$

$$\partial \Pi_{ib}^a/\partial \overset{0}{g}_{st,\,l} = -\tfrac{1}{2}\overset{0}{g}{}^{aj}\,(\delta_{ijb}^{stl} + \delta_{jbi}^{stl} - \delta_{ibj}^{stl}). \tag{44}$$

On substituting this into (40) we obtain

$$\vartheta^{st} = \tfrac{1}{2}\overset{0}{\nabla}_l\,(A^{st,\,t} + A^{lt,\,s} - A^{st,\,l}) = \overset{0}{g}{}^{st}b_{|\,lm}^{lm} + \overset{0}{g}{}^{lm}b_{|\,lm}^{st} - \overset{0}{g}{}^{tm}b_{|\,lm}^{sl} - \overset{0}{g}{}^{sm}b_{|\,lm}^{tl} = \overset{0}{\nabla}_l\overset{0}{\nabla}_m\,(\overset{0}{g}{}^{st}b^{lm} + \overset{0}{g}{}^{lm}b^{st} - \overset{0}{g}{}^{tm}b^{sl} - \overset{0}{g}{}^{sl}b^{tm}) \equiv \overset{0}{\nabla}_l\overset{0}{\nabla}_m B^{sm,\,tl}. \tag{45}$$

The relation (29) is satisfied in virtue of the symmetry properties of $B^{sm,tl}$. Since

$$B^{sm,\,tl} = -B^{ms,\,tl} = -B^{sm,\,lt}, \tag{46}$$

then because the derivatives commute we have

$$\overset{0}{\nabla}_s\vartheta^{st} = \overset{0}{\nabla}_s\overset{0}{\nabla}_l\overset{0}{\nabla}_m B^{sm,\,tl} = 0. \tag{47}$$

In the case of a harmonic gauge the energy-momentum tensor assumes a very simple form (cf., the result of Papapetrou.[6]):

$$\vartheta^{ik} = \overset{0}{g}{}^{lm}\overset{0}{\nabla}_l\overset{0}{\nabla}_m b^{ik} = -\overset{0}{\Box}b^{ik}, \tag{48}$$

and if we investigate ϑ^{ik} in accordance with formula (38), then (48) play the role of field equations.

On carrying out analogous calculations for the spin tensor we obtain

$$S_l^{st} = S_l^{[st]} = b_{|\,i}^{tk}\,(\overset{0}{g}{}^{is}\overset{0}{g}_{kl} - b^{is}b_{kl}) - b_{|\,i}^{sk}\,(\overset{0}{g}{}^{it}\overset{0}{g}_{kl} - b^{it}b_{kl}). \tag{49}$$

6. ENERGY OF A CLOSED SYSTEM

If material processes occur principally within a restricted volume of space and sufficiently slowly (so that gravitational radiation can be directed) then on going far away from this volume the space gradually goes over into Minkowski space.

We shall establish a harmonic correspondence such that the point in comparison space corresponding to the center of mass of the system should describe a timelike straight line. As time we shall choose the Cartesian coordinate in the direction of this straight line, and we shall choose the remaining (spacelike) coordinates orthogonal to time in the comparison space:

$$\overset{0}{g}{}^{00} = -c^{-2}, \quad \overset{0}{g}{}^{0i} = 0 \quad \text{for } i \neq 0. \tag{50}$$

Then at distances from the center of mass large compared to the dimensions of the region b^{00} can be expanded into a series in $1/r$:

$$b^{00} = -c^{-1}\,[1 + a/r + O\,(r^{-2})], \tag{51}$$

$$\vartheta^{00} = \Delta b^{00} - c^{-2}\partial^2 b^{00}/\partial t^2 \approx \Delta b^{00}, \tag{52}$$

while the energy of the system is given by

$$E = c\int\frac{c^4}{16\pi k}\,\vartheta^{00}\,dU = \frac{c^5}{16\pi k}\int\Delta b^{00}\,dU = \frac{c^5}{16\pi k}\oint\nabla b^{00}\,d\boldsymbol{\sigma}. \tag{53}$$

On substituting (51) into (53) and on calculating the energy inside a sphere with $r \to \infty$, we obtain

$$E = \frac{c^5}{16\pi k}\,\frac{1}{c}\,4\pi r^2\,\frac{a}{r^2} = \frac{ac^4}{4k}. \tag{54}$$

If we set $a = 4km/c^2$, then formula (51) determines the mass (m) as the source of the gravitational field, while formula (54) determines it as the quantity defining the total energy of the system:

$$E = mc^2, \tag{55}$$

as is already required by the special theory of relativity.

For a static spherically-symmetric space (i.e., a space possessing a three dimensional rotation group about one point and a one dimensional group of time translations) the general form of the metric in the spherical polar system of coordinates has the following form:

$$ds^2 = U(r)\, c^2\, dt^2 - U^{-1}(r)\, [a^2(r)\, dr^2 + b^2(r)\, (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)]. \tag{56}$$

The metric of the comparison space is given by

$$\overset{0}{ds^2} = c^2\, dt^2 - dr^2 - r^2\, (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2). \tag{57}$$

In the case of harmonic correspondence the following condition is satisfied[7]:

$$d\,(b^2/a)/dr - 2ar = 0. \tag{58}$$

In particular, the metric of the space around a single charged particle has in the case of harmonic correspondence the form

$$ds^2 = Uc^2dt^2 - U^{-1}dr^2 - (r+\alpha)^2\, (\sin^2\vartheta\, d\varphi^2 + d\vartheta^2), \tag{59}$$

$$U = (r-\alpha)/(r+\alpha) + \varepsilon^2/(r+\alpha)^2,$$
$$\alpha = km/c^2, \quad \varepsilon^2 = kq^2/c^4. \tag{60}$$

For $\epsilon^2 = 0$ it goes over into the metric calculated by Fock[7] for an uncharged particle:

$$ds^2 = \frac{r-\alpha}{r+\alpha} c^2\, dt^2 - \frac{r+\alpha}{r-\alpha} dr^2 - (r+\alpha)^2\, (\sin^2\vartheta\, d\varphi^2 + d\vartheta^2). \tag{61}$$

Since

$$g^{00} = -c^2/U, \quad \Pi = c\,(r+\alpha)/r^2,$$
$$-b^{00} = c^{-1}\,[1 + 4\alpha/r + O(r^{-2})], \tag{62}$$

then the energy of a charged particle is equal to

$$E = \alpha c^4/k = mc^2 \tag{63}$$

and does not depend on the charge (on ϵ^2), if ϵ and α are independent.

We recall that the arguments given above hold only for spaces which are topologically equivalent to a Minkowski space, i.e., not for the exact metrics of Nordström and Schwarzschild, but for particles which have these metrics at distances which are at least greater than the gravitational radii, but whose internal structure does not permit a deviation of the space from a pseudoeuclidean topology.

7. EINSTEIN-ROSEN WAVES

From formula (48) it can be seen that in order for transport of energy to occur (ϑ^{0i} with $i \neq 0$ should differ from zero) it is necessary that in the case of a harmonic gauge and in terms of coordinates in which $\overset{0}{g}{}^{0i} = 0$ for $i \neq 0$ the following expressions should hold: $b^{0i} \neq 0$ and $\overset{0}{\Box} b^{0i} \neq 0$. It is exactly this situation which contains the reason for the paradox of the Einstein-Rosen waves discovered by Rosen (cf., [8]), that cylindrical waves do not transmit energy and, moreover, if electromagnetic waves are also present the total energy flux is still equal to zero.

The metric for the cylindrical waves has the form

$$ds^2 = e^{u-v}\,(dr^2 - dt^2) + r^2 e^{-v}\, d\varphi^2 + e^v\, dz^2, \tag{64}$$

where u and v depend on r and t. Calculations yield

$$\Pi = \sqrt{\overset{0}{g}/g} = e^{u-v}, \quad b^{00} = \Pi g^{00} = -1, \quad b^{11} = \Pi g^{11} = 1,$$
$$b^{0i} = 0, \quad i \neq 0, \tag{65}$$

from which it follows that the gauge is harmonic: $b^{ii}_{|i} = 0$ for any arbitrary i (there is no summation).

However, we also have $\vartheta^{00} = 0$, $\vartheta^{0i} = 0$ for any i, which follows immediately from (65) and (48). This means that the general form of the metric has already been chosen in such a way that energy is neither contained nor transported. But the fact that the waves are real is manifested in their spin. Thus $S^{t\varphi}_{\varphi}$, $S^{r\varphi}_{\varphi}$, S^{tz}_{z}, S^{rz}_{z} all differ from zero. Indeed, on taking into account the fact that $b^{\varphi\varphi} = e^u/r^2$, it can be easily verified that in accordance with (49) we have:

$$S^{t\varphi}_{\varphi} = b^{\varphi\varphi}_{|t}\,(\overset{0}{g}{}^{tt}\overset{0}{g}_{\varphi\varphi} - b^{tt} b_{\varphi\varphi}) = r^{-2}e^u\,(e^{-u} - 1)\, r^2\, \partial u/\partial t$$
$$= (1 - e^u)\, \partial u/\partial t \neq 0 \quad \text{etc.} \tag{66}$$

We note that the vanishing of the energy-momentum tensor is not an exceptional case in the general theory of relativity, since formula (48) does not require a positive definite value for the energy density and even admits a negative density.

---

[1] I. I. Gutman, JETP 37, 1639 (1959), Soviet Phys. JETP 10, 1162 (1960).

[2] Ya. I. Pugachev, Thesis, Moscow State Univ., 1955.

[3] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1962.

[4] E. Noether, Math. Phys. Kl. 2, 235 (1918) (cf. Russ. Transl.: E. Noether. In the collection

of articles "Variational Principles of Mechanics," Moscow, 1961, p. 611).

[5] I. A. Schouten and D. J. Struik, Introduction to New Methods of Differential Geometry (Russ. Transl. 1, Gostekhizdat, 1939).

[6] S. Gupta, Proc. Phys. Soc. (London) **A65**, 608 (1952) (cf. Russ. Transl.: S. Gupta. In the collection of articles "Newest Problems in Gravitation," Moscow, 1961).

[7] V. A. Fock, Teoriya prostranstva, vremeni i tyagoteniya (Theory of Space, Time, and Gravitation), Fizmatgiz, 1961, p. 280.

[8] J. Weber and J. Wheeler, Revs. Modern Phys. **29**, 505 (1957) (cf. Russ. Transl. in the collection of articles "Newest Problems in Gravitation," Moscow, 1961).

Translated by G. Volkoff
308