MOMENTS OF INERTIA OF BETA AND GAMMA VIBRATIONAL BANDS

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The difference of the moments of inertia of the β and γ vibrational bands and the ground state are calculated on the basis of a microscopic vibrational model. In the adiabatic approximation for vibrations, with accuracy to a certain function, the result is the same as that obtained by phenomenological calculations performed with the Hamiltonian of O. Bohr. Nonadiabaticity of the vibrations yields the result that the difference of the moments of inertia changes sign near the boundary of stable deformation.

1. Two types of well investigated collective levels exist in the region of atomic nuclei with stable deformation—rotational and vibrational. The latter include, in particular, the β and γ vibrational levels which, like the ground-state level of the atomic nucleus, have a rotational structure. The interaction between the β and γ oscillations and the rotation cause the experimental values of the moments of inertia of the ground and vibrational bands to differ by 10—25%.

The difference $J_K - J_0$ between the moments of inertia of the first vibrational state (β and γ oscillation level) and the ground state was calculated in the framework of the hydrodynamic model by Faessler and Greiner[1]; it was found that $J_K - J_0 < 0$ in the entire region of deformed nuclei. Actually, from the presently available experimental data we can conclude that $J_K - J_0$ is negative only at the start and the end of the region of stable deformation, and is positive in the middle of this region.

It will be shown in the present paper that the change in sign of $J_K - J_0$ is the consequence of pairing in the system, on the one hand, and of the violation of the adiabaticity of the β and γ oscillations in the middle of the deformed nuclei region, on the other. The oscillations for a system of particles of one sort with pairing and residual quadrupole-quadrupole interaction are described in this case microscopically[2], while the rotation is considered in the framework of the forced-rotation model[3].

2. We consider a system of particles described by a model Hamiltonian of the form

$$H = H_0 + H_s - \frac{\varkappa}{2} \sum_{\mu=0,\ \pm 2} q_\mu (12)\, q_{-\mu} (1'2')\, a_1^+ a_2 a_{1'}^+ a_{2'},$$

where H_0 is the Hamiltonian of the self-consistent field of the nucleus; its eigenfunctions and eigenvalues are $\varphi_\lambda(\mathbf{r})$ and ϵ_λ, respectively; H_S is the usual pairing-interaction Hamiltonian; κ is the positive quadrupole-quadrupole interaction constant; $q_\mu(\mathbf{r})$ is the operator of quadrupole moment with projection μ, and a_λ^+ and a_λ are the operators for the creation and annihilation of the particles in the Heisenberg representation.

To calculate the moment of inertia we change, as is customary, to a coordinate frame that rotates together with the nucleus, in which the Hamiltonian of the nucleus has the form

$$H' = H - \Omega M_x,$$

where Ω is the angular velocity of rotation about the x axis, perpendicular to the symmetry axis z of the nucleus, and M_x is the projection of the angular momentum on this axis. Since the moment of inertia of the system is given by

$$J = -\Omega^{-1} \partial \langle H' \rangle / \partial \Omega,$$

where $\langle \ldots \rangle$ denotes averaging over the wave functions of the Hamiltonian H', the difference in the moments of inertia of the first vibrational and ground states will have the form

$$J_K - J_0 = -\Omega^{-1} \partial \{ \langle H' \rangle_K - \langle H' \rangle_0 \} / \partial \Omega. \qquad (1)$$

Here $\langle \ldots \rangle_K$ denotes averaging over the state with one phonon, and $\langle \ldots \rangle_0$ denotes averaging over the ground state. We note that we introduce the phonons in the rotating coordinate system.

It is easiest to calcultae $\langle H' \rangle_K - \langle H' \rangle_0$ with the aid of the Bethe-Salpeter equation for two-particle Green's functions. If we neglect the logarithmically small terms which do not change the value of $J_K - J_0$ significantly, we can determine $\langle H' \rangle_K - \langle H' \rangle_0$ by confining ourselves to an equation for only one two-particle Green's function

$$G(12t;\ 34t') = \langle\Phi_0 | T(a_1 a_2^+ a_3 a_4^+) | \Phi_0\rangle,$$

which has the form

$$G(12t;\ 34t') = G_0(12t;\ 34t') - i\varkappa\int d\tau \sum_\mu q_\mu(56')\, q_{-\mu}(65')$$
$$\times [G(15t-\tau)\, G(6'2\tau - t)$$
$$- F(25't-\tau)\, F^+(15t-\tau)]\, G(5'6\tau;$$

Writing G in the form of a product of the wave functions $\varphi_K(12\omega)$ describing the bound state of two particles with projection of the angular momentum on the symmetry axis of the nucleus K with energy ω, we can obtain, by going over to the Fourier representation in time, a homogeneous equation in φ:

$$\varphi_K(12\omega) = -\varkappa \sum_{3,4,\mu} q_\mu(34)\int\frac{d\varepsilon}{2\pi i}[G(13\varepsilon+\omega)\, G(42\varepsilon)$$
$$- F^+(31\varepsilon+\omega)\, F(42\varepsilon)] \sum_{3'4'} q_{-\mu}(4'3')\,\varphi_\mu(3'4'\omega). \qquad (2)$$

The functions G and F are determined from the equations (see [4])

$$(\varepsilon - H_0(\mathbf{r}) - V(\mathbf{r}))\, G(\mathbf{r}\mathbf{r}'\varepsilon) = \delta(\mathbf{r}-\mathbf{r}') + i\Delta(\mathbf{r})\, F(\mathbf{r}\mathbf{r}'\varepsilon),$$
$$(\varepsilon + H_0(\mathbf{r}) - V(\mathbf{r}) - 2\varepsilon_0)\, F(\mathbf{r}\mathbf{r}'\varepsilon) = -i\Delta^*(\mathbf{r}) G(\mathbf{r}\mathbf{r}'\varepsilon),$$
$$\Delta(\mathbf{r}) = \gamma(\mathbf{r})\int\frac{d\varepsilon}{2\pi i} F^+(\mathbf{r}\mathbf{r}\varepsilon), \qquad (3)$$

where $V(\mathbf{r}) = -\Omega M_X$ and ϵ_0 is the Fermi surface energy.

As follows from (2), rotation leads to an interaction between the β and γ oscillations, but if the inequality $\omega_\beta, \omega_\gamma \gg \omega_\beta - \omega_\gamma \gg 1/J_0$ is satisfied, then it can be shown that the terms that cause the coupling of the β and γ oscillations in (2) are at least $1/J_0(\omega_\beta - \omega_\gamma)\rho_0\omega_{\beta,\gamma}$ times smaller than the diagonal terms (ρ_0 is the level density at the Fermi surface). Discarding these terms, we obtain an equation for the excitation energy $\omega_K = \langle H'\rangle_K - \langle H'\rangle_0$ with projection K:

$$1 = -\varkappa\sum q_{21}^K q_{34}^K \int\frac{d\varepsilon}{2\pi i}\{G(13\varepsilon+\omega)\, G(42\varepsilon)$$
$$- F^+(31\varepsilon+\omega)\, F(42\varepsilon)\}. \qquad (4)$$

Generally speaking, for β oscillations this equation is not accurate because of the additional terms derived from H_S, but it can be shown that they are $A^{-1/3}$ times smaller than the main terms in (4).

Thus, to determine the sought-for value of $J_K - J_0$ it is necessary to find the correction to the excitation energy ω_K', proportional to Ω^2 (there is naturally no correction proportional to Ω).

Gathering in (4) all terms proportional to Ω^2 we obtain an equation for ω_K':

$$\omega_K\sum |q_{12}^K|^2\,\Pi_1(12\omega_K) = -\sum\{q_{21}^K q_{32}^K[\Pi_2(123;\omega_K)$$
$$+ \Pi_2(123;-\omega_K)] + q_{21}^K q_{34}^K \Phi_2(1234;\omega_K)\}; \qquad (5)$$
$$\Pi_1(12\omega_K) = \int\frac{d\varepsilon}{2\pi i}\left\{G(2\varepsilon)\frac{\partial}{\partial\varepsilon} G(1\varepsilon+\omega_K)\right.$$
$$\left. + F(2\varepsilon)\frac{\partial}{\partial\varepsilon} F(1\varepsilon+\omega_K)\right\},$$
$$\Pi_2(123\omega_K) = \int\frac{d\varepsilon}{2\pi i}\left\{G''(13\varepsilon)\, G(2\varepsilon+\omega_K)\right.$$
$$\left. + F''(13\varepsilon)\, F(2\varepsilon+\omega_K)\right\}$$
$$\Phi_2(1234\omega_K) = \int\frac{d\varepsilon}{2\pi i}\left\{G'(13\varepsilon+\omega_K)\, G'(42\varepsilon)\right.$$
$$\left. - F'^+(31\varepsilon+\omega_K) F'(42\varepsilon)\right\};$$
$$G(\lambda\varepsilon) = \frac{\varepsilon + (\varepsilon_\lambda - \varepsilon_0)}{\varepsilon^2 - E_\lambda^2}, \qquad F(\lambda\varepsilon) = -\frac{i\Delta}{\varepsilon^2 - E_\lambda^2},$$
$$E_\lambda = \sqrt{(\varepsilon_\lambda - \varepsilon_0)^2 + \Delta^2}. \qquad (6)$$

The first and second order corrections G'F' and G''F'' for the functions G and F are determined here from the system (3).

We note that Eq. (5) does not contain the quadrupole-quadrupole interaction constant κ, so that $J_K - J_0$ is a function of ω_K, β, and of the pair correlation energy Δ.

Since all the functions in front of the matrix elements in (5) have a sharp maximum at the Fermi surface at fixed differences $\epsilon_1 - \epsilon_2 = k_1$, $\epsilon_1 - \epsilon_3 = k_2$, etc., in order to find ω_K' we can use the quasi-classical technique developed by Migdal[4] for the calculation of such sums. In the calculation of the integrals with respect to $d\epsilon d\epsilon_\lambda$ it is convenient to use Feynman's covariant integration technique. Replacing ϵ by ip^0 and $\epsilon_\lambda - \epsilon_0$ by p_λ', we get

$$\int\frac{d\varepsilon}{2\pi i} d\varepsilon_\lambda = \int\frac{d^2p_\lambda}{2\pi}.$$

3. For simplicity we shall continue the calculations by choosing for the self-consistent nuclear potential H_0 an anisotropic oscillator, for which the single-particle level energy is

$$\varepsilon_\lambda = \omega_x(n_x + n_y) + \omega_z n_z,$$

where $\omega_X = \omega_Y$ and ω_Z are the frequencies of the oscillator along the corresponding axes, while n_X, n_Y, and n_Z are the quantum numbers.

The matrix elements of the quadrupole moment operators

$$q_0 = r^2 Y_{20} = \sqrt{5/16\pi}\,(2z^2 - x^2 - y^2),\ q_{\pm2} = r^2 Y_{2\pm2}$$
$$= \sqrt{15/32\pi}\,(x \pm iy)^2$$

differ from zero for transitions within a given shell $N = n_x + n_y + n_z$ and through the shell $N' = N \pm 2$. The operator of time differentiation of the angular momentum $\dot{M}_x = (\omega_y^2 + \omega_z^2)yz$, to which it is convenient to reduce $V(r)$ in (3), also has transitions within the shell and to the neighboring shell $N' = N \pm 1$.

The transitions for which the principal quantum number N changes by one or two are far in the sense that their corresponding energy difference is $\epsilon_1 - \epsilon_2 \gg 2\Delta$. It can be shown that the contribution of near transitions, i.e., transitions in which the number N does not change, is $A^{2/3}$ times larger than the contribution of the far ones, so that we shall neglect far transitions from the very outset. In addition, the energy difference contributing to the near transitions of the operators q_0 and $q_{\pm 2}$ in the oscillator potential is equal to zero.

Substituting G′, G″, F′, and F″, which are determined from (3), in (6) and using the expressions obtained in [4,5] for the corrections Δ′ and Δ″ to the pair correlation, we get after integrating with respect to d^2p_1 the following expression for ω_K':

$$\omega_K'\omega_K\sum{}' |q_{12}^K|^2 \delta(\varepsilon_1) = \Omega^2\sum{}' M_{14}^x M_{43}^x q_{32}^K q_{21}^{-K} F_1\left(x_{14}; \frac{\omega_K}{2\Delta}\right)\delta(\varepsilon_1) - \Omega^2\sum{}' M_{14}^x q_{43}^K M_{32}^x q_{21}^{-K} F_2\left(x_{14}; \frac{\omega_K}{2\Delta}\right)\delta(\varepsilon_1), \quad (7)$$

where $x_{14} = (\epsilon_1 - \epsilon_2)/2\Delta$, and F_1 and F_2 are some dimensionless functions such that $F_1(x; 0) = F_2(x; 0)$, while the prime at the summation sign denotes that the summation over the indices 2, 3, and 4 is carried out in (7) only with respect to the near transitions of the operators M_x and q_K.

Replacing the summation over the quantum numbers by integration and using (1) we obtain after some manipulation

$$\frac{J_\beta - J_0}{J_0} = \frac{1}{\omega_\beta J_0}\left\{-\frac{3}{2}\left[F_1\left(\nu; \frac{\omega_\beta}{2\Delta}\right) + F_2\left(\nu; \frac{\omega_\beta}{2\Delta}\right)\right] + \frac{8J_r}{5\rho_0}\left[F_2\left(\nu; \frac{\omega_\beta}{2\Delta}\right) - F_1\left(\nu; \frac{\omega_\beta}{2\Delta}\right)\right]\right\},$$

$$\frac{J_\gamma - J_0}{J_0} = \frac{1}{\omega_\gamma J_0}\left\{-\frac{1}{2}\left[F_2\left(\nu; \frac{\omega_\gamma}{2\Delta}\right) + F_1\left(\nu; \frac{\omega_\gamma}{2\Delta}\right)\right] + \frac{8J_r}{5\rho_0}\left[F_2\left(\nu; \frac{\omega_\gamma}{2\Delta}\right) - F_1\left(\nu; \frac{\omega_\gamma}{2\Delta}\right)\right]\right\}, \quad (8)$$

where $\nu = (\omega_y - \omega_z)/2\Delta$ and J_r is the rigid-body moment of inertia.

The explicit form of F_1 and F_2 is rather complicated, but to investigate the qualitative behavior of $J_K - J_0$ it is sufficient to consider two limiting cases. If $\omega_K/2\Delta \ll 1$, i.e., the oscillations are adiabatic, then

$$\frac{J_\beta - J_0}{J_0} = \frac{1}{\omega_\beta J_0}\left\{-3f_0(\nu) + \left(\frac{\omega_\beta}{2\Delta}\right)^2 \frac{J_r}{5\rho_0} f_1(\nu)\right\},$$

$$\frac{J_\gamma - J_0}{J_0} = \frac{1}{\omega_\gamma J_0}\left\{-f_0(\nu) + \left(\frac{\omega\gamma}{2\Delta}\right)^2 \frac{J_r}{5\rho_0} f_1(\nu)\right\};$$

$$f_0(x) = \frac{3}{4x^2(1+x^2)^2}\{-2 - 5x^2 + 6\alpha x^4 + g(x)(2 + 5x^2 + 2\alpha x^4 - 4\alpha x^6)\},$$

$$f_1(x) = \frac{1}{x^2(1+x^2)^3}\{9 + 18x^2 + 40x^4 + 16x^6 - 3g(x)(3 + 8x^2)\},$$

$$\alpha(x) = \frac{g(x)}{x^2 g(x) + \ln 20}, \qquad g(x) = \frac{\operatorname{arg\,sinh} x}{x\sqrt{1+x^2}}. \quad (9)$$

In obtaining this expression we neglected in f_0 the terms proportional to a^2. In addition, for the sake of simplicity we left out of the second term the quantities with Δ′ and Δ″, the contribution of which does not exceed 30% of the calculated terms.

The sign and the order of magnitude of the first terms in (9) is the same as in the phenomenological model of interaction between the oscillations and the rotation[1]. This effect is connected with the increase in the mean value of the square of the quadrupole moment (and consequently also the deformation of the nucleus) in the excited vibrational state. Its appearance is due to the fact that rotation of the nucleus is accompanied by a change in the pair correlation energy, and this leads to a change in the oscillation frequency. It turns out here that a small change in Δ does not cause the oscillation frequency to change in the adiabatic approximation.

In the other limiting case $(2\Delta - \omega)/2\Delta \ll 1$ the first term of (8) can be neglected, and we obtain

$$(J_\beta - J_0)/J_0 = (J_\gamma - J_0)/J_0 = 8J_r/5\rho_0\Delta\nu_1^2 J_0. \quad (10)$$

Thus, it follows from (9) and (10) that as $\omega/2\Delta$ increases from 0 to 1 the quantity $J_K - J_0$, which is negative when $\omega/2\Delta \ll 1$, changes sign at some value $(\omega/2\Delta)_0$. Let us trace the variation of $J_K - J_0$ for the β and γ oscillations in the region of the rare-earth elements. The value of $\omega/2\Delta$ for the β and γ oscillations changes from 0.3 at the start and end of this region to 0.6 at its middle. $J_\beta - J_0$ reverse sign when $(\omega/2\Delta)_0 = 0.3$ and 0.4, respectively. The theoretical values of this quantity, determined from (8) with $\nu = 0.8$, corresponding to the start of the stable deformation region, are the same for the β and γ oscillations and are equal to 0.2.

We note that the reversal of the sign of $J_K - J_0$ is essentially related to the presence of

Comparison of experimental and theoretical values of $(J_K - J_0)$ calculated by formula (8).

Nucleus	ν_n	$\frac{\omega_\beta}{2\Delta}$	$\frac{\omega_\gamma}{2\Delta}$	$(J_\beta - J_0)/J_0$		$(J_\gamma - J_0)/J_0$	
				theory	experiment	theory	experiment
Sm^{152}	0.83	0.27	0.42	0.01	—0.06	0.06	—0.14
Gd^{154}	0.77	0.27	0.39	0.03	—0.10	0.10	—0.12
Du^{160}	1.12	—	0.51	—	—	0.12	0.05
Er^{166}	1.10	0.91	0.49	0.34	~ 0	0.12	0.10
Er^{168}	1.40	—	0.53	—	—	0.09	0.07
W^{184}	0.89	—	0.53	—	—	0.17	0.07
Os^{186}	0.72	—	0.42	—	—	0.10	—0.29
Th^{228}	0.89	—	0.61	—	—	0.20	0.13
Th^{230}	0.97	0.42	0.51	0.08	0.26	0.14	—
U^{232}	1.13	0.49	0.61	0.11	~0	0.18	0.05
U^{234}	1.16	0.61	0.79	0.14	0.22	0.25	0.28

pairing. Formal calculations of this quantity in a model without pairing show that $(J_K - J_0)/J_0 \sim 1/\omega_K J_0$ always and that it is negative.

The foregoing calculations were made for particles of the same kind. For a real nucleus, the excitation energy ω_K should be obtained from the coupled system of equations describing the oscillations of the protons and neutrons. By way of illustration, the table presents the results of a comparison of the calculations for one kind of particles (neutrons) with experiment. The theoretical values of $(J_K - J_0)/J_0$ were calculated in the oscillator-potential model[4], and the level density at the Fermi surface for particles and the rigid-body moment of inertia were calculated from the formulas

$$p_0 = \tfrac{3}{76} A,\ \text{MeV}^{-1},\quad J_T = \tfrac{2}{5} AR_0^2(1 + 0.31\,\beta),$$

$$R_0 = 1.2 \cdot 10^{-13} A^{1/3},\ \text{cm}.$$

It is seen from the table that at the start and end of the rare-earth region we have theoretically $J_K - J_0 > 0$, which apparently is connected with the crudeness of our model.

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[1] A. Faessler and W. Greiner, Z. Physik **168**, 425 (1962).

[2] Yu. T. Grin' and D. F. Zaretskiĭ, Izv. AN SSSR, ser. fiz. **24**, 1169 (1960), Columbia Tech. Transl. p. 1169. D. R. Bes, Mat. fys. Med. Dan. Vid. Selsk. **33**, No 2. 1961.

[3] D. Inglis, Phys. Rev. **96**, 1059 (1954).

[4] A. B. Migdal, JETP **37**, 249 (1959), Soviet Phys. JETP **10**, 176 (1960).

[5] Yu. T. Grin' and I. M. Pavlichenkov, JETP **43**, 465 (1962), Soviet Phys. JETP **16**, 333 (1963).

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