THE DEFINITION OF THE SYMMETRY GROUP OF A QUANTUM SYSTEM. THE ANISOTROPIC OSCILLATOR

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The usual definition of the symmetry group as the group of unitary operators which commute with the Hamiltonian is insufficient. The concepts of maximal, minimal, incomplete and excessive symmetry groups are introduced. A prescription is given for the construction of the minimal symmetry group. It is sufficient for this purpose to choose a group with the necessary dimensionalities of its irreducible representations. As an example, a solution is found of the problem proposed by Jauch and Hill, that of finding the symmetry group for the anisotropic oscillator.

1. THE DIFFERENT TYPES OF SYMMETRY GROUPS

WE may consider the symmetry group of a quantum mechanical system to be the group of unitary operators commuting with H, the energy operator of the system. But such a definition is insufficient and requires more exact statement.

We shall call such a group consisting of all such unitary operators the maximal symmetry group S. It is then easy to see that this group is "too general," and is therefore not of interest. In the simplest case, when all the energy levels E_n are nondegenerate and numbered by an index n, every unitary operator U which commutes with H will be diagonal in the energy representation, its matrix elements will have the form $U_{nn'} = \delta_{nn'} \exp(i\alpha_n)$, and thus the maximal symmetry group will be isomorphic to the direct product of one-parameter cyclic (one-dimensional unitary) groups—one for each energy level of the system.

If the energy levels are degenerate, with degree of degeneracy λ_n for level E_n, the matrix of U will be quasidiagonal in the energy representation; unitary matrices U_n of dimension λ_n will be along the diagonal, and the maximal group will be isomorphic to the direct product of the unitary groups of dimension λ_n, which give independent unitary transformations in the subspaces of eigenfunctions corresponding to a given energy level:

$$S = U_1 \times U_2 \times \ldots \times U_n \times \ldots .$$

From these remarks we see that the group S is trivial in nature and in no way "explains" the degeneracy, but rather merely "describes" it. Of the whole set of irreducible representations of the maximal symmetry group, only a small part is realized in a quantum mechanical system.

In fact, the irreducible representations of a direct product of groups are the direct products of the irreducible representations of the group factors. Consequently to each energy level there corresponds a very special form of irreducible representation of the group S, where we choose the identity representation for all except one of the group factors, while for that one we take the representation by the unitary matrices themselves. All the other representations of the maximal group do not correspond to any energy level of the system. Such groups, which contain "superfluous" representations, may be called "excessive." We see that the maximal group is "extremely" excessive. An example of an excessive group is Fock's[1] four-dimensional rotation group, with respect to the discrete part of the hydrogen spectrum: the representations of the four-dimensional rotation group have dimensionality $(2l+1)(2l'+1)$, where l and l' are integral or half-integral, whereas in the hydrogen atom only the case of $l = l'$ is realized (four-dimensional spherical functions).

Usually when one speaks of the symmetry group of a system one means some subgroup of the maximal group S. By the minimal complete or simply minimal symmetry group Σ we mean a subgroup of S for which 1) to each irreducible representation of the group there corresponds at least one energy level of the system; 2) every subgroup of the group Σ no longer completely explains the degree of degeneracy of all the levels, i.e., there

is at least one energy level to which there corresponds a reducible representation of the subgroup, i.e., every subgroup of Σ is an incomplete symmetry group. An example of an incomplete symmetry group is the rotation group in the case of the hydrogen atom.

Thus, for example, from this point of view the n-dimensional unitary group defined in various papers [2,3] for the n-dimensional isotropic oscillator will not be minimal; but the n-dimensional unimodular unitary group will be minimal.

The conditions defining the minimal symmetry group are obviously sufficient to explain the degeneracy. But in general the symmetry group defined in this way is not unique, and one can sometimes construct several minimal groups, where these groups of operators are not isomorphic, i.e., they differ by more than a unitary transformation. This is especially easy to follow on an example where the degree of degeneracy of all levels is finite, so that the minimal group contains a finite number of elements. For a unique definition one requires auxiliary conditions, related to a more detailed description of the system, for example locality of the transformations of the group in coordinate, momentum, or some other space, i.e., the requirement that the operators of the group have the form

$$U\psi(x, y, z) = C(x, y, z)\psi[f_1(x, y, z), \quad f_2(x, y, z), \quad f_3(x, y, z)].$$

This requirement is satisfied by rotations, reflections, translations, various similarity transformations, i.e., by almost all the symmetry operators which are usually used in physics.

From these various arguments we arrive at the following method for constructing the minimal complete symmetry group of a system.

1. We determine the degree of degeneracy of the energy levels of the system.

2. We look for a group for which one finds irreducible unitary representations with just these dimensionalities and no others. In general this part of the solution is not unique.

3. We construct unitary (or infinitesimal Hermitean, if the group is continuous) operators, each of which gives a unitary transformation in the subspace of the given energy level according to an irreducible representation of the appropriate dimensionality. This part of the problem is solved uniquely, to within a common unitary transformation of all the operators.

4. We express these operators in terms of canonical variables, going over from the energy representation to the one which interests us.

Of course such a formulation of the problem, where one knows the energy levels and wave functions, while the symmetry properties of the system are unknown, is artificial in most cases. Usually it is very much easier to establish the symmetry of the system than to solve the Schrödinger equation. That this is not always the case is shown by the examples of the hydrogen atom and the isotropic oscillator.

We shall now show that by using the apparently trivial scheme described here, one can solve the problem of the symmetry group of the anisotropic oscillator with commensurable frequencies. Numerous attempts have been made to solve this problem (cf., for example, [2]).

2. THE ANISOTROPIC OSCILLATOR

Let us consider an anisotropic oscillator with the ratio 1:2 in the frequencies. We use the system of units in which $h = m = \omega_1 = 1$, $\omega_2 = 2$. The energy operator then has the form

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + x_1^2 + 4x_2^2).$$

The wave functions $\psi_{n_1 n_2} = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$ of the system are determined by the two quantum numbers n_1 and n_2, which take on values $0, 1, 2, \ldots$, while the energy is

$$E = n_1 + 2n_2 + {}^3/_2 = n + {}^3/_2; \quad n = 0, 1, 2, \ldots .$$

It is easy to see that the first two levels are nondegenerate, there are two wave functions corresponding to the next two, three each for the next two, etc. To the level with quantum number n there correspond the $[n/2] + 1$ functions $\psi_{n,0}$, $\psi_{n-2,1}, \ldots \psi_{n-2k,k} \ldots$, (the sign [ ] denotes the integral part of the number).

A group having representations with these dimensionalities is well known. It is the two-dimensional unitary unimodular group, which is the covering group of the rotation group.

The infinitesimal operators of this group can be chosen as usual:

$$L_\pm = L_1 \pm iL_2, \quad L_3, \quad L_iL_j - L_jL_i = iL_k,$$

where i, j, k are a cyclic permutation of the numbers 1, 2, 3. If the representation has dimensionality $2l + 1$ (where l is integral or half-integral), one can choose the basis vectors to satisfy the equations

$$L_3\varphi_m = m\varphi_m, \quad m = -l, -l+1, \ldots, l;$$

$$L_-\varphi_m = \alpha_{lm}\varphi_{m-1},$$

$$L_+\varphi_{m-1} = \alpha_{lm}\varphi_m, \; \alpha_{lm} = \sqrt{(l+m)(l-m+1)}.$$

Assuming that the set of functions $\psi_{n,0}$, $\psi_{n-2,1}$, ... form such a basis, where the first function corresponds to the lowest value of m, and the last to the largest, we get

$$l = [n/2]/2, \qquad m = n_2 - l.$$

Thus we have defined the operators L by giving their effect on any of the functions $\psi_{n_1 n_2}$.

In order to express $L_\pm$ and L_3 explicitly in terms of canonical variables, we introduce the creation and annihilation operators:

$$b_1 = 2^{-1/2}(x_1 + ip_1), \quad b_1^+ = 2^{-1/2}(x_1 - ip_1);$$
$$b_2 = x_2 + ip_2/2, \quad b_2^+ = x_2 - ip_2/2,$$

whose effect on the wave functions is defined by the equations

$$b_k\psi_{n_k} = \sqrt{n_k}\,\psi_{n_k-1}, \quad b_k^+\psi_{n_k-1} = \sqrt{n_k}\,\psi_{n_k},$$
$$b_k^+ b_k\psi_{n_k} = n_k\psi_{n_k}, \qquad k = 1, 2.$$

Using all these formulas, we get for even n:

$$L_+' = 2^{-1/2} b_1 (b_1^+ b_1)^{-1/2} b_1 b_2^+; \qquad L_-' = 2^{-1/2} b_1^+ (b_1^+ b_1)^{-1/2} b_1^+ b_2;$$
$$L_3' = (2b_2^+ b_2 - b_1^+ b_1)/4;$$

and for odd n:

$$L_+'' = 2^{-1/2} b_1 (b_1 b_1^+)^{-1/2} b_1 b_2^+; \qquad L_-'' = 2^{-1/2} b_1^+ (b_1 b_1^+)^{-1/2} b_1^+ b_2;$$
$$L_3'' = (2b_2 b_2^+ - b_1 b_1^+)/4.$$

Using the projection operator for states with even n

$$P = \cos^2\frac{\pi}{2}(b_1^+ b_1),$$

we can write in general form for each of the pairs L_+', L_+''; L_-', L_-''; L_3', L_3'':

$$L = L' \cos^2\frac{\pi}{2}(b_1^+ b_1) + L'' \sin^2\frac{\pi}{2}(b_1^+ b_1).$$

From the method of construction it is clear that these operators commute with H and satisfy the necessary commutation relations.

In the classical approximation the quantities L' and L'' coincide, since they differ only in the order of the factors, and we get the classical integrals of motion of this system which were found in the paper of Hill and Jauch.[2] These authors tried to generalize their result to the quantum region, but did not succeed, apparently because they did not treat even and odd values of n separately.

We have thus shown that the degeneracy in the anisotropic oscillator can be explained using the two-dimensional unitary unimodular group. The fact that this same group explains the degeneracy for the case of the isotropic oscillator was shown earlier.[2,3]

The very simple example considered here can be generalized to the case of any rational ratio of the frequencies and to an oscillator with any number N of degrees of freedom. In the latter case the minimal symmetry group will be the N-dimensional unitary unimodular group.

The anisotropic oscillator and, in particular, the case of any rational frequency ratio ω_1/ω_2 will be treated in more detail in a paper of L. A. Il'kaeva, to appear in the "Vestnik" of the Leningrad University.

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