

SOUND IN LIQUID HELIUM II NEAR ABSOLUTE ZERO

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Oscillations in helium II are considered for the low temperature regions in which collisions between excitations are unimportant. It is shown that the velocity of sound in this case increases with the temperature as $T^4 \ln(\text{const}/T)$, while the absorption follows the T^6 law and is proportional to the sound frequency.

THE character of sound propagation in liquid helium depends essentially on the value of the dimensionless parameter $\omega\tau$, where ω is the frequency of the sound and $1/\tau$ is the relaxation frequency of the system, determined by the interaction of the elementary excitations. The value of τ increases monotonically with decrease in temperature.

In the region of comparatively high temperatures, the inequality $\omega\tau \ll 1$ is satisfied. Here the oscillations of the density of the liquid take place slowly in comparison with the relaxation time and all quantities, in particular the pressure, are practically equal to their equilibrium values. The sound velocity in this region is determined in the well-known way from the compressibility, while the absorption is small and is proportional to τ . Upon decrease in temperature, τ becomes of the order of $1/\omega$. This means that upon propagation of sound in the liquid intensive processes take place in the approach to equilibrium. This region of temperatures is therefore characterized by absorption and by dispersion of the sound velocity.

Finally, for still lower temperatures ($\omega\tau \gg 1$), the relaxation processes generally cease to play any sort of a role and the sound again becomes weakly damped. The sound vibrations in liquid He³ in the region $\omega\tau \gg 1$ were considered by Landau.^[1] We shall consider sound in He⁴ close to $T = 0$.

For ordinary frequencies (10^7 – 10^8 cps) the inequality

$$\omega\tau \gg 1 \quad (1)$$

begins to be satisfied in the region of temperatures in which the contribution of the roton to all phenomena can be neglected. We note that collisions between phonons can play the decisive role even upon satisfaction of the inequality (1). Let us consider this by an example of sound absorption. We

write down the connection of the phonon energy ϵ and its momentum p in the form

$$\epsilon(p) = cp(1 - \gamma p^2), \quad (2)$$

where c is the sound velocity at absolute zero.

A quantum of the sound wave cannot be absorbed by a free phonon with the dispersion law (2), since such a process is forbidden by the laws of conservation of energy and momentum. However, if the phonon possesses an energy uncertainty $\hbar/\tau \sim c\gamma p^2(\hbar\omega/c)$, then the indicated process will take place with an appreciable probability. Since $\gamma p^2 \gg 1$, the inequality (1) is satisfied here. In what follows we consider sufficiently low temperatures ($\omega\tau \gg 1/\gamma p^2$) and neglect collisions between phonons. We also assume that $\hbar\omega \ll kT$. This allows us to consider the sound wave classically.

In the temperature region under consideration, the free path of the phonons is significantly larger than the wavelength of the sound. Therefore, it is not possible to apply hydrodynamic equations to the normal part, and one must make use of the kinetic equation for the phonon distribution function $n(\mathbf{p}, \mathbf{r}, t)$, in which one can neglect the collision integral:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial r} \frac{\partial H}{\partial p} - \frac{\partial n}{\partial p} \frac{\partial H}{\partial r} = 0. \quad (3)$$

Here $H = \epsilon(p) + \mathbf{p} \cdot \mathbf{v}_S$, where $\epsilon(p)$ is the energy of the phonon and depends on the liquid density ρ , while \mathbf{v}_S is the velocity of the superfluid part. The quantities ρ and \mathbf{v}_S play the role of external conditions for the phonons.

In order to obtain a complete set of equations, it is necessary to write down two other equations for ρ and \mathbf{v}_S . These equations have the form^[2]

$$\partial\rho/\partial t + \text{div } \mathbf{j} = 0, \quad (4)$$

$$\partial\mathbf{v}_S/\partial t + \nabla(\mu + \mathbf{v}_S^2/2) = 0, \quad (5)$$

where \mathbf{j} is the momentum of a unit volume of the liquid, and $\mu = \partial E / \partial \rho$, E being the energy per unit volume in the system of coordinates in which $\mathbf{v}_S = 0$. The derivative with respect to the density is taken for a constant distribution of excitations, i.e., for constant $n(\mathbf{p})$.

To determine \mathbf{j} we note that in the system where $\mathbf{v}_S = 0$ the momentum is

$$\int \mathbf{p} n d\tau \quad (d\tau = d^3p / (2\pi\hbar)^3).$$

Using the well-known Galilean transformation formula, we get

$$\mathbf{j} = \rho \mathbf{v}_S + \int \mathbf{p} n d\tau. \quad (6)$$

For the energy E we have, obviously,

$$E = E_0 + \int \epsilon n d\tau,$$

where E_0 is the energy when $T = 0$.

Taking the derivative and making use of (4), (5), and (6), we get the desired equations:^[3]

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}_S + \int \mathbf{p} n d\tau) = 0, \quad (7)$$

$$\frac{\partial \mathbf{v}_S}{\partial t} + \nabla \left(\mu_0 + \int \frac{\partial \epsilon}{\partial \rho} n d\tau + \frac{v_S^2}{2} \right) = 0, \quad (8)$$

where μ_0 is the chemical potential at absolute zero.

We set $n = n_0 + n'$, $\rho = \rho_0 + \rho'$, where n_0 and ρ_0 are equilibrium values. Assuming n' , ρ' , and \mathbf{v}_S to be small quantities proportional to $e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$, and linearizing Eqs. (3), (7), (8), we get

$$(\omega - k v) n' + (v k) \frac{\partial n_0}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial \rho} \rho' + \rho v_S \right) = 0,$$

$$\omega \rho' - k \rho v_S - k \int \rho n' d\tau = 0,$$

$$-\omega v_S + k \left(\frac{c^2}{\rho} + \int n_0 \frac{\partial^2 \epsilon}{\partial \rho^2} d\tau \right) \rho' + k \int \frac{\partial \epsilon}{\partial \rho} n' d\tau = 0, \quad (9)$$

where $v = \partial \epsilon / \partial p$. We have left out the zero subscript on the density and have used the fact that $d\mu_0 = dp_0 / \rho = c^2 dp / \rho$.

We introduce a spherical set of coordinates with the polar axis along the \mathbf{k} vector. The angle between \mathbf{k} and \mathbf{p} is denoted by θ . From the first of Eqs. (9) we find n' :

$$n' = -\frac{\partial n_0}{\partial \epsilon} \rho \frac{v \cos \theta}{\omega/k - v \cos \theta} \left(\frac{\partial \epsilon}{\partial \rho} \rho' + v_S \cos \theta \right). \quad (10)$$

Substituting (10) in the first two equations of (9), and carrying out the integration over θ , we get¹⁾

¹⁾Here we keep only terms containing the logarithm, which is large inasmuch as $v \approx c$ and, as we shall see, $\omega \approx kc$.

$$\begin{aligned} & \left[-\frac{\omega}{k} - \frac{\partial c}{\partial \rho} \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \left(\frac{\omega}{kv} \right)^2 \ln \frac{\omega + kv}{\omega - kv} \right] v_S \\ & + \left[\frac{c^2}{\rho} - \left(\frac{\partial c}{\partial \rho} \right)^2 \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \frac{\omega}{kv} \ln \frac{\omega + kv}{\omega - kv} \right] \rho' = 0, \\ & \left[\rho - \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \left(\frac{\omega}{kv} \right)^3 \ln \frac{\omega + kv}{\omega - kv} \right] v_S \\ & - \left[\frac{\omega}{k} + \frac{\partial c}{\partial \rho} \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \left(\frac{\omega}{kv} \right)^2 \ln \frac{\omega + kv}{\omega - kv} \right] \rho' = 0. \quad (11) \end{aligned}$$

We have neglected the term with n_0 in the first of Eqs. (9), since, as is easily seen, it is small in comparison with the terms containing n' .

The system (11) is a system of two linear homogeneous equations. The condition for the existence of a nontrivial solution is defined by setting the corresponding determinant equal to zero:

$$\begin{aligned} \left(\frac{\omega}{k} \right)^2 - c^2 &= - \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \ln \frac{\omega + kv}{\omega - kv} \cdot \frac{c^2}{\rho} (u + 1)^2, \\ u &= \frac{\rho}{c} \frac{\partial c}{\partial \rho}. \quad (12) \end{aligned}$$

For $T = 0$, the term on the right hand side of (12) vanishes and one gets $\omega = ck$, as one should. At low temperatures, we assume $\omega = ck + k\delta c$, $\delta c \ll c$. For δc we have then

$$\delta c = -\frac{c}{2\rho} (u + 1)^2 \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \ln \frac{2}{3\gamma \rho^2}. \quad (13)$$

The integral here is transformed in the following fashion:

$$- \int_0^\infty \frac{p^4 dp}{4\pi^2 \hbar^3} \frac{\partial n_0}{\partial \epsilon} \ln \frac{2}{3\gamma \rho^2} = \frac{T}{c} \frac{\partial}{\partial T} \int_0^\infty \frac{p^3 dp}{4\pi^2 \hbar^3} n_0 \ln \frac{2}{3\gamma \rho^2}.$$

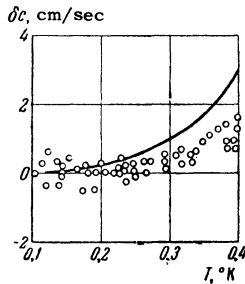
The integrand has a maximum for $\bar{p} \approx 3kT/c$. The slowly changing logarithm can be taken out from under the integral sign, with $p = \bar{p}$ set in it. The remaining integral is easily calculated. As a result, we get

$$\delta c = \frac{\pi^2}{30\hbar^3 \rho} \left(\frac{kT}{c} \right)^4 (u + 1)^2 \ln \left[\frac{2}{27\gamma} \left(\frac{c}{kT} \right)^2 \right]. \quad (14)$$

Thus at sufficiently low temperatures the sound velocity is larger than at absolute zero. This circumstance is important, since in the opposite case the logarithm in Eq. (12) would become complex and this would mean that the sound is strongly damped. Substituting numerical values of all parameters in (14), in particular, $(\partial c / \partial \rho)(\rho / c) \approx 1.8$, $\gamma = 3 \times 10^{37}$,^[4] we get

$$\delta c \approx 20 \cdot T^4 \ln(67/T^2). \quad (15)$$

The dependence $\delta c(T)$ given by Eq. (15) is shown in the drawing. The points denote the experimental data of Whitney and Chase^[5] for the temperature below 0.4° K.



In ^[5] the measurement of the sound velocity is carried out in the temperature range from 0.1 to 1.7°K. Upon decrease in temperature, the sound velocity increases. However, at $T = 0.65^\circ\text{K}$, the character of the temperature dependence changes and increase alternates with decrease. It seems to us that this is evidence that at temperatures below 0.65°K we have the situation described above.

We now consider the question of sound absorption in the region of temperatures under study. As has already been pointed out above, the absorption of a sound quantum by thermal phonons as the result of a three phonon process is forbidden by the laws of conservation of energy and momentum. Therefore the absorption takes place fundamentally by means of a four-phonon process, i.e., a process as the result of which the sound quantum and the

thermal phonon are converted into two thermal phonons. The time rate of change of the number of phonons N in a sound wave is determined by the equation

$$\partial N / \partial t = -N / \tau. \quad (16)$$

The time τ characterizing the four-phonon process was previously computed^[4] in connection with the problem of the time for establishing equilibrium in a phonon gas. The following result was obtained:

$$\frac{1}{\tau} = \frac{(u+1)^4 6!}{(12\hbar^2 \rho)^2 c \gamma (2\pi\hbar)^3} \left(\frac{kT}{c}\right)^6 \frac{\hbar\omega}{c}. \quad (17)$$

From this, it is easy to determine the acoustic absorption coefficient:

$$\alpha = \frac{1}{\tau c} = \frac{(u+1)^4 6!}{(12\hbar^2 \rho c)^2 \gamma (2\pi\hbar)^3} \frac{\hbar\omega}{c} \left(\frac{kT}{c}\right)^6. \quad (18)$$

Thus, for sufficiently low temperatures, the sound absorption is proportional to the frequency and to the sixth power of the temperature.

¹ L. D. Landau, JETP **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957).

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³ I. M. Khalatnikov, JETP **23**, 8 (1952).

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⁵ W. M. Whitney and C. E. Chase, Phys. Rev. Lett. **9**, 243 (1962).