DIRECT NUCLEAR REACTIONS AND INTERACTION IN INITIAL AND FINAL STATES

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A formal scheme for taking into account the interaction in initial and final states in direct nuclear reactions is developed. Use is made of the unitarity of the S matrix and the analytic properties of the reaction amplitude as a function of energy. The solution of the singular integral equation for the partial reaction amplitude is found. It is shown that the presence of an essential singularity of the type e^{-ikR} connected with the size of the nucleus does not alter the result. The solution of the problem is given by the product of two factors, one of which $[\rho_{ll'}(E) = \rho_l(E)\rho_{l'}(E)]$ contains all singularities connected with the interaction in the initial and final states while the singularities of the other are determined by the mechanism of the reaction. The function $\rho_l(E)$ is calculated for a rectangular well. The problem of bound states of the system is considered. The solution for the partial reaction amplitude satisfies physical requirements which determine a unique solution and has the correct behavior near threshold. The solution also takes into account compound nucleus resonance effects in direct nuclear reactions. A boundary value problem is proposed which is equivalent to, but more perspicuous than, the dispersion method.

1. INTRODUCTION

A new approach to the theory of direct nuclear reactions is being developed now[1-3] which makes use of the unitarity of the S matrix and the analytic properties of the amplitudes of the direct processes. The interaction in the initial and final states, which is usually taken into account by the method of distorted waves, is actually connected with the singularities of the amplitude in the energy variable. The method of distorted waves is not sufficiently well-founded. It makes use of optical model wave functions which, strictly speaking, describe well only the scattering phases, i.e., the asymptotic form of the wave function. Moreover, the method of distorted waves presupposes the validity of perturbation theory, which can hardly be justified, since, firstly, nuclear interactions are not small and, secondly, the wave functions of the initial and final states are not orthogonal for a complex potential. In the dispersion method the basic assumptions are formulated more clearly and the numerical calculations are apparently simpler than in the method of distorted waves.

In a preceding paper[3] the authors have considered the solution of the singular Omnes-Muskhelishvili integral equation for taking into account the interaction in the final state. By assuming a model for the phases $\delta_l(E)$, $\tan\delta_l(E) = \sqrt{E}\,Q(E)/P(E)$ [$Q(E)$ and $P(E)$ are arbitrary polynomials], the solution was reduced to a single quadrature determined by the mechanism of the reaction and the scattering phase.

In the present paper we consider the interaction in the initial as well as the final states. We investigate the role of an essential singularity of the type e^{-ikR} (k is the wave number, R is the radius of the nucleus) in the reaction amplitude. The function

$$\rho_l(E) = \exp\left\{\frac{E-E_1}{\pi}\int_0^\infty \frac{\delta_l(E')\,dE'}{(E'-E_1)(E'-E)}\right\} \tag{1}$$

is calculated for the case of scattering by a square well.

The problem of bound states of the system is considered. A boundary value problem is proposed which is equivalent to, but more perspicuous than, the dispersion method.

2. FORMULATION OF THE PROBLEM

Let us consider the amplitude for the reaction $A + x \rightarrow B + y$ as a function of the nonrelativistic variables s, t, and u, of which only two are independent. Let us separate from this amplitude M a part $M^{(0)}$ containing all pole terms and the terms which have branch points in t and u, and introduce the variables E and z, where E is the kinetic energy of the outcoming particle in the center-of-mass system (c.m.s.) and $z = \cos\vartheta$, where ϑ is the angle between the momenta of the particles x and y in the same system.

Let us now write a dispersion relation in the energy on the sheet $\operatorname{Im}\sqrt{E} > 0$ (z fixed) for the function $M'(E,z) = M(E,z) - M^{(0)}(E,z)$, which has only a cut from E_0 to ∞:

$$E_0 = \begin{cases} -Q, & Q>0 \\ 0, & Q<0 \end{cases}, \qquad Q = m_y + m_B - m_x - m_A .$$

Expressing, as usual, the discontinuity of the amplitude across the cut through its absorptive part, we find

$$M'(E,z) = \frac{1}{\pi}\int_{E_0}^{\infty} \frac{A(E',z)}{E'-E-i\eta}\,dE'.$$

We assume for simplicity that no subtractions are needed.

Let us make an expansion into partial waves and use the unitarity condition, retaining in it only those terms which correspond to elastic scattering of x on A and y on B. We obtain the following singular integral equation for the partial amplitude $M_{ll'}(E)$ (l and l' are the angular momenta of the relative motion in the initial and final states; the difference between these is determined solely by the mechanism of the reaction in the approximation that the elastic scattering amplitude is diagonal in l):

$$M_{ll'}(E) = M_{ll'}^{(0)}(E) + \frac{1}{\pi}\int_{E_0}^{\infty} \frac{M_{ll'}(E')\,h_{l'}^{*}(E') + M_{ll'}^{*}(E')\,f_l(E')}{E'-E-i\eta}\,dE'; \tag{2}$$

$$h_l = \begin{cases} e^{i\delta_l}\sin\delta_l, & E>0 \\ 0, & E<0 \end{cases}, \quad f_e = \begin{cases} e^{i\varphi_l}\sin\varphi_l, & E+Q>0 \\ 0, & E+Q<0 \end{cases}, \tag{2a}$$

where $\varphi_l(E+Q)$ and $\delta_l(E)$ are the scattering phase shifts in the initial and final states.

Thus the problem consists in the solution of Eq. (2). In the most general case, when the amplitude for elastic scattering contains terms which are not diagonal in the orbital angular momentum, we obtain a system of integral equations. However, the restriction to the diagonal terms of the elastic scattering amplitude is apparently a sufficiently good approximation.

3. ESSENTIAL SINGULARITY OF THE AMPLITUDE AT INFINITY

In general, the amplitude for a direct process as a function of energy can have an essential singularity at infinity of the type e^{-ikR} connected with the radius of the nucleus, R. We shall show by an example, in which only the final state interaction is taken into account (the generalization to the case of interaction in initial and final states involves no difficulties and does not alter the result), that this singularity of the amplitude does not affect the solution of the problem, i.e., the integral along the large circle can be neglected all the same.

Let us multiply the amplitude $M_l(E)$ by e^{ix} ($x = kR$). One could also take $e^{ikR'}$ ($R' \geq R$) as a multiplier, but the final result is independent of R'. The function $M_l(E)e^{ix}$ vanishes for $|E| \to \infty$ ($\operatorname{Im}\sqrt{E} > 0$), and the dispersion relation can be written in the form

$$M_l(E)\,e^{ix} = M_l^{(0)}(E)\,e^{ix} + \frac{1}{\pi}\int_0^{\infty} \frac{M_l(E')\,e^{ix'}H_l^{*}(E') - M_l^{(0)}(E')\sin x'}{E'-E-i\eta}\,dE'; \tag{3}$$

$$H_l^{*}(E) = e^{-ix}\,[h_l^{*}(E)\,e^{-ix} + \sin x]. \tag{4}$$

Equation (3) is reduced to a boundary value problem by considering the integral of the Cauchy type (see, e.g., [4])

$$F(E) = \frac{1}{2\pi i}\int_0^{\infty} \frac{M_l(E')\,e^{ix'}H_l^{*}(E') - M_l^{(0)}(E')\sin x'}{E'-E}\,dE', \tag{5}$$

where E is complex.

The discontinuity of the function F(E) across the real axis is equal to

$$F(E_+) - F(E_-) = M_l(E)\,e^{ix}H_l^{*}(E) - M_l^{(0)}(E)\sin x. \tag{6}$$

We have from (3)

$$M_l(E) = M_l^{(0)}(E) + e^{-ix}\,2iF(E_+). \tag{7}$$

Substituting (7) in (6), we find

$$F(E_+)\,[1 - 2iH_l^{*}(E)] - F(E_-) = M_l^{(0)}(E)\,h_l^{*}(E)\,e^{-ix}. \tag{8}$$

Using the equality

$$1 - 2iH_l^{*}(E) = \exp\{-2i(\delta_l^{*} + x)\}, \tag{9}$$

we obtain the solution in the form

$$M_l(E)\,e^{ix} = M_l^{(0)}e^{ix} + \frac{\widetilde{\rho}_l(E_+)}{\pi}\int_0^{\infty} \frac{M_l^{(0)}(E')\,h_l^{*}(E')\,e^{-ix'}}{\widetilde{\rho}_l(E'_-)(E'-E-i\eta)}\,dE' + \theta(E)\,\widetilde{\rho}_l(E_+), \tag{10}$$

where $\theta(E)\widetilde{\rho}_l(E_+)$ is the solution of the homogeneous problem; $\theta(E)$ is a function the restrictions on which will be discussed later, and

$$\widetilde{\rho}_l(E) = \exp\left\{\frac{E-E_1}{\pi}\int_0^{\infty} \frac{\delta_l^{*}(E') + x'}{(E'-E_1)(E'-E)}\,dE'\right\}, \tag{11}$$

where E is complex.

Using the equality

$$P\int_0^\infty \frac{\sqrt{E'}\,dE'}{(E'-E_1)(E_1-E)} = 0 \qquad (E_1 > 0), \tag{12}$$

we find

$$\tilde{\rho}_l(E_\pm) = \rho_l(E_\pm)\,e^{\pm ix}, \tag{13}$$

where ρ_l is given by (1). Substituting (13) in (10) and canceling the common factor e^{ix}, we obtain the solution

$$M_l(E) = M_l^{(0)}(E) + \frac{\rho_l(E_+)}{\pi}\int_0^\infty \frac{M_l^{(0)}(E')\,h_l^*(E')\,dE'}{\rho_l(E'_-)(E'-E-i\eta)}$$

$$+\,\theta(E)\,\rho_l(E_+), \tag{14}$$

which is the same as in the case of an amplitude without an essential singularity at infinity.

4. CHOICE OF SOLUTION

Let us consider now the meaning of the function $\theta(E)$. It must belong to the class of functions without singularities in the finite part of the complex plane, i.e., it must be an entire function. As was shown above, the integral along the large circle can be discarded even in the case when the amplitude has an essential singularity at infinity of the type e^{-ikR}. Therefore, the function $\theta(E)$ can also have an essential singularity of this type.

The integral equation has an infinite manifold of solutions from which a unique solution must be chosen. To single out the solution, we use the following requirements: 1) $M_l \to M_l^{(0)}$ if $\delta_l \to 0$ and 2) $M_l \to 0$ if $M_l^{(0)} \to 0$. Both requirements are physically reasonable. Indeed, the vanishing of δ_l for all energies implies that the scattering of that particular partial wave can be neglected, i.e., the amplitude must go to $M_l^{(0)}$. On the other hand, the amplitude M_l must go to zero when $M_l^{(0)} \to 0$, since $M_l^{(0)}$ is connected with the mechanism of the reaction itself.

In [3] the solution with $\theta(E) = 0$ was chosen. This choice satisfies the above-mentioned requirements if $M \to 0$ on the large circle. In the general case the solution of the homogeneous equation has to be taken into account and $\theta(E)$ must be determined by the requirements discussed above.

5. INTERACTION IN INITIAL AND FINAL STATES

Let us now consider (2), in which the interaction in the initial as well as final states is taken into account. Denoting the integral in (2) by $2i\Phi(E_+)$, we reduce the equation to the boundary value problem

$$\Phi(E_+)\left[1 - 2i\frac{f_l + h_{l'}^*}{1 + 2if_l}\right] - \Phi(E_-) = \frac{f_l + h_{l'}^*}{1 + 2if_l}\,M_{ll'}^{(0)}. \tag{15}$$

Here we have used the following relation, which is valid in the two-particle approximation of elastic scattering:

$$\mathrm{Im}\,M_{ll'} = M_{ll'}h_{l'}^* + M_{ll'}^* f_l. \tag{16}$$

Since

$$1 - 2i\,(f_l + h_{l'}^*)/(1 + 2if_l) = \exp\{-2i\,(\delta_{l'}^* + \varphi_l)\}, \tag{17}$$

we can rewrite (15) in the form

$$\Phi(E_+)\exp\{-2i\,(\delta_{l'}^* + \varphi_l)\} - \Phi(E_-)$$

$$= M_{ll'}^{(0)}\exp\{-i\,(\delta_{l'}^* + \varphi_l)\}\sin(\delta_{l'}^* + \varphi_l). \tag{18}$$

The difference between this and the case where only the final state interaction was taken into account consists in the replacement of the phase $\delta_{l'}^*$ by the sum of phases $\delta_{l'}^* + \varphi_l$. Thus the solution of the problem has the previous form (14) with ρ_l replaced by the product of the functions ρ_l and $\rho_{l'}$, referring to the interaction in the initial and final states, respectively. We recall [see (2a)] that the lower limits of the integrals of $\delta_{l'}^*$ and φ_l are 0 and $-Q$, respectively.

6. CALCULATION OF THE FUNCTION $\rho_l(E)$

In contrast to the method of distorted waves, our method does not require the knowledge of the wave functions. The interaction in the initial and final states comes in through the scattering phase shifts only. The phase shifts can be found from experiment by a phase analysis. However, a phase analysis at high energies becomes unfeasible in view of the large number of partial waves. It has been shown in particular cases that the optical model phase shifts which best agree with experiment are very close to the minimizing values of the phase analysis.[5] We may therefore hope to obtain the "experimental" phase shifts $\delta_l(E)$ with the help of the optical model.

The simplest case is the square well for which

$$S_l = e^{2i\delta_l} = -\frac{yj_{l+1}(y)\,h_l^{(2)}(x) - xh_{l+1}^{(2)}(x)\,j_l(y)}{yj_{l+1}(y)\,h_l^{(1)}(x) - xh_{l+1}^{(1)}(x)\,j_l(y)}, \tag{19}$$

where j_l, $h_l^{(1)}$, and $h_l^{(2)}$ are the spherical Bessel and Hankel functions, $x = kR$, $y = \sqrt{2\mu(E+U)/\hbar^2}\,R$, μ is the reduced mass, U is the depth of the well $(U > 0)$, and R is the radius of the well.

S_l is conveniently written in the form

$$S_l = \frac{e^{-ix}\,[P(E) + 2ixQ(E)]}{e^{ix}\,[P(E) - 2ixQ(E)]},$$

$$P(E)=\alpha_l(E)\sum_{n=0}^{E[(l+1)/2]}(-1)^n\frac{[2(l+1-n)]!}{(l+1-2n)!\,(2n)!}(2x)^{2n}$$
$$-2\gamma_l(E)\sum_{n=0}^{E[l/2]}(-1)^n\frac{[2(l-n)]!}{(l-2n)!\,(2n)!}(2x)^{2n};$$
$$Q(E)=\alpha_l(E)\sum_{n=0}^{E[l/2]}(-1)^n\frac{(2l+1-2n)!}{(l-2n)!(2n+1)!}(2x)^{2n}$$
$$-2\gamma_l(E)\sum_{n=0}^{E[(l-1)/2]}(-1)^n\frac{(2l-1-2n)!}{(l-2n-1)!\,(2n+1)!}(2x)^{2n},$$
$$l>0,\quad Q(E)=\alpha_0(E),\quad l=0. \tag{20}$$

For a square well

$$\alpha_l(E)=j_l(y)/2^{l+1}y^l,\quad \gamma_l(E)=j_{l+1}(y)/2^{l+1}y^{l-1}. \tag{21}$$

It follows from (20) and (21) that $y\to x$ and $S_l\to 1$ for $E\to\infty$ (on the real axis). We may conclude from quasiclassical considerations that $\delta_l(E)\to 0$ for $E\to\infty$ and shall therefore assume that $\delta_l(\infty)=0$.

In order to compute $\rho_l(E)$ [see (1)], we write δ_l in the form

$$\delta_l(E)=\frac{1}{2i}\ln\frac{e^{-ix}[P(E)+2ixQ(E)]}{e^{ix}[P(E)-2ixQ(E)]} \tag{22}$$

and consider the integral along the contour C on the sheet $\operatorname{Im}\sqrt{E}>0$ of the function

$$[2\pi i(E'-E)(E'-E_1)]^{-1}\ln\{e^{ix'}[P(E')-2ix'Q(E')]\}.$$

The contour C avoids all singularities of this function including the poles at E and E_1 on the real axis and the branch points of the logarithm given by the equation

$$P(E)-2ixQ(E)=0. \tag{23}$$

All roots E_ν of (23) on the sheet $\operatorname{Im}\sqrt{E}>0$ are poles of S_l corresponding to bound states. The integral along the large circle can be neglected for a square well, since $e^{ix}[P(E)-2ixQ(E)]\to 1$ for $|E|\to\infty$ on the sheet $\operatorname{Im}\sqrt{E}>0$. We note that, as the radius increases, the function $e^{-ix}[P(E)+2ixQ(E)]$ increases on the large circle as e^{-2ix}, i.e., S_l has an essential singularity at infinity in the case of a square well.

We find

$$\rho_l(E)=\prod_{\nu=1}^{n}\left(1-\frac{E_\nu}{E}\right)\Big/ e^{ix}[P(E)-2ixQ(E)], \tag{24}$$

where n is the number of bound states for given l. In [3] we have found an expression for the function $\rho_l(E)$ under the assumption that $\delta_l(0)=0$ and the phase shift is given by

$$\tan\delta_l(E)=\sqrt{E}\,Q(E)/P(E) \tag{25}$$

[Q(E) and P(E) are arbitrary polynomials]. The expression for the phase shifts in the case of the square well (22) differs from (25) in an inessential manner. We have therefore retained the notations Q(E) and P(E). In our case Q(E) and P(E) are entire functions of E, which does not, however, alter the method of calculation.

The above-given expression for $\rho_l(E)$ permits a simpler proof of Levinson's theorem[6] for the case of a square well [$\delta_l(0)-\delta_l(\infty)=n\pi$, where n is the number of bound states for given l], since it is easy to find $\delta_l(0)$ with the help of (24). Indeed, it is seen from (20), (21), and (24) that for $E\to 0$

$$\rho_l(E)\sim E^{-n}. \tag{26}$$

On the other hand, an estimate of the behavior of the Cauchy-type integral near the left end of the integration contour[4] leads to the expression

$$\rho_l(E)\sim E^{-\delta_l(0)/\pi}. \tag{27}$$

Comparing (26) and (27), we find $\delta_l(0)=n\pi$ and hence $\delta_l(0)-\delta_l(\infty)=n\pi$ [we have chosen the branch of $\delta_l(E)$ for which $\delta_l(\infty)=0$].

In the general case one can write

$$\rho_l(E)=\prod_{\nu=1}^{n}\left(1-\frac{E_\nu}{E}\right)\Big/ f(-k), \tag{28}$$

where f(k) is the Jost function (for the definition of the Jost function see, e.g., the paper by Newton [7]). We note that the function $\rho_l(E)$ is normalized such that $\rho_l(E)\to 1$ for $E\to\infty$.

7. THE BOUNDARY VALUE METHOD

The problem of the interaction in the initial and final states is connected with a singular integral equation which is solved by reducing it to a boundary value problem for a certain auxiliary function. The boundary value problem can be formulated directly for the amplitude M(E) (the index l will be omitted from now on), if its singularities are known. We note that the knowledge of the singularities is necessary also for writing down the dispersion relation. For simplicity we shall only consider the case of the final state interaction.

The amplitude has a right-hand cut L_0 along the positive real axis connected with the interaction in the final state and the singularities of the function $M^{(0)}(E)$ determined by the mechanism of the reaction. The boundary value problem has the form

$$M(E_+)-M(E_-)=2i\bar{\chi}(E), \tag{29}$$

where $M(E_\pm)$ are the values of the function M(E) on the left and right sides of the cut, if we define a beginning and an end for each cut; $\bar{\chi}(E)$ is the discontinuity of $M(E)$, which has a definite value on each of the cuts. Let us solve the problem (29) under the assumption that $\bar{\chi}(E)$ contains the required amplitude $M(E)$ only along L_0, where we can write

$$\bar{\chi}(E) = M(E_+)h^*(E) + \Psi(E); \tag{30}$$

$\Psi(E)$ represents the remaining terms in the unitarity condition.

It is convenient to rewrite (29) in the form

$$M(E_+)G(E) - M(E_-) = 2i\chi(E), \tag{31}$$

where the functions $\chi(E)$ and $G(E)$ are considered known, with

$$\chi(E) = \Psi(E), \qquad G(E) = e^{-2i\delta^*(E)}$$

along L_0, and

$$\chi(E) = \bar{\chi}(E)', \qquad G(E) = 1 \tag{32}$$

on the remaining cuts.

The solution of (31) is

$$M(E_+) = \rho(E_+)\left[\frac{1}{\pi}\int\limits_L \frac{\chi(E')dE'}{\rho(E'_-)(E'-E-i\eta)} + \theta(E)\right]. \tag{33}$$

The line L includes all cuts. The two-particle approximation of elastic scattering implies the neglect, in (33), of the integral along L_0 as compared with the integrals along all other cuts.

The equivalence of (14) and (33) is easily shown, if $M^{(0)} \to 0$ on the large circle. Here we must take into account that $h^*(E)/\rho(E_-)$ is the discontinuity of the function $-[2i\rho(E)]^{-1}$ along L_0. Considering now the integral along the contour C (which avoids all cuts L_k) of the function $M^{(0)}(E')/[\rho(E')(E'-E)]$, it is easy to express the integral along the part of the contour which follows the cut L_0 in terms of the integral along the contour around the left-hand cuts L_k $(k \neq 0)$. Then (14) and (33) coincide if

$$\Delta M(E) = M^{(0)}(E_+) - M^{(0)}(E_-) = 2i\chi(E)$$

$$\text{along } L_k \ (k \neq 0). \tag{34}$$

Expression (33) shows that the solution has the form of a product of two factors, one of which is connected with the interaction in the final state while the other has only singularities connected with the mechanism of the reaction. An analogous result has been obtained by Bosco,[8] who considered the analytic properties of the amplitude in the method of distorted waves. In the special case $M^{(0)} = \text{const}$ the solution coincides with the solution of the homogeneous problem. Jackson[9] has considered the relation between the method of distorted waves and the dispersion method for this simple case.

8. BOUND STATES AND BEHAVIOR NEAR THRESHOLD

Let us now investigate our solution as to the behavior of the amplitude near the threshold of the reaction. If the system has n bound states for given l, then $\rho_l(E) \sim E^{-n}$ for $E \to 0$. Since the reaction amplitude must be finite at the origin, we obtain additional conditions on $M^{(0)}$ or on the multiplier of $\rho_l(E)$ in (33).

Let $M(E) \to 0$ on the large circle. Then

$$M(E) = \frac{\rho(E)}{\pi}\int\limits_L \frac{\chi(E')\,dE'}{\rho(E'_-)(E'-E)}. \tag{35}$$

If bound states are present which are formally included in the integral (35) [near the pole $E = E_\nu$ we have $\chi(E) = r_\nu\delta(E-E_\nu)$], we can rewrite (35) by taking out explicitly the pole terms in E:

$$M(E) = \rho(E)\left[\frac{1}{\pi}\int\limits_{L'} \frac{\chi(E')\,dE'}{\rho(E'_-)(E'-E)} + \sum_{\nu=1}^{n} \frac{r_\nu}{\rho(E_\nu)(E_\nu - E)}\right], \tag{36}$$

where L' no longer contains poles corresponding to bound states for the given l.

The condition that the amplitude be finite at $E = 0$ implies the relations

$$\frac{1}{\pi}\int\limits_{L'} \frac{\chi(E')\,dE'}{\rho(E'_-)(E')^k} + \sum_{\nu=1}^{n} \frac{r_\nu}{\rho(E_\nu)E_\nu^k} = 0, \quad k = 1, 2, \ldots, n. \tag{37}$$

The relations (37) connect the discontinuities along the cuts with the residues r_ν and are the generalizations of the analogous formulas of Bosco[10] for the elastic scattering amplitude $h(E)/k$.

Bosco considered the dispersion relation for a scattering amplitude which vanishes at infinity. However, for potentials which vanish for $r > R$ (R is the radius of the potential) the scattering amplitude has no singularities in the finite part of the complex E plane except the right-hand cut and has an essential singularity at infinity. In this case one must consider the Cauchy integral of the function $e^{ikR'}h(E)/k\rho(E)$ $(R' \geq 2R)$, which leads to the following expression for the residue of the scattering amplitude at the pole $E = E_0$ (for simplicity of writing we consider the case where there is only one bound state):

$$r = \frac{e^{-ik_0R'}E_0\rho(E_0)}{\pi}\int\limits_0^\infty \frac{h(E')\sin k'R'}{k'E'\rho(E'_+)}\,dE'. \tag{38}$$

9. CONCLUSION

In the present paper we have considered a formal scheme for taking into account the interaction in the initial and final states in direct nuclear reactions. Use has been made of the unitarity of the S matrix and the analytic properties of the reaction amplitude as a function of energy. The problem of establishing the reaction amplitude from its singularities has been formally divided into two parts: 1) the singularities in t and u determine the mechanism of the reaction described by the function $M^{(0)}$, which has definite discontinuities along the left-hand cuts in the complex energy plane, and 2) the singularities in s are, except for the poles, connected with the interaction in the initial and final states.

We have investigated only the second part of the problem. The mechanism of the process described by the function $M^{(0)}$ has been left out of consideration. The function $M^{(0)}$ and its discontinuities along the left-hand cuts were assumed known.

If the two parts of the problem are considered independently, it must be assumed that the presence of the cut L_0 does not alter the discontinuities of the amplitude on the left-hand cuts. The amplitude without initial and final state interaction corresponds to the Born approximation B(E), which is given by graphs having singularities in t and u and pole-type graphs in s. The inclusion of the pole-type graphs in s guarantees the correct behavior of the amplitude for $E \rightarrow 0$.

At sufficiently high energy, when the threshold effects are unimportant, the pole-type graphs in s can be neglected. We assume here that

$$M(E) \rightarrow B(E) \text{ as } E \rightarrow \infty . \tag{39}$$

However, if the discontinuities along the different cuts are independent, i.e., if $\Delta M = \Delta B$ on the left-hand cuts, condition (39) is not satisfied, as can be seen from (33). The contradiction is avoided by assuming that the discontinuities along the left-hand cuts depend on the presence of the right-hand cut L_0, so that, for example,

$$\Delta M(E) = \rho(E)\,\Delta B(E) \tag{40}$$

along L_k ($k \neq 0$). Then the solution has the simple form

$$M(E) = \rho(E)B(E). \tag{41}$$

The amplitude (41) has all required singularities in the complex energy plane, satisfies the physical requirements which single out a unique solution, and has the correct behavior near the threshold of the reaction.

It should be noted that we have established the reaction amplitude from its singularities on the physical sheet, i.e., we have not explicitly taken into account the complex poles of the amplitude corresponding to unstable states, which lie on the unphysical sheet. However, the presence of the complex poles has an essential effect on the scattering phase which enters in the solution. As can be seen from (33), the reaction amplitude has complex poles corresponding to resonances of the scattering matrix which are contained in the function $\rho(E)$. Thus our solution includes resonance effects of the compound nucleus in direct processes.

In conclusion we use this opportunity to express our sincere gratitude to Prof. I. S. Shapiro for his interest in this work and useful comments and also to L. D. Blokhintsev and E. I. Dolinskiĭ for a discussion of the results.

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