CONSTRUCTION OF A LOCAL QUANTUM FIELD THEORY WITHOUT ULTRAVIOLET DIVERGENCES

G. V. EFIMOV

Joint Institute for Nuclear Research

Submitted to JETP editor January 26, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 2107-2117 (June, 1963)

It is suggested that the difficulties of local theories in scalar quantum field theory connected with the presence of ultraviolet divergences in the perturbation approach are a result of the circumstance that the interaction Lagrangians usually considered grow more rapidly with respect to the scalar fields than the free field Lagrangian, i.e., than φ^2. It is found possible to introduce nonlinear local interaction Lagrangians obeying certain conditions which do not give rise to ultraviolet divergences in second-order perturbation theory. The correction to the mass of the scalar particle is computed.

INTRODUCTION

The problems connected with the presence of ultraviolet divergences in quantum field theories with local interactions play a basic role in the formulation of a consistent and closed theory of microparticles.

The removal of the divergences from the scattering matrix by renormalization does not get rid of the difficulties of the theory, since the infinities are transferred from the matrix elements to the interaction Lagrangian. The whole formulation of the problem of quantum field theory in terms of a Schrödinger equation has therefore a rather provisional character. This has now led the majority of physicists to regard it as necessary to abandon the Hamiltonian and to work only with the S matrix, which is constructed on the basis of the general principles of unitarity, causality, and the spectral conditions. The successes of the development of these ideas in the field of dispersion relations are well known.

Attempts to avoid the divergences usually involve dropping the requirement of the locality of the interaction, which gives rise to new specific difficulties which have not yet been completely resolved.

In the present paper we attempt, on the example of a one-component scalar field, to avoid the difficulties of quantum field theory connected with the ultraviolet divergences by introducing a local interaction Lagrangian which is essentially nonlinear in the scalar fields and satisfies certain requirements. The free field Lagrangian remains unaltered, so that there will be no difficulties in the quantization of the theory. In this respect our model of a quantum field theory is essentially different from the classical nonlinear theory of Born, [1] which no one has yet been able to quantize.

A complete discussion of the second order of perturbation theory is presented. It was found possible to choose such a nonlinear local interaction that no ultraviolet divergences appear in this order. The author is aware of the fact that it is impossible to draw definite conclusions on the possibility of constructing a finite local theory without investigating the higher approximations.

In Sec. 1 we consider the S matrix by the method of functional integration and formulate the basic assumption concerning the possible forms of the interaction Lagrangian. In Sec. 2 the necessary conditions for the absence of ultraviolet divergences in second order perturbation theory are investigated in detail. A class of possible interaction Lagrangians is found in Sec. 3. As an example, we calculate the correction to the mass of the scalar particle in Sec. 4.

1. BASIC ASSUMPTION

We shall consider a one-component scalar meson field. The Lagrangian density is written in the form

$$L(x) = L_0(x) + L_I(x), \tag{1.1}$$

$$L_0(x) = -\frac{1}{2}\left(\mu^2\varphi^2(x) - \frac{\partial\varphi(x)}{\partial x_\nu}\frac{\partial\varphi(x)}{\partial x^\nu}\right), \tag{1.2}$$

$$L_I(x) = -gU(\varphi(x)), \tag{1.3}$$

where $U(\alpha)$ is some function of α, for example,

$U(\alpha) = \alpha^4$ for the self-interaction of the scalar field usually considered.

The S matrix is in the interaction representation

$$S = T \exp\left\{- ig \int d^4x U(\varphi(x)) \right\}; \qquad (1.4)$$

it is a functional of the operator $\varphi(x)$. As usual, we shall assume that the meson operators $\varphi(x)$ in the interaction Lagrangian $L_I(x)$ are normally ordered.

Following Feynman, [2] we assume that the S matrix (1.4) as a functional of the operator $\varphi(x)$ can be written as a superposition of exponential functionals:

$$S = TC \int \delta\Phi \int \delta \Lambda \exp\left\{i \int d^4x\Phi(x) \varphi (x) - i \int d^4x\Phi(x) \Lambda (x)\right\} \exp\left\{- ig \int d^4xU (\Lambda (x))\right\}, \qquad (1.5)$$

where C is a normalization constant chosen such that S = 1 if g = 0. The functional integration goes over the space of the real scalar functions $\Phi(x)$ and $\Lambda(x)$.

Let us rewrite (1.5) in terms of normal operator products with the help of Wick's theorem, written in the form [3]

$$T = N \exp\left\{\frac{1}{2} \iint d^4x_1 d^4x_2 \Delta (x_1 - x_2) \frac{\delta^2}{\delta\varphi (x_1) \delta\varphi (x_2)}\right\}, \qquad (1.6)$$

$$\Delta (x_1 - x_2) = \langle 0 \mid T (\varphi (x_1) \varphi (x_2)) \mid 0\rangle. \qquad (1.7)$$

We obtain

$$S = C \int \delta\Phi : \exp\left\{i \int d^4x\Phi(x) \varphi (x)\right\} : \exp\left\{-\frac{1}{2} \iint d^4x_1 d^4x_2 \Phi(x_1) \Delta (x_1 - x_2) \times \Phi (x_2)\right\} \int \delta\Lambda \exp \left\{- i \int d^4x\Phi (x) \Lambda (x) - ig \int d^4xU (\Lambda (x))\right\}. \qquad (1.8)$$

In this expression we can take the functional integral over Φ, since it is of Gaussian form. We find

$$S = C \int \delta\Lambda : \exp\left\{- \frac{1}{2} \iint d^4x_1 d^4x_2 [\Lambda (x_1) - \varphi (x_1)] \Delta^{-1} (x_1 - x_2) \times [\Lambda (x_2) - \varphi (x_2)]\right\} : \exp \left\{- ig \int d^4xU (\Lambda (x))\right\}, \qquad (1.9)$$

where the function $\Delta^{-1} (x_1 - x_2)$ is defined by the equation

$$\int d^4y \Delta (x_1 - y) \Delta^{-1} (y - x_2) = \delta^{(4)} (x_1 - x_2).$$

Let us investigate (1.9) in more detail. If the function $U(\Lambda)$ grows faster with Λ than Λ^2 (as, for example, in the Hurst-Thirring model, $L_I = g\varphi^3$ or $L_I = g\varphi^4$), the convergence of the integral over Λ is not determined by the quadratic term in Λ arising from the time ordering, but by the interaction Lagrangian $U(\Lambda)$. Therefore, the integral (1.9) will not be analytic in the coupling constant g and cannot be expanded in a power series in g.

This qualitative result is in agreement with the known view that the perturbation series are asymptotic in the coupling constant. [4]

The following consideration also confirms this conclusion. The free meson field represents a set of oscillators, i.e., particles moving in a potential well of the form φ^2. Any interaction which changes the asymptotic form of this well at infinitely large values of φ (for example, φ^4) will, by all appearances, lead to a new system of eigenfunctions in this modified well which is cardinally different from the system of oscillator eigenfunctions and may even be orthogonal to the latter in the limit of an infinite number of degrees of freedom. [5]

In view of all that has been said above, we make the following assumption:

In the quantum theory of a scalar meson field only such interactions are admissible for which the function $U(\varphi)$ in the interaction Lagrangian increases less rapidly with φ than φ^2.

If this principle is valid, then the S matrix for interactions satisfying this principle can be expanded in powers of the interaction and one may also hope that the basic difficulties of present day quantum field theory have been removed.

Anticipating our later discussion, we note that the vanishing of $U(\varphi)$ with increasing φ alone is not a sufficient condition for the absence of ultraviolet divergences.

2. FORMULATION OF THE PROBLEM

In order to choose the interaction Lagrangian $U(\varphi)$, we consider the S matrix in second order perturbation theory, since this order contains the most important divergence of the theory—the correction to the self-mass of the particle.

Using (1.4) and (1.6), we can write the S matrix in second order in the form

$$S_2 = \frac{(- ig)^2}{2} \iint d^4x_1 d^4x_2 \times \exp \left\{\frac{1}{2} \iint d^4y_1 d^4y_2 \Delta (y_1 - y_2) \frac{\delta^2}{\delta\varphi (y_1) \delta\varphi (y_2)}\right\} \times U (\varphi (x_1)) U (\varphi (x_2)) = \frac{(- ig)^2}{2} \iint d^4x_1 d^4x_2 \times \exp \left\{\Delta (x_1 - x_2) \frac{\partial}{\partial\alpha_1 \partial\alpha_2}\right\} U (\alpha_1) U (\alpha_2)\Big|_{\substack{\alpha_1=\varphi(x_1)\\ \alpha_2=\varphi(x_2)}}. \qquad (2.1)$$

Concerning the function $U(\alpha)$, we make the following assumptions.

1. $U(\alpha)$ is continuous and has no singularities on the real axis, and can be expanded in a Taylor series around the point $\alpha = 0$ with some radius of convergence ρ:

$$U(\alpha) = \sum_{n=0}^{\infty} \frac{u_n}{n!} \alpha^n. \qquad (2.2)$$

2. $U(\alpha)$ satisfies the following condition at infinity:

$$\lim_{\alpha \to \infty} | \alpha^{-2} U(\alpha) | = 0. \qquad (2.3)$$

It will turn out in the following that these conditions on $U(\alpha)$ are not sufficient for the absence of ultraviolet divergences in the theory.

The transition to the normal product in (2.1) reduces to the determination of the function

$$F(\Delta, \alpha_1, \alpha_2) = \exp\left\{\Delta \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2}\right\} U(\alpha_1) U(\alpha_2)$$

$$= \sum_{m_1, m_2=0}^{\infty} F_{m_1 m_2}(\Delta) \frac{\alpha_1^{m_1} \alpha_2^{m_2}}{m_1! \, m_2!}, \qquad (2.4)$$

with $U(\alpha)$ given. The variables α_1 and α_2 can be regarded as c numbers.

The functions $F_{m_1 m_2}(\Delta)$ in the expansion of $F(\Delta, \alpha_1, \alpha_2)$ in powers of α_1 and α_2 are related to the radiation operators and are the coefficients of the expansion of the S matrix in terms of normal products of the operator $\varphi(x)$. We quote the usual definition of the radiation operator $R_{m_1 m_2}(\Delta)$:

$$R_{m_1 m_2}(x_1 - x_2) = \langle 0 | S | 0 \rangle^{-1} \left\langle 0 \left| \frac{\delta^{m_1+m_2} S}{\delta \varphi^{m_1}(x_1) \, \delta \varphi^{m_2}(x_2)} \right| 0 \right\rangle. \qquad (2.5)$$

The expansion of the operator (2.4) is equivalent to the solution of the following partial differential equation:

$$\frac{\partial}{\partial \Delta} F(\Delta, \alpha_1, \alpha_2) = \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} F(\Delta, \alpha_1, \alpha_2) \qquad (2.6)$$

with the initial condition

$$F(\Delta, \alpha_1, \alpha_2)|_{\Delta=0} = U(\alpha_1) U(\alpha_2). \qquad (2.7)$$

The existence of a solution of this equation even in some neighborhood of the point ($\alpha_1 = 0$, $\alpha_2 = 0$) is very problematical and depends in an essential way on the region of variation of the complex variable Δ and on the initial function $U(\alpha)$.

However, for us it suffices to satisfy this equation by the formal power series (2.4) which is not required to converge in any neighborhood of the point $\alpha_1 = 0$, $\alpha_2 = 0$ and to use an initial condition which is weaker than (2.7), viz.,

$$\lim_{\Delta \to 0} F_{m_1 m_2}(\Delta) = u_{m_1} u_{m_2}, \qquad (2.8)$$

where u_m is defined by (2.2). Here it is essential to define the region of variation of Δ and the path in the complex Δ plane along which Δ tends to zero in (2.8), since equations of the type (2.6) have a singularity at the point $\Delta = 0$.

The adequacy of such a solution follows from the fact that the amplitudes for physical processes are determined only by the functions $F_{m_1 m_2}(\Delta)$ and not by the function $F(\Delta, \alpha_1, \alpha_2)$. Physical requirements as, for example, unitarity and causality, are imposed on the $F_{m_1 m_2}(\Delta)$, not on $F(\Delta, \alpha_1, \alpha_2)$. Moreover, although α_1 and α_2 can at the present stage be regarded as numerical variables, they are actually operators, $\alpha_1 = \varphi(x_1)$ and $\alpha_2 = \varphi(x_2)$, and the series (2.4) is an expansion of the S matrix in normal products of these operators. It follows from this that there is no reason to require convergence of the series (2.4) in any neighborhood of the point $\alpha_1 = 0$, $\alpha_2 = 0$.

Let us turn to the determination of the region of variation of Δ, which is carried out within the framework of quantum field theory. At the same time we shall find the conditions on the radiation operators $F_{m_1 m_2}(\Delta)$ which remove the ultraviolet divergences from the amplitudes for various processes in perturbation theory. For this purpose we derive an expression for $F_{m_1 m_2}(\Delta)$ in the form of a formal expansion in Δ by substituting (2.2) for $U(\alpha)$ in (2.4) and expanding in Δ:

$$F_{m_1 m_2}(\Delta) = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} u_{n+m_1} u_{n+m_2} = \sum_{n=n_0}^{\infty} r_n \Delta^n. \qquad (2.9)$$

The coefficients r_n have been introduced for later convenience.

The matrix elements of the scattering matrix are Fourier transforms of the radiation operators (2.9). The problem reduces to the determination of the following integral:

$$K(p^2) = i \int d^4 x e^{ipx} F(\Delta(x)). \qquad (2.10)$$

In the formalism of perturbation theory, where the radiation operator for the interactions usually considered is simply a polynomial in $\Delta(x)$ in second order, the integral (2.10) is calculated in momentum space and, since the integrals usually diverge at large momenta, $\Delta(x)$ is regularized (for example, by the Pauli-Villars method). In our case, where $F(\Delta(x))$ is of a rather complicated form, it is more convenient to do the calculations in coordinate space.

As usual, we will take for $\Delta(x)$ the causal function (1.7) regularized according to Pauli-Villars and make the transition to the limit, removing the regularity, only in the final expressions. For simplicity we shall not introduce a special index to denote the regularized function $\Delta(x)$. We have [6]

$$\Delta(x) = \frac{1}{i(2\pi)^4}\int d^4k e^{ikx}\left(\frac{1}{\mu^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)$$

$$= \frac{1}{4\pi^2}\int_0^\infty d\alpha e^{-i\alpha x^2}\left(\exp\left\{-i\frac{\mu^2 - i\varepsilon}{\alpha}\right\} - \exp\left\{-i\frac{M^2 - i\varepsilon}{\alpha}\right\}\right)$$

$$= \theta(\lambda)\,\Delta_1(\sqrt{\lambda}) + \theta(-\lambda)\,\Delta_2(\sqrt{-\lambda}). \qquad (2.11)$$

Here

$$\Delta_1(\sqrt{\lambda}) = i\,[\mu H_1^{(2)}(\mu\sqrt{\lambda}) - MH_1^{(2)}(M\sqrt{\lambda})]/8\pi\sqrt{\lambda},$$

$$\Delta_2(\sqrt{-\lambda}) = [\mu K_1(\mu\sqrt{-\lambda}) - MK_1(M\sqrt{-\lambda})]/4\pi^2\sqrt{-\lambda},$$

where $\lambda = x^2 = t^2 - \mathbf{r}^2$, M is the regularizing mass, and $H_1^{(2)}(z)$ and $K_1(z)$ are the known Bessel functions.

First of all we define the integral

$$K^{(n)}(p^2) = i\int d^4x e^{ipx}\Delta^n(x). \qquad (2.12)$$

In the calculation of this integral it is convenient to go over to a Euclidean metric in coordinate space. It is easily seen from the parametric representation of the causal function (2.11) that $\Delta(x)$ can be continued into the region of complex times $t = x_0 + ix_4$ or into the region of complex coordinates $\mathbf{r} = \mathbf{x} + i\boldsymbol{\rho}$. The contour of integration in the complex t plane must be displaced from $(-\infty, +\infty)$ to $(+i\infty, -i\infty)$ and in the complex $\mathbf{r}$ plane, from $(-\infty, +\infty)$ to $(-i\infty, +i\infty)$.

The choice of continuing in the time or in the coordinates depends on whether the momentum vector p in (2.12) is time- or spacelike. In the first case we must continue in the coordinates, in the second, we must continue in time.

Making use of what has just been said, it is now easy to reduce the four dimensional integral (2.12) to a single integral. We have, for $p^2 > 0$ $(p = \sqrt{p^2} = \sqrt{p_0^2 - \mathbf{p}^2})$,

$$K^{(n)}(p^2) = \frac{4\pi^2}{p}\int_0^\infty d\beta\beta^2 J_1(\beta p)\,\Delta_1^n(\beta). \qquad (2.13)$$

and for $p^2 < 0$ $(|p| = \sqrt{-p^2} = \sqrt{\mathbf{p}^2 - p_0^2})$,

$$K^{(n)}(p^2) = \frac{4\pi^2}{|p|}\int_0^\infty d\beta\beta^2 J_1(\beta|p|)\,\Delta_2^n(\beta), \qquad (2.14)$$

where $J_1(z)$ is a Bessel function. We note that the integral (2.14) is real, since the function $\Delta_2(\beta)$ is real.

Let us consider the integral (2.13) in more detail. If the momentum $p < \mu n$, the integral must be real, since it corresponds to a graph with two vertices and n intermediate particles. Such a graph is real for energies below the threshold for the production of n real particles. The integral (2.13) can in this case be written in a form which makes its reality explicit, by deforming the contour of integration from $(0, +\infty)$ to $(0, -\infty)$. We find

$$K^{(n)}(p^2) = \frac{4\pi^2}{p}\int_0^\infty d\beta\beta^2 I_1(\beta p)\,\Delta_2^n(\beta)\,|\,(p < \mu n). \qquad (2.15)$$

If $p > \mu n$, the integral (2.11) is complex. Let us divide it into its real and imaginary parts. The imaginary part of the integral $K^{(n)}(p^2)$ does not contain ultraviolet divergences in accordance with the unitarity condition, since the region of integration is limited by the momentum p. All divergences are contained in the real part.

Let us now determine the integral (2.10) which, for $p^2 < 0$, is taken to be equal to

$$K(p^2) = \frac{4\pi^2}{|p|}\int_0^\infty d\beta\beta^2 J_1(\beta|p|)\,F(\Delta_2(\beta)). \qquad (2.16)$$

If $p^2 > 0$ and $\mu n_0 \le \mu N \le p \le (N+1)\mu$, the integral (2.10) is defined as

$$K(p^2) = \frac{4\pi^2}{p}\left\{\int_0^\infty d\beta\beta^2 I_1(\beta p)\left[F(\Delta_2(\beta)) - \sum_{n=n_0}^{N} r_n\Delta_2^n(\beta)\right]\right.$$
$$\left. + \int_0^\infty d\beta\beta^2 J_1(\beta p)\left[\sum_{n=n_0}^{N} r_n\,\mathrm{Re}\,\Delta_1^n(\beta) + i\sum_{n=n_0}^{N} r_n\,\mathrm{Im}\,\Delta_1^n(\beta)\right]\right\}. \qquad (2.17)$$

If the function $F(\Delta)$ is such that $\Delta^{-2}F(\Delta) \to 0$ for $\Delta \to \infty$, the integral of it is free from ultraviolet divergences. The divergences in the two real sums (2.17) cancel each other, while the imaginary part of the integral is always finite.

We note that it is not possible to go to the limit $M \to \infty$ directly in (2.17), since for $M = \infty$ the functions $\Delta_1(\beta)$ and $\Delta_2(\beta)$ behave differently at the point $\beta = 0$. For the limit $M \to \infty$ it is necessary to change the integration contour of the integral of $\mathrm{Re}\,\Delta\, n/1(\beta)$ from the ray $(0, \infty)$ to the path shown in Fig. 1, where a is some finite quantity, for example, $a = 1/\mu$. In the resulting expression the limit $M \to \infty$ can now be taken. The whole expression will be finite if the radiation operator $\Delta^{-2}F(\Delta)$ vanishes with increasing Δ.

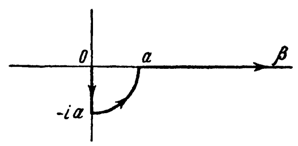

FIG. 1

To avoid ultraviolet divergences in the expressions for the amplitudes it is therefore necessary that the radiation operators $F_{m_1m_2}(\Delta)$, regarded as functions of the real variable Δ (where $\Delta > 0$), have the following properties:

1. The functions $F_{m_1m_2}(\Delta)$ must increase less rapidly than Δ^2 as Δ goes to infinity, i.e.,

$$\lim_{\Delta\to\infty} \Delta^{-2}F_{m_1m_2}(\Delta) = 0. \qquad (2.18)$$

2. The functions $F_{m_1m_2}(\Delta)$ must satisfy the initial condition (2.8).

3. The functions $F_{m_1m_2}(\Delta)$ must not have any singularities in the interval $0 < \Delta < \infty$. Indeed, the presence of an isolated point x_0 in space at which the radiation operator has a singularity would be in contradiction with the homogeneity of space.

3. CHOICE OF THE INTERACTION $U(\alpha)$

Thus the problem reduces to the following: find functions $U(\alpha)$ satisfying (2.2) and (2.3) for which the function $F_{m_1m_2}(\Delta)$ defined by the formal series (2.9) satisfies conditions 1, 2, and 3, enumerated at the end of Sec. 2.

Let us consider the series (2.9) for $m_1 = m_2$, i.e., when its coefficients are manifestly positive. The following cases are possible: if the series converges everywhere, then $\Delta^{-2}F_{m_1m_2}(\Delta)$ increases with Δ, and if the series has a finite radius of convergence ρ, then the function $\Delta^{-2}F_{m_1m_2}(\Delta)$ can decrease with increasing Δ only if $F_{m_1m_2}(\Delta)$ has a singularity on the real axis.

Hence $F_{m_1m_2}(\Delta)$ cannot fulfill the above-mentioned conditions if the series (2.9) converges in some neighborhood of the point $\Delta = 0$. From this it follows also that the required $U(\alpha)$ cannot decrease rapidly at infinity, e.g., exponentially.

There remains a last possibility: the series (2.9) diverges for arbitrary values of Δ, i.e., it is asymptotic.

In order to find a function $F_{m_1m_2}(\Delta)$ which has an essential singularity at $\Delta = 0$ and can be expanded into the asymptotic series (2.9), we return to the original expression (2.4). We make the change of variables

$$\alpha_1 = \beta_1 + \beta_2, \qquad \alpha_2 = \beta_1 - \beta_2. \qquad (3.1)$$

Then

$$F(\Delta, \alpha_1, \alpha_2) = \exp\left\{\frac{\Delta}{4}\left(\frac{\partial^2}{\partial\beta_1^2} - \frac{\partial^2}{\partial\beta_2^2}\right)\right\} U(\beta_1+\beta_2)\, U(\beta_1-\beta_2). \qquad (3.2)$$

Let us use the operator equation

$$\exp\left\{\pm\frac{\Delta}{4}\frac{\partial^2}{\partial\beta^2}\right\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx \exp\left\{-x^2 + \sqrt{\pm\Delta}\, x\frac{\partial}{\partial\beta}\right\}. \qquad (3.3)$$

On the right-hand side of (3.3) we have the displacement operator for the variable β. Substituting (3.3) in (3.2) and returning to the variables α_1 and α_2, we obtain

$$F(\Delta, \alpha_1, \alpha_2) = \frac{1}{\pi}\iint_{-\infty}^{\infty} dx_1\, dx_2\, e^{-x_1^2-x_2^2} U(\alpha_1 + \sqrt{\Delta}(x_1 + ix_2)) \times U(\alpha_2 + \sqrt{\Delta}(x_1 - ix_2)). \qquad (3.4)$$

Let us now make the change of variables

$$x_1 = \frac{1}{\sqrt{\Delta}}\left(y_1 - \frac{\alpha_1+\alpha_2}{2}\right), \quad x_2 = \frac{1}{\sqrt{\Delta}}\left(y_2 - \frac{\alpha_1-\alpha_2}{2i}\right). \qquad (3.5)$$

We obtain finally

$$F(\Delta, \alpha_1, \alpha_2) = \frac{1}{\pi}\iint_{-\infty}^{\infty} dy_1\, dy_2 \,|\, U(y_1 + iy_2)\,|^2 \frac{1}{\Delta} \times \exp\left\{-\frac{[(y_1+iy_2)-\alpha_1]\,[(y_1-iy_2)-\alpha_2]}{\Delta}\right\}. \qquad (3.6)$$

It should be noted that this method of solution is formal and must be put on a better foundation, since in taking the residues we have worried little about the existence of singularities of the function $U(\alpha)$ and have freely deformed the contour of integration in the complex plane.

Let us consider the solution $F(\Delta, \alpha_1, \alpha_2)$ in more detail. The integral (3.6) may contain singularities connected with possible singularities of $U(\alpha)$ in the complex α plane. We require that these singularities of the function $U(\alpha)$ are of the integrable type, i.e., that the integral of $|U(\alpha)|^2$ over an arbitrary finite region in the complex α plane exists. Under this assumption the integral (3.6) exists for abritrary values of α_1 and α_2 and $\Delta > 0$. It is easily verified that it satisfies condition (2.6).

Thus the integral (3.6) is the solution of (2.6) in the whole region of variation of the variables α_1 and α_2 for $\Delta > 0$.

It is now necessary to show that the solution (3.6) satisfies the initial condition. It turns out that (3.6) does not fulfill the initial condition (2.7) in the whole region of variation of the variables α_1 and α_2. There exists a region (for example, $\alpha_1 > 0$, $\alpha_2 < 0$) where the initial condition is not satisfied.

However, as already said in Sec. 2, it suffices to satisfy the weaker initial condition (2.8) instead of (2.7).

We find an expression for $F_{m_1m_2}(\Delta)$ from (3.6) by expanding in α_1 and α_2:

$$F_{m_1m_2}(\Delta) = \sum_k \frac{(-)^{m_1+m_2+k}\, m_1!\, m_2!}{(k-m_1)!\,(k-m_2)!\,(m_1+m_2-k)!}\left(\frac{1}{\Delta}\right)^k \times Y_{k-m_1,\, k-m_2}(\Delta), \qquad (3.7)$$

where the summation goes over all allowed values of k, and

$$Y_{n_1n_2}(\Delta)=\int_0^\infty dxe^{-x}\frac{1}{2\pi}\int_0^{2\pi}d\vartheta\,(\sqrt{x\Delta}e^{i\vartheta})^{n_1}\times(\sqrt{x\Delta}e^{-i\vartheta})^{n_2}|U(\sqrt{x\Delta}e^{i\vartheta})|^2. \quad (3.8)$$

Here we have gone over to an integration over polar coordinates.

It can be shown that $F_{m_1m_2}(\Delta)$ as given by (3.7) satisfies the initial condition (2.8).

Therefore, if $U(\alpha)$ is such that (3.7) satisfies also the condition (2.18), then the conditions imposed on $F_{m_1m_2}(\Delta)$ are all fulfilled and the problem is solved.

Below we shall consider interaction functions of the form

$$U(\alpha)=\alpha^s/(1+f\alpha^2)^\gamma, \quad (3.9)$$

where f is a second coupling constant of the dimension of the square of a length, γ is a positive parameter, and s is a positive integer. For f = 0 and s = 1, 2, 3, 4 we obtain the known interaction Lagrangians for the scalar field.

Substituting (3.9) in (3.8) and integrating, we find

$$Y_{n_1n_2}(\Delta)=\frac{(-)^r}{2^{2\gamma-1}}\frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}\Delta^{s+l-r}f^r\frac{\Gamma(s+l-r+1)}{2^{s+l}}\times\sum_{q=0}^{[(s+l-r)/2]}\frac{(-)^q(2s+2l-2q)!}{q!(s+l-q)!(s+l-r-2q)!}A^r_{2\gamma+r,\,s+l-q+1/2}(f^2\Delta^2), \quad (3.10)$$

where $n_1+n_2=2l$, $|n_1-n_2|=2r$, and l and r are integers. If (n_1+n_2) is odd, then $Y_{n_1n_2}\equiv 0$. The function $A^r{}_{\sigma,\mu}(z)$ is given by

$$A^r_{\sigma,\mu}(z)=\int_0^\infty\frac{dt\,t^{\sigma-1}J_r(t)}{(1+zt^2)^\mu}. \quad (3.11)$$

Let us write down the asymptotic behavior of the function A for large and small values of z:

$$A^r_{\sigma,\mu}(z)=z^{-(\sigma+r)/2}\frac{\Gamma(\mu-(\sigma+r)/2)\,\Gamma((\sigma+r)/2)}{2^{r+1}\Gamma(\mu)\,\Gamma(r+1)}\left[1+O\left(\frac{1}{z}\right)\right] \quad (z\gg 1). \quad (3.12)$$

$$A^r_{\sigma,\mu}(z)=2^{\sigma-1}\frac{\Gamma((\sigma+r)/2)}{\Gamma((r-\sigma)/2+1)}[1+O(z)] \quad (z\ll 1). \quad (3.13)$$

The function A is not analytic in the point z = 0.

The presence of the factor $\Gamma(1-\gamma)$ in (3.10) indicates that the expression obtained, regarded as an analytic function of the parameter γ, has poles for positive integer values of γ. The requirement of the integrability of $U(\alpha)$ mentioned earlier implies that $0<\gamma<1$.

Let us now direct our attention to the growth of the function $F_{m_1m_2}(\Delta)$ as Δ increases. The maximal power of Δ is easily seen to be $\Delta^{s-2\gamma}$, which corresponds to a correction to the vacuum energy. Recalling the condition (2.18) we obtain

$$s-2\gamma<2 \text{ or } \gamma>s/2-1. \quad (3.14)$$

The same restriction is obtained from (3.9), using (2.3). Thus, for example, it suffices to take $\gamma = 3/4$ for an interaction with s = 3 ($L_I = -g\varphi^3$).

In this section we have derived formulas for the calculation of any process in second order perturbation theory with the interaction Lagrangian (3.9). Divergences connected with high energies do not occur anywhere.

4. SELF-INTERACTING SCALAR FIELD

As an example, let us consider the modification of the usually considered "self-interacting" one-component scalar field with the interaction Lagrangian

$$L_I(x)=-g\varphi^3(x). \quad (4.1)$$

As is known, this interaction leads to divergences in perturbation theory, which can be removed by mass renormalization.

In view of what has been said in the preceding sections, it is necessary to consider an interaction Lagrangian of the form

$$L_I(x)=-g\varphi^3(x)(1+f\varphi^2(x))^{-3/4}, \quad (4.2)$$

which goes over into (4.1) for f = 0.

Let us find the correction to the mass of the scalar particle due to the interaction (4.2). This correction is in second order perturbation theory determined by the integral

$$\delta\mu^2=i\int d^4xe^{ipx}R_{11}(x), \quad (4.3)$$

where $p^2=\mu^2$, and $R_{11}(x)$ is given by (2.5). Substituting the second order S matrix in (2.5), we find

$$R_{11}(x)=-g^2[F_{11}(\Delta(x))+\delta^{(4)}(x)\int d^4yF_{20}(\Delta(y))], \quad (4.4)$$

where F_{11} and F_{20} are defined by (2.4) and (3.7). The first term in (4.4) corresponds to the graph of Fig. 2 and the third term, to the graph of Fig. 3. The vertices are labeled by numbers and the dashed areas between them correspond to the sum of all possible internal lines which would be obtained if the Lagrangian (4.2) were expanded in powers of f and a Feynman graph associated with each term of this expansion.

Let us substitute (4.4) in (4.3) and use (2.17).

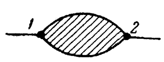

FIG. 2

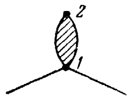

FIG. 3

We find

$$\delta\mu^2 = -g^2\Big[\frac{4\pi^2}{\mu}\int_0^\infty d\beta\beta^2 I_1(\mu\beta)\, F_{11}(\Delta_2(\beta))$$

$$+ 2\pi^2\int_0^\infty d\beta\beta^3 F_{20}(\Delta_2(\beta))\Big]. \qquad (4.5)$$

Substituting the functions F from (3.7) and (3.10) in (4.5), we obtain finally

$$\delta\mu^2 = -g^2\,[35\,b_4 - 35\,b_3 + 6\,b_2 - 7\,d_3 + 3\,d_2]; \qquad (4.6)$$

$$b_K = 3\,\frac{\Gamma(^1/_4)\,(2\pi)^{3/2}}{\mu}\int_0^\infty d\beta\beta^2 I_1(\beta\mu)\Big(\frac{\mu}{4\pi^2\beta}K_1(\mu\beta)\Big)^2$$

$$\times \int_0^\infty \frac{dt\, t^{1/2} J_0(t)}{\Big[1 + f^2t^2\Big(\frac{\mu}{4\pi^2\beta}K_1(\mu\beta)\Big)^2\Big]^{\varkappa+1/2}},$$

$$d_K = 15 f\Gamma\Big(\frac{1}{4}\Big)(2\pi^3)^{1/2}\int_0^\infty d\beta\beta^3\Big(\frac{\mu}{4\pi^2\beta}K_1(\mu\beta)\Big)^3$$

$$\times \int_0^\infty \frac{dt\, t^{3/2} J_1(t)}{\Big[1 + f^2t^2\Big(\frac{\mu}{4\pi^2\beta}K_1(\mu\beta)\Big)^2\Big]^{\varkappa+1/2}}. \qquad (4.7)$$

Each of these integrals is convergent. We have already taken the limit $M = \infty$.

We call attention to the fact that the integrals (4.7) have a singularity at $f = 0$ and can, therefore, not be expanded in f. This implies that the divergence in the theory with the Lagrangian (4.1) is connected precisely with the expansion in f.

Thus it appears that the correction to the mass of the scalar particle with the local interaction Lagrangian (4.2) is finite in second order perturbation theory.

CONCLUSION

This investigation has confirmed the assumption that the presence of ultraviolet divergences in the quantum theory of the meson field is connected with the behavior of the interaction Lagrangian for large fields. In the present paper we have only considered the second approximation of perturbation theory in complete detail. The final answer to the problem of the possibility of constructing a local field theory without ultraviolet divergences can be given only after the higher approximations have been investigated.

We note that the disappearance of the ultraviolet divergences in the amplitudes for physical processes is connected with the nonanalyticity in the new coupling constant f. If the constant f is regarded as very small and we put $f = 1/M^2$, we obtain the usual divergent expressions if the parameter M tends to infinity ($f \to 0$).

We have considered only the one-component scalar field. However, it is easy to generalize our result to other types of fields and interactions involving scalar particles (mesons). For example, instead of the usual nucleon-meson interaction Lagrangian $L_I = g(\overline{\psi}\Gamma\psi)\,\varphi$ it suffices to choose $L_I = g(\overline{\psi}\Gamma\psi)\,(1 + f\,\varphi^2)^{-1/2}$. The matrix elements calculated from this Lagrangian will be free from divergences in second order in g.

From the physical point of view the introduction of interaction Lagrangians with an infinite number of powers of the meson operator φ means that processes involving the exchange of an arbitrarily large number of particles can occur already in first order of perturbation theory. The matrix element for the process will be simply a constant, which is in agreement with the basic assumption of the statistical theory of multiple production.

The author expresses his deep gratitude to Prof. D. I. Blokhintsev and Academician N. N. Bogolyubov for their interest and valuable comments and to L. G. Zastavenko and I. T. Todorov for useful advice.

Note added in proof (May 11, 1963): The author learned recently that similar results have been obtained by E. S. Fradkin (Nucl. Phys., in press).

[1] M. Born, Proc. Roy. Soc. **A143**, 410 (1934), M. Born and L. Infeld, Proc. Roy. Soc. **A144**, 425 (1934).

[2] R. P. Feynman, Phys. Rev. 84, 108 (1951).

[3] S. Hori, Progr. Theor. Phys. 7, 578 (1952).

[4] F. J. Dyson, Phys. Rev. 85, 631 (1952).

[5] L. van Hove, Acad. Roy. de Belgique, Bulletin 37, 1055 (1951).

[6] N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, N. Y. (1959).

Translated by R. Lipperheide