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STABILITY OF ROTATION OF A SUPERFLUID LIQUID

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It is shown that the region of stability of stationary motion of a superfluid liquid rotating between two coaxial cylinders is broader than that corresponding to an ideal classical liquid. This is due to the stabilizing effect of quantized vortex lines. The result obtained is in agreement with experiment.

1. It is shown in the theoretical study of Chandrasekhar and Donnelly[1] and in the experimental investigation of Donnelly[2] that when He II is made to rotate between two coaxial cylinders two critical rotation speeds are observed. The first (on the low side) is interpreted by the cited authors as a manifestation of the Rayleigh instability of the laminar rotation of the superfluid component. The second critical speed determines the boundary of the Taylor instability (see, e.g., [2], Sec. 28) of the normal component of He II. Both critical speeds turn out to be higher than that expected for classical liquids (ideal or viscous with kinematic viscosity $\nu = \nu_n$). This circumstance is attributed to the additional stabilization of the rotation of both components by the friction between them.

In the study cited[1] the mutual friction was the only manifestation of the presence of quantized Onsager-Feynman vortices in the rotating He II. Yet it is clear that the elastic properties of these vortices should also play an important role in the stabilization of the rotation. In fact, let us consider a liquid element which has randomly strayed from its equilibrium rotation trajectory. The vortex filaments passing through it are likewise displaced, and their bending gives rise to a rectifying force f [1)] which tends to return the element to the equilibrium trajectory. This force f should be included, along with the centrifugal force and the radial pressure gradient, in the force balance, and this leads to a shift of the stability boundary towards the higher speeds of rotation.

As is well known[4,5], the elastic properties of the vortices are described in the hydrodynamics of the rotating superfluid component by the term $\nu_s[\omega \times \mathrm{curl}(\boldsymbol{\omega}/\omega)]$, which was not taken into consideration in [1]. Obviously, it is precisely the term, which does not vanish at low temperatures when the mutual friction is insignificant, which should determine the different stability criteria of the superfluid component of He II, on the one hand, and of the classical ideal liquid, on the other. This difference remains also in the case of a purely superfluid liquid (T = 0), when there is no mutual friction. In other words, the stability of rotating liquid helium at low temperatures is described by a parameter ν_s, which characterizes the internal properties of the superfluid component (and not its interaction with the normal component). It is natural to expect the quantity ν_s to play a role analogous to that of viscosity [2)], which (as is well known[3]) broadens the region of stable rotation.

2. In connection with the foregoing, let us consider the stability of rotation of a superfluid liquid (T = 0) between two infinite coaxial cylinders with radii R_1 and R_2, rotating with angular velocities ω_{01} and ω_{02} (the subscripts 1 and 2 pertain to the internal and external cylinders, respectively).

The equations of hydrodynamics of a rotating superfluid liquid have the form[4-6]

$$\partial \mathbf{v}_s/\partial t + (\mathbf{v}_s\nabla)\mathbf{v}_s + \nu_s[\boldsymbol{\omega}, \mathrm{rot}(\boldsymbol{\omega}/\omega)] = -\rho_s^{-1}\nabla p,$$
$$\mathrm{div}\,\mathbf{v}_s = 0. \qquad (1)^*$$

A stationary solution of this system, satisfying the boundary conditions of the problem under consideration, is

$$v_{sr}^{(0)} = v_{sz}^{(0)} = 0, \qquad v_{s\varphi}^{(0)} = Ar + B/r,$$
$$p^{(0)} = \tfrac{1}{2}A\rho_s r^2 + 2AB\rho_s \ln r - \tfrac{1}{2}\rho_s B r^{-2} + \mathrm{const} \qquad (2)$$

1)It is easy to see (see for example [4], Sec. 3.2) that $f = m\nu_s|\boldsymbol{\omega}|K$, where m is the mass of the liquid element, $\nu_s = \epsilon/\rho_s\Gamma$, ϵ is the tension in the vortex filament, Γ the circulation quantum, $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}_s$, and K the curvature of the vortex filament.

2)It is appropriate to note in this connection that ν_s plays a role analogous to viscosity also in the damping of a disc in rotating He II (see [4], Sec. 4.6, where $\eta_s = \rho_s\nu_s$).

*$[\boldsymbol{\omega}, \mathrm{rot}(\boldsymbol{\omega}/\omega)] = \omega \times \mathrm{curl}(\boldsymbol{\omega}/\omega)$.

we use a cylindrical coordinate system (r, φ, z). In (2) we have

$$A = \frac{\omega_{02}R_2^2 - \omega_{01}R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{\omega_{01} - \omega_{02}}{R_2^2 - R_1^2} R_1^2 R_2^2. \tag{3}$$

We introduce as usual small perturbations of the stationary mode of rotation, in the form of waves defined by

$$\mathbf{v}_s - \mathbf{v}_s^{(0)} = \mathbf{w}_s(r) e^{i(kz+\alpha t)}, \quad p - p^{(0)} = \rho_s \sigma(r) e^{i(kz+\alpha t)}, \tag{4}$$

where k is the wave number (real) and α is the complex frequency ($\alpha = \Omega + i\gamma$).

Substituting (4) and (2) in the system (1) we obtain the following system of equations, linearized in w_s and in σ,

$$i\alpha w_{sr} - (2\omega_0 + \nu_s k^2) w_{s\varphi} + d\sigma/dr = 0,$$

$$(2A + \nu_s k^2) w_{sr} + i\alpha w_{s\varphi} + ik\nu_s \, dw_{sz}/dr = 0,$$

$$i\alpha w_{sz} + ik\sigma = 0, \quad r^{-1} d(r w_{sr})/dr + ik w_{sz} = 0, \tag{5}$$

where $\omega_0 = A + B/r^2$. The boundary conditions have the form

$$w_{sr}(R_1) = w_{sr}(R_2) = 0. \tag{6}$$

Eliminating $w_{s\varphi}$, w_{sz}, and σ from (5) we obtain an equation for the function w_{sr}

$$\left[\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r + \frac{2Ak^2(2\omega_0 + \nu_s k^2)}{\alpha^2 - \nu_s k^2(2\omega_0 + \nu_s k^2)} - k^2\right] w_{sr} = 0. \tag{7}$$

3. In solving the equation (7) we confine ourselves to the case of close cylinders [$R_2 - R_1 \ll \frac{1}{2}(R_1 + R_2)$]. In addition, we replace ω_0 by $\bar{\omega}_0 = \frac{1}{2}(\omega_{01} + \omega_{02})$, something we can do only if the cylinders rotate in the same direction. The same approximations were made in [1]. As a result, Eq. (7) assumes the form

$$\frac{d^2 w_{sr}}{dr^2} - k^2\left[1 - \frac{2A(2\bar{\omega}_0 + \nu_s k^2)}{\alpha^2 - \nu_s k^2(2\bar{\omega}_0 + \nu_s k^2)}\right] w_{sr} = 0. \tag{8}$$

A nontrivial solution of this equation under boundary conditions (6) is

$$w_{sr}^{(n)} = A' \sin[\varkappa_n (r - R_1)], \tag{9}$$

where A' is an arbitrary constant and

$$\varkappa_n = \pi n/(R_2 - R_1), \quad n = 1, 2, \ldots . \tag{10}$$

The corresponding characteristic equation has the form

$$\varkappa_n^2 + k^2 - \frac{2Ak^2(2\bar{\omega}_0 + \nu_s k^2)}{\alpha^2 - \nu_s k^2(2\bar{\omega}_0 + \nu_s k^2)} = 0, \quad n = 1, 2, \ldots . \tag{11}$$

Determining from (11) the square of the complex frequency α^2

$$\alpha^2 = \nu_s k^2 (2\bar{\omega}_0 + \nu_s k^2)\left[1 + 2A/\nu_s(k^2 + \varkappa_n^2)\right], \tag{12}$$

we see that α can be either purely real ($\alpha_{1,2} = \pm\Omega$—neutral stability), or purely imaginary ($\alpha_{1,2} = \pm i\gamma$—instability).

Thus, the stability condition is the inequality $\alpha^2 \geq 0$ or

$$A \geqslant -\tfrac{1}{2}\nu_s(\varkappa_n^2 + k^2), \quad n = 1, 2, \ldots \tag{13}$$

Since $\kappa_{n\,\min} = \kappa_1$ and $k_{\min} = 0$, the stability criterion is

$$A \geqslant -\tfrac{1}{2}\nu_s \varkappa_1^2 \tag{14}$$

or, using (3) and (10),

$$\omega_{02}R_2^2 - \omega_{01}R_1^2 > -\tfrac{1}{2}\pi^2 \nu_s (R_2 + R_1)/(R_2 - R_1). \tag{15}$$

The inequality (14) [or (15)] replaces in the case of a superfluid liquid the well known Rayleigh condition $A > 0$ ($\omega_{02}R_2^2 > \omega_{01}R_1^2$) for an ideal liquid, and illustrates the statement made at the end of Sec. 1 concerning the broadening of the stability region of the rotation of a superfluid liquid by the stabilizing action of the Onsager-Feynman vortices.

Let us compare the result obtained with the experimental data[2]. The experiment was performed with $R_1 = 1.9$ cm, $R_2 = 2$ cm, and $\omega_{02} = 0$. At these values of the parameters, (15) yields the following value for the critical period P_s of the rotation of the inner cylinder:

$$P_s \approx 0.12 \nu_s^{-1} \text{ sec.}$$

In many experiments the values obtained for ω_s were 8.5×10^{-4}[6], 8×10^{-4}[7], and 9.7×10^{-4} cm^2/sec[8]. The corresponding values of P_s are 140, 150, and 120 seconds. The experimental value of P_s measured by Donnelly[2] at the lowest temperature used in his experiment ($T = 1.35°$ K), at which the He II can be regarded as almost completely superfluid ($\rho_s \gg \rho_n$) is ~140 sec, which is in sufficiently good agreement with our result.

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