ATOMIC PHOTOEFFECT AT HIGH ENERGIES

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It is shown that in the extreme relativistic case the main contribution to the photoeffect cross section is from the wave function of the ejected electron, which is in the form of a Schrödinger-equation solution with energy E equal to the momentum p. The error in the cross section can in this case be of the order of E^{-2}. It is proven that at high energies calculations based on the solution of the Schrödinger equation and on distorted plane waves lead to identical results. An analytical formula is obtained for the small-angle cross section. The limiting nuclear charge Z beyond which the peak of the angular distribution coincides with zero angle is also determined.

THE main difficulty in the calculation of the relativistic atomic photoeffect lies in the absence of a closed analytic expression for the wave function of the ejected electron. In the high-energy region, many authors [1-4] used the approximate Furry-Sommerfeld-Maue (FSM) function [5,6] or the asymptotic form of this function in the form of a distorted plane wave [2]. The principal arguments in favor of the use of this function at higher energies were advanced by Pratt [2]. There is no rigorous proof, however, that in the extreme relativistic case the main contribution to the amplitude is made by the FSM function or by part of this function.

We use here for the wave function of the ejected electron a representation in the form of an expansion in powers of αZ, the first term of which is the nonrelativistic wave function [7] 1). We shall show with the aid of this representation that at high energies the principal part of the amplitude is contained in the matrix element with the first term of the series, and the remaining terms are of order $\varepsilon^{-i\alpha Z-1}$ with respect to the principal part.

We shall show further that the results of the calculation with this function coincide with the results obtained using a distorted plane wave [2-4]. By the same token, we shall prove rigorously the sufficiency of the use of the distorted plane wave in calculation of the photoeffect at high energies. At small angles [accurate to terms of order $(\varepsilon\vartheta)^2$] we obtain an analytic formula for the photoeffect cross section. This enables us to investigate in detail the behavior of the angular distribution of the photoelectrons near small angles.

1. It is shown in [8] that the amplitude of the photoeffect can be represented in the form (see formula (11))

$$\langle\varphi_p|\hat{A}|\varphi_b\rangle = \sqrt{2/\pi}\,N_p N_b \bar{u}_p T u_b, \qquad (1)$$

$$T = \langle\varphi_p|V_{i\mu}|\,k\rangle\,\hat{e}\overleftarrow{\Gamma}_\lambda, \quad \mu = \lambda + \eta, \quad \eta = m\alpha Z, \quad \hat{e} = \gamma_i e_i, \qquad (2)$$

$$\langle \mathbf{f}|V_{i\mu}|\,\mathbf{k}\rangle = \frac{1}{2\pi^2}\frac{1}{(\mathbf{f}-\mathbf{k})^2+\mu^2}. \qquad (3)$$

The operator Γ_λ is defined by formula (5a) of [8], **p** and **k** are the momenta of the electron and the photon, and **e** is the polarization of the γ quantum. The energy conservation law is given by

$$k + m\gamma = E = p + o(\varepsilon^{-1}), \quad \gamma = \sqrt{1-\alpha^2 Z^2}, \quad \varepsilon^{-1} = m/E. \qquad (4)$$

The wave function of the ejected electron $\langle\varphi_p|$ can be represented in the form (see [7,8])

$$\langle\varphi_p| = \sum_{n=0}^{\infty}(\alpha Z)^n\langle\varphi_p^n|, \qquad (5)$$

$$\langle\varphi_p^0| = -\frac{\partial}{\partial\nu}\langle\Phi_p(i\nu)|_{\nu\to 0}, \qquad (5a)$$

$$\langle\varphi_p^{n+1}| = \frac{\tilde{\nabla}_p}{2i\xi}\langle\Phi_p(i\nu)|(\hat{V}_0 G)^n|_{\nu\to 0}, \quad n \geqslant 0; \quad \tilde{\nabla} = \boldsymbol{\alpha}\nabla. \qquad (5b)$$

The function $\langle\Phi_p(i\nu)$ has in momentum space the form

$$\langle\Phi_p(i\nu)|\mathbf{k}\rangle = \frac{(q^2+\nu^2)^{i\xi-1}}{[(\mathbf{q}+\mathbf{p})^2-(p+i\nu)^2]^{i\xi}} = (-2p)^{-i\xi}(q^2+\nu^2)^{i\xi-1}$$
$$\times\left(i\nu - \mathbf{q}\frac{\mathbf{p}}{p} - \frac{q^2+\nu^2}{2p}\right)^{-i\xi}, \qquad (6)$$

where $\mathbf{q} = \mathbf{k} - \mathbf{p}$, $\xi = \alpha ZE/p$, and ∇_p in (5b) does not act on **q**.

1)This function is nonrelativistic only in form, but the relations between the energy E and the momentum p are relativistic in this function: $E^2 = p^2 + m^2$ ($\hbar = c = 1$).

Using (5)—(5b), we obtain the following expression for T in (2)[2])

$$T = (T_0 + \alpha Z\mathcal{T})\,\hat{e}\overleftarrow{\Gamma}_\lambda;\quad \mathcal{T} = \sum_{n=0}^{\infty} (\alpha Z)^n T_{n+1}, \tag{7}$$

$$T_0 = -\frac{\partial}{\partial \nu}\langle \Phi_p (i\nu) | V_{i\mu} | \mathbf{k}\rangle, \tag{7a}$$

$$T_{n+1} = \frac{\widetilde{\nabla}_p}{2i\xi}\langle \Phi_p (i\nu) | (\widetilde{V}_0 G)^n V_{i\mu} | \mathbf{k}\rangle. \tag{7b}$$

With the aid of formulas (A.2)—(A.6) of the appendix to a paper by one of the authors [9] we can obtain the following identity

$$\langle \mathbf{f} | V_0 G V_{i\mu} | \mathbf{k}\rangle = O_1 \langle \mathbf{f} | V_{i\mu_1} | \mathbf{B}_1\rangle,\quad i\mu_1 = \Lambda_1 + i\lambda_1, \tag{8}$$

$$O_1 \equiv \frac{1}{2}\int_0^1 dy_1 \left\{ \frac{i(\widetilde{B}_1 - E - \gamma_4 m)}{\Lambda_1}\Big|_{\lambda_1=0} + \widetilde{\nabla}_{B_1}\int_0^\infty d\lambda_1 \right\}, \tag{8a}$$

$$\Lambda_1^2 = (p^2 - k^2 y_1)(1 - y_1) - \mu^2 y_1$$
$$= p^2 (1 - y_1)^2 + (p^2 - k^2)\, y_1 (1 - y_1) - \mu^2 y_1,$$
$$\mathbf{B}_1 = \mathbf{k} y_1,\quad \widetilde{B} = \boldsymbol{\alpha}\mathbf{B}\quad (p^2 - k^2 = 2mk\gamma - \eta^2). \tag{8b}$$

The symbol $\langle \mathbf{f} \mid V_{i\mu} \mid \mathbf{B}\rangle$ is defined by formula (3).

Applying identity (8) in succession (see the analogous procedure in [7]) we obtain

$$\langle \mathbf{f} | (V_0 G)^n V_{i\mu} | \mathbf{k}\rangle = O_n \ldots O_2 O_1 \langle \mathbf{f} | V_{i\mu_n} | \mathbf{B}_n\rangle, \tag{9}$$

$$O_k = \frac{1}{2}\int_0^1 dy_k \left\{ \frac{i(\widetilde{B}_k - E - \gamma_4 m)}{\Lambda_k}\Big|_{\lambda_k=0} + \widetilde{\nabla}_{B_k}\int_0^\infty d\lambda_k \right\}, \tag{9a}$$

$$i\mu_n = \Lambda_n + i\lambda_n,$$

$$\Lambda_k^2 = (p^2 - B_{k-1}^2 y_k)(1 - y_k) - \mu_{n-1}^2 y_k$$
$$= p^2 (1 - y_1 y_2 \ldots y_k)^2 + (p^2 - k^2)$$
$$\times (1 - y_1 y_2 \ldots y_k)\, y_1 y_2 \ldots y_k - \mu^2 y_1 y_2 \ldots y_k$$
$$+ (2i\lambda_1\Lambda_1 - \lambda_1^2)\, y_2 \ldots y_k + \ldots$$
$$+ (2i\lambda_{k-1}\Lambda_{k-1} - \lambda_{k-1}^2)\, y_k,$$
$$\mathbf{B}_k = \mathbf{k} y_1 y_2 \ldots y_k. \tag{9b}$$

Using (9) we can represent the matrix element in (7b) in the form

$$\langle \Phi_p (i\nu) | (V_0 G)^n V_{i\mu} | \mathbf{k}\rangle = O_n \ldots O_2 O_1 \langle \Phi_p (i\nu) | V_{i\mu_n} | \mathbf{B}_n\rangle. \tag{10}$$

Further, in the appendix of [7] [formulas (A.8)—(A.10)] the following identity is proved:

$$\langle \Phi_p (i\nu) | V_{i\mu} | \mathbf{B}\rangle = \int_{\nu+\mu}^{\infty} \langle \Phi_p (i\lambda) | \mathbf{B}\rangle\, d\lambda. \tag{11}$$

At high energies, the nonvanishing photoeffect cross section is concentrated near the small angles satisfying the inequality

$$s = 2(pk - \mathbf{pk}) \approx \varepsilon\vartheta \lesssim 1 \ll \varepsilon,\quad \varepsilon = E/m,$$
$$\cos\vartheta = \mathbf{pk}/pk. \tag{12}$$

For these angles and energies we can rewrite (6) in the form

$$\langle \Phi_p (i\nu) | \mathbf{k}\rangle = (-2p)^{-i\xi} (z^2 + m^2 s^2 + \nu^2)^{i\xi - 1} (z + i\nu)^{-i\xi}$$
$$+ o(\varepsilon^{-1}),$$
$$z = p - k = m\gamma + o(\varepsilon^{-1}). \tag{13}$$

Confining ourselves to small angles and to the high energies (12), we can readily verify with the aid of (11) and (6) that for arbitrary μ and k

$$\frac{\widetilde{\nabla}_p}{2i\xi}\langle \Phi_p (i\nu) | V_{i\mu} | \mathbf{k}\rangle = \frac{\widetilde{p}}{p}\frac{1}{2m\varepsilon}\langle \Phi_p (i\nu) | V_{i\mu} | \mathbf{k}\rangle$$
$$+ o\left(\frac{1}{\varepsilon^2}, \frac{s}{\varepsilon^2}, \frac{s^2}{\varepsilon^2}\right). \tag{14}$$

Thus, the presence of ∇_p in (7b) leads [see (10)] to the appearance of a smallness of order ε^{-1}, whereas the derivative with respect to ν in (7a) does not lead to any smallness whatever, as can be seen from (13)[3]).

To compare the relative values of (10) and (11) it is necessary to go in (10) to the limit $\varepsilon \to \infty$ ($p \to k + m\gamma$), but in this case those terms of (9a) containing Λ will have logarithmic divergences at the lower limits, corresponding to the presence of $\ln^n\varepsilon$ in (10). Therefore these terms must be summed out prior to going to the limit. To this end, proceeding as in [7], we make in (9) the change of variables $x_k = 1 - y_1 \ldots y_k$, after which we break up all the integrals into parts with $x_k < a$ and $x_k > a$, $\varepsilon^{-1} \ll a \ll 1$. Then expression (10) (see [7]) can be represented in the form

$$\langle \Phi_p (i\nu) | (V_0 G)^n V_{i\mu} | \mathbf{k}\rangle = \sum_{k=0}^{n} A_k(a)\, B_{n-k}(a), \tag{15}$$

$$A_n(a) = \left[\frac{i(\widetilde{k} - E)}{2k}\right]^n \int_0^a \frac{dx_1}{\Lambda_1} \ldots \int_{x_{n-1}}^a \frac{dx_n}{\Lambda_n}\Big|_{\lambda_1 = \ldots \lambda_n = 0}$$
$$= \frac{1}{n!}\left[\frac{i(\widetilde{k} - E)}{2k}\right]^n \ln^n \frac{a\varepsilon}{\gamma}, \tag{15a}$$

$$B_n(a, \vartheta, \mu) = \frac{1}{2^k}\int_{x_{n-1}}^1 \frac{dx_n}{1 - x_n}\{\ldots\}_n \ldots \int_{x_1}^1 \frac{dx_2}{1 - x_2}\{\ldots\}_2$$
$$\times \int_a^1 \frac{dx_1}{1 - x_1}(1 - x_n)\langle \Phi_p (i\nu) | V_{i\mu_n} | \mathbf{B}_n\rangle, \tag{15b}$$

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[2])The gradient with respect to p acts in (7b) only on $\langle\Phi_p(i\nu)|$ and not on the p contained in G.

[3])We note that at large angles $\vartheta \sim 1$ ($s \sim \varepsilon$) Eq. (14) contains no smallness whatever and turns out to be of the same order as (7a).

$$B_0(\vartheta, \mu) = \langle \Phi_p(i\nu) | V_{i\mu} | \mathbf{k} \rangle;$$

$$\{\ldots\}_n = \left\{ \frac{i(\widetilde{B}_n - E - \gamma_4 m)}{\Lambda_n} \Big|_{\lambda_n = 0} + \widetilde{\nabla}_{B_n} \int_0^\infty d\lambda_n \right\}. \quad (15c)$$

Substituting (15) in (17) and taking (14) into account, we get

$$\mathcal{T} = -\frac{\widetilde{p}}{p} \frac{1}{\varepsilon} \frac{1}{2m} \left(\frac{\varepsilon}{\gamma} \right)^{i\alpha Z (\widetilde{k} - E)/2k} \mathcal{B}(\alpha Z, \vartheta, \mu), \quad (16)$$

$$\mathcal{B}(\alpha Z, \vartheta, \mu) = a^{i\alpha Z (\widetilde{k} - E)/2k} \sum_{n=0}^{\infty} (\alpha Z)^n B_n(a, \vartheta, \mu). \quad (17)$$

We can now go to the limit as $\varepsilon \to 0$ in all the braces of (15b). With the aid of (11), (12), and (9b) it is also easy to verify that in this limit $\langle \Phi_p(i\nu) | V_{i\mu_k} | B_k \rangle$ is of the same order as $B_0(\vartheta, \mu)$ in the parameter ε^{-1}, and consequently of the same order as T_0, i.e., all the $\mathcal{B}(\alpha Z, \vartheta, \mu)$ (17) have the same order in ε^{-1} as T_0.

In the case of photoelectrons emitted forward ($\mathbf{k} \to \mathbf{p}$), using the equality [see (1)]

$$\bar{u}_p f(\widetilde{k}) = \bar{u}_p f(\widetilde{p}) = \bar{u}_p f(-1), \quad (18)$$

we can rewrite (16) in the form

$$\mathcal{T} = \varepsilon^{-1} (\varepsilon/\gamma)^{-i\alpha Z} \mathcal{B}(\alpha Z, 0, \mu)/2m. \quad (19)$$

We see that $\mathcal{T}$ is of the order of $\varepsilon^{-i\alpha Z - 1}$ with respect to T_0. All the logarithmic terms appearing in $\mathcal{T}$ constitute the expansion of the phase factor $\varepsilon^{-i\alpha Z}$. However, owing to the absence of such a phase factor in T_0, the logarithms of ε do not vanish in the cross section [see (51) and (52) of [8]].

Thus, in the calculation of the photoeffect in the extreme relativistic case, we can use only the first term in (7), corresponding to a matrix element with nonrelativistic wave function (5a). The resultant error in the cross section is of order ε^{-2}, since the interference term (19) and (7a) will not be significant, owing to the strong oscillations of (19). We note also that at large angles $\vartheta \sim 1$ ($s \sim \varepsilon$) the use of the function (5a) alone is insufficient, for in accordance with footnote 3) (16) is of the same order as (7a).

The derivation presented is valid for all shells of the atoms [all that changes is the form of Γ_λ in (7)]. The foregoing statements apply also to an equal degree to the one-photon positron annihilation, which is a process related to the photoeffect (it is merely necessary to reverse the signs of the momentum and the energy of the ejected electron, and to rewrite the energy conservation law (4) in the form $k = E_+ + m\gamma$).

2. From the facts proved in the preceding section it follows [see (1), (7a), (11), and (13)], that the expression for the matrix element of the photoeffect on the K shell has in the extreme relativistic case the following form (see also formula (13a) in the authors' earlier paper [8]):

$$T = \hat{e}\Gamma_\lambda \langle \Phi_p(i\mu) | \mathbf{k} \rangle$$

$$= \hat{e}\Gamma_\lambda \left(-\frac{\partial}{\partial \eta} \right) (-2p)^{-i\xi} \int_0^\infty d\lambda \lambda^\sigma \frac{(z^2 + m^2 s^2 + \mu^2)^{i\xi - 1}}{(z + i\mu)^{i\xi}}$$

$$= \hat{e}(-2p)^{-i\xi} \sigma \Gamma_\eta I, \quad (20)$$

$$I = \int_0^\infty d\lambda \lambda^{\sigma - 1} \frac{(z^2 + m^2 s^2 + \mu^2)^{i\xi - 1}}{(z + i\mu)^{i\xi}}, \quad (21)$$

where

$$\Gamma_\eta = -\partial/\partial\eta + {}^1/_2 \varkappa \widetilde{\nabla}_k, \quad {}^1/_2 \varkappa = \sigma/\alpha Z,$$

$$\sigma = 1 - \gamma, \gamma = (1 - \alpha^2 Z^2)^{1/2},$$

$$\eta = m\alpha Z, \quad \xi = \alpha Z E/p \cong \alpha Z, \quad \widetilde{\nabla} = \boldsymbol{\alpha}\nabla. \quad (22)$$

The integral I in (21) is transformed by making the change of variable $x = (z + i\eta)/(i\lambda + z + i\eta)$:

$$I = i^{-\sigma} (-1)^{1 - i\xi} (\gamma + i\eta)^{\sigma + i\xi - 2} J, \quad (23)$$

$$J = \int_0^1 x^{-\sigma - i\xi + 1} (1 - x)^{\sigma - 1} (1 - xz_1)^{i\xi - 1} (1 - xz_2)^{i\xi - 1} dx; \quad (24)$$

$$z_1 = \frac{z - \sqrt{z^2 + m^2 s^2}}{z + i\eta}; \quad z_2 = \frac{z + \sqrt{z^2 + m^2 s^2}}{z + i\eta}. \quad (25)$$

The integral J (23) can be expressed in terms of the Appel double hypergeometric series [10] (formula 9.180)

$$J = \sum_{n=0}^{\infty} \frac{(1 - i\xi)_n}{n!} z_1^n \int_0^1 x^{n - \sigma - i\xi + 2 - 1} (1 - x)^{\sigma - 1} (1 - xz_2)^{i\xi - 1} dx$$

$$= \frac{\Gamma(2 - i\xi - \sigma)\Gamma(\sigma)}{\Gamma(2 - i\xi)}$$

$$\times \sum_{n=0}^{\infty} z_1^n \frac{(1 - i\xi)_n (2 - i\xi - \sigma)_n}{(2 - i\xi)_n n!} {}_2F_1(2 - i\xi - \sigma + n, 1 - i\xi;$$

$$n + 2 - i\xi; z_2) = \frac{\Gamma(2 - i\xi - \sigma)\Gamma(\sigma)}{\Gamma(2 - i\xi)} \quad (26)$$

$$F_1(2 - i\xi - \sigma, 1 - i\xi, 1 - i\xi; 2 - i\xi; z_1, z_2),$$

$$(a)_n = \Gamma(a + n)/\Gamma(a). \quad (27)$$

Substituting (26) and (23) in (20) and carrying out the differentiations contained in the operator Γ_η, we arrive at an expression for the scattering amplitude, coinciding with that obtained by Nagel [4], who used a distorted plane wave for the wave function of the ejected electron. Thus, the sufficiency of the use of the distorted plane wave is proved [the use of expression (13) in (20) leads to an error of order ε^{-1}].

We now consider the amplitude (20) for small s. To this end we expand (21) in powers of s^2 and make in the integral a change of variable $x = i\lambda/(z + i\mu)$. We then obtain with the aid of the integral representation for the hyperfunction (see [10], formula 9.111)

$$T = \sigma(-2p)^{-i\xi}\,\hat{e}\Gamma_\eta \sum_{n=1}^{\infty} (ms)^{2(n-1)} \frac{(1-i\xi)_{n-1}}{(n-1)!} J^n, \quad (28)$$

$$J^n = \int_0^\infty \lambda^{\sigma-1} d\lambda \frac{(z-i\mu)^{i\xi-n}}{(z+i\mu)^n} = i^{-\sigma}(z+i\eta)^{\sigma-n}(z-i\eta)^{i\xi-n}$$
$$\times \frac{\Gamma(\sigma)\Gamma(2n-i\xi-\sigma)}{\Gamma(2n-i\xi)}\, {}_2F_1\left(n-i\xi, \sigma;\ 2n-i\xi;\ \frac{2z}{z-i\eta}\right). \quad (29)$$

The operations contained in Γ_η can be readily carried out with the aid of the following formulas [see (22)]:

$$\Gamma_\eta J^n = -\left(\frac{\partial}{\partial\eta} + \frac{\varkappa}{2}\frac{\widetilde{k}}{k}\frac{\partial}{\partial z}\right) J^n,$$
$$\Gamma_\eta s^2 = \varkappa p\left(\frac{\widetilde{k}}{k} - \frac{\widetilde{p}}{p}\right), \quad \widetilde{k} = \boldsymbol{\alpha}\mathbf{k}. \quad (30)$$

Confining ourselves to terms of order s^2, carrying out operations (30), and using the Gauss recurrence formulas for the hyperfunction (see 9.137 (12) of [10]), we can express (28) in terms of one hyperfunction of the form

$$\Phi_1 = {}_2F_1(1-i\xi, \sigma;\ 2-i\xi;\ 2z/(z-i\eta)). \quad (31)$$

Substituting the thus obtained expression for T in (11) and (1) of [8] and summing over the electron polarizations, we obtain the following expression for the differential cross section of the photoeffect

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha\varepsilon}{m^2}\frac{1+\gamma}{\gamma^3}\frac{|\Gamma(\gamma-i\xi)|^2}{\Gamma(2\gamma)} e^{-\pi\xi+2\xi \arcsin\xi}\, \xi^3 (2\xi)^{2(\gamma-1)} Q, \quad (32)$$

$$Q = |w|^2 + s^2\Bigg\{\frac{\xi^2}{(1-2\gamma)^2+\xi^2}\Bigg|1 - \frac{1-i\xi}{1+\gamma}\left(1+\gamma-i\xi+i\xi\frac{2+\gamma-i\xi}{2\gamma}\right)w\Bigg|^2$$
$$+ \mathrm{Re}\, w^*\Bigg[\frac{i\xi}{2\gamma}\left(1+\gamma-i\xi+\frac{i\xi}{1+\gamma}\frac{2-i\xi}{\gamma}\right)(2+\gamma-i\xi) - \frac{1}{\gamma}\left(1+\gamma+i\xi+\frac{2-i\xi}{2\gamma}\right)$$
$$+ \left(\frac{2-i\xi}{2\gamma^2} - \left(1+\gamma-i\xi+\frac{i\xi}{1+\gamma}\frac{2-i\xi}{2\gamma}\right)\right.$$
$$\left.\times\left(1+\gamma-i\xi+i\xi\frac{2+\gamma-i\xi}{2\gamma}\right)\right)\frac{1-i\xi}{1-i\xi-2\gamma}(1-w)\Bigg]\Bigg\}, \quad (32a)$$

$$w = 1 - \frac{1-i\xi-2\gamma}{1-i\xi}\left(\frac{\xi-i\gamma}{\xi+i\gamma}\right)^{-\gamma}\Phi_1. \quad (32b)$$

When s = 0 formulas (32) coincide with the results obtained by Weber and Mullin.[3]

In spite of the fact that (32) contains only known functions and only one hyperfunction (31), the formula presented for the cross section is not very suitable for specific calculations. Therefore, using the analyticity of the functions, we expand (32) in powers of $\xi = \alpha Z$. The main difficulty lies in this case in the expansion of the hyperfunction (31). With the aid of the known transformation formula (see 9.132(2) and 9.137 in [10]) we can represent the function (11) in the form

$$\Phi_1 = -\frac{1-i\xi}{2\gamma}\left(\frac{\xi-i\gamma}{\xi+i\gamma}\right)^{\gamma}\Bigg[1 + \frac{i\xi(\gamma-i\xi)}{2\gamma(1-\gamma+i\xi)}\left(\frac{2\gamma}{\gamma+i\xi}\right)^{\gamma}\Phi_2$$
$$+ (\gamma-i\xi)\left(-\frac{2\gamma}{\gamma-i\xi}\right)^{i\xi}\left(\frac{\xi-i\gamma}{\xi+i\gamma}\right)^{-\gamma}\frac{\Gamma(1-i\xi)\Gamma(i\xi-\gamma)}{\Gamma(1-\gamma)}\Bigg], \quad (33)$$

$$\Phi_2 \doteq {}_2F_1(\sigma, i\xi+\sigma;\ 1+i\xi+\sigma;\ (\gamma-i\xi)/2\gamma). \quad (34)$$

As shown in the appendix, the expansion of (34) in the arguments σ and ξ has the form

$$\Phi_2 = 1 + a\sigma\left\{\sum_{n=0}^{\infty}(-1)^n a^n L_{n+2}(z) + \sum_{n=1}^{\infty}(-1)^{n+1}\sigma^n N_{n+1}(z)\right\},$$

$$N_n(z) = \frac{1}{n!}\int_0^z \frac{\ln^n(1-x)}{x}dx$$
$$= \frac{1}{n!}\ln z \ln^n(1-z) + (-1)^n L_{n+1}(1)$$
$$- \sum_{k=1}^{n}\frac{(-1)^k}{(n-k)!}\ln^{n-k}(1-z)\, L_{k+1}(1-z),$$

$$L_n(z) = \int_0^z \frac{L_{n-1}(x)}{x}dx = \sum_{k=1}^{\infty}\frac{z^k}{k^n}, \quad a = \sigma + i\xi, \quad z = \frac{\gamma-i\xi}{2\gamma}. \quad (35)$$

With the aid of (35) it is possible in principle to calculate the cross sections (32) in powers of αZ up to an arbitrary order. We present below the expansion of (32a) up to terms of order ξ^3

$$Q = \xi^2(A + s^2 B), \quad (36)$$

$$A = \tfrac{1}{4}\xi^2\,[(1+\pi^2/4) - \pi\xi(1+\ln 2-\pi^2/8) - \xi^2[3\pi^2/8 - \pi^4/64 - \tfrac{3}{2} + \tfrac{1}{3}(\pi^2 - \tfrac{1}{2}\ln^2 2)\ln 2 + L_3(\tfrac{1}{2})]$$
$$= 0{,}87\xi^2(1-0{,}42\xi-3{,}45\xi^2), \quad (36a)$$

$$B = 1 - \pi\xi/2 - \tfrac{1}{2}\xi^2(5\pi^2/8-1) + \tfrac{1}{4}\xi^3\pi(2-3\pi^2/8 + \tfrac{9}{2}\ln 2 + \ln^2 2) = 1 - 1{,}57\xi - 2{,}58\xi^2 + 1{,}49\xi^3. \quad (36b)$$

The first two terms of (36a) coincide with the principal terms of the expansion of the forward cross section in powers of ε^{-1}, as obtained by the authors in [8]. The first two terms of the coefficient B of s^2 agree with the formula of Gavrila [11] taken for the extreme relativistic case.

The sign of B determines the position of the peak of the angular distribution. The equation $B = 0$ is satisfied for some $Z = Z_{cr}$. $B > 0$ when $Z < Z_{cr}$ and consequently the maximum of the angular distribution lies to the right of the zero angle. $B < 0$ when $Z > Z_{cr}$ and the maximum coincides with the zero angle. From (36b) we find that $Z_{cr} \cong 56$.

The quantity Z_{cr} is an indirect index of the rate of convergence of the series in ξ. The smaller Z_{cr}, the worse the convergence. As follows from the plots presented in [8], Z_{cr} increases with decreasing energy, thus evidencing an improvement in the convergence of the series on going into the region of low energies. We note that in spite of the poor convergence of the series for A and B separately, the series for the ratio of A to B converges sufficiently well. This justifies to some degree the authors' assumption [8] of satisfactory convergence of the cross section ratio at zero angle to the maximum.

From formulas (20)—(26) for the amplitude and the cross section we cannot draw any conclusions concerning the behavior of the angular distribution in the large-angle region $\vartheta \sim 1$ ($s \sim \mathcal{E}$), for in this case (see footnote 3)) the discarded terms of the wave function of the ejected electron turn out to be of the same order as the terms used. However, from considerations of monotonicity of the angular distribution, the photoeffect cross section should be negligibly small at high energies in this region.

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APPENDIX

The expansion of the hyperfunction ${}_2F_1$ (σ, a; 1 + a; z) in the parameters σ and a has the form

$$
\begin{aligned}
{}_2F_1(\sigma, a; 1+a; z) = 1 + \sigma a\Big[\sum_{n=1}^{\infty}\frac{z^n}{n^2} - a\sum_{n=1}^{\infty}\frac{z^n}{n^3} \\
+ a^2\sum_{n=1}^{\infty}\frac{z^n}{n^4} - \ldots + \sigma\sum_{n=1}^{\infty}\frac{z^{n+1}}{(n+1)^2}\sum_{m=1}^{n}\frac{1}{m} \\
+ \sigma^2\sum_{n=1}^{\infty}\frac{z^{n+2}}{(n+2)^2}\sum_{m=1}^{n}\frac{1}{m+1}\sum_{k=1}^{m}\frac{1}{k} + \ldots\Big].
\end{aligned} \tag{A.1}
$$

Using the readily verified formula (see also [10], 1.516)

$$
\sum_{k=1}^{\infty}\frac{x^{k+s}}{k+s}\sum_{n_1=1}^{k}\frac{1}{(n_1+s-1)}\cdots\sum_{n_s=1}^{n_{s-1}}\frac{1}{n_s} = \frac{(-1)^{s+1}}{(s+1)!}\ln^{s+1}(1-x), \tag{A.2}
$$

we find that

$$
\sum_{k=1}^{\infty}\frac{z^{k+s}}{(k+s)^2}\sum_{n_1=1}^{k}\frac{1}{(n_1+s-1)}\cdots\sum_{n_s=1}^{n_{s-1}}\frac{1}{n_s} = \frac{(-1)^{s+1}}{(s+1)!}\int_0^z\frac{\ln^{s+1}(1-x)}{x}dx. \tag{A.3}
$$

Substituting (A.3) in (A.1) we arrive at formula (35).

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