RANDOM FORCE METHOD IN TURBULENCE THEORY

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A method for the statistical description of the motion of fluid particles in turbulent flow is proposed, based on the decay law for equilibrium velocity fluctuations and the associated Langevin stochastic equation. The decay law is generalized by taking into account the internal time scale of the fluctuations; this ensures the existence of finite accelerations. In contrast to the Brownian movement, in turbulence theory interest attaches to the correlations for periods that are small compared with the relaxation time of the fluid particles and for distances that are small compared with the mixing length (analogous to the mean free path of a molecule). Consideration of such time intervals and distances yields both the well-known relations derived in similarity theory (particularly the $\frac{2}{3}$-law) and some new results, the correlation functions of the relative motion of fluid particles. Coefficients relating the characteristics of single-particle and two-particle descriptions of turbulence are also determined.

A well-developed turbulent fluid flow is a system with a very large number of degrees of freedom; turbulence must therefore be investigated mainly by statistical methods. It may be considered that the basic laws of turbulent flow do not depend on the detailed structure of the Navier-Stokes equation. The most important characteristics of this equation are its nonlinearity, which ensures an exchange of energy between movements of different scales, and the existence of viscosity as a dissipative factor.

In [1] Kolmogorov advanced the similarity hypothesis, according to which the structure of turbulent flow on scales that are very small compared with the external turbulence scale L is determined by only two parameters, the kinetic energy of dissipation ϵ and the kinematic viscosity ν. The similarity hypothesis enables us by dimensional reasoning alone to obtain many results, the most important of which is the Kolmogorov-Obukhov $\frac{2}{3}$-law. [1,2]

In the present paper turbulence is investigated by using the statistical description of the time evolution of fluctuations, a method developed in molecular statistics (in the spirit of the theories of Langevin [3] and Onsager [4]). It has been possible, starting with the decay law of equilibrium fluctuations and the associated Langevin equation, to derive relations arising out of the similarity hypothesis (particularly the $\frac{2}{3}$-law) together with several new relations that cannot be written on the basis of dimensional considerations alone.

We note some of the fundamental aspects of the way in which molecular statistical methods are applied to turbulence.

1. It is more convenient to start with the Lagrangian description of turbulence, excluding the convective term in the equation of motion. It is also easier to select individual degrees of freedom—the coordinates and velocities of specified fluid particles that can be followed—and to account for all other degrees of freedom statistically.

2. The relaxation time for the velocities of fluid particles in turbulent flow cannot, as a rule, be considered small; also, the mixing length (which is analogous to the mean free path of a molecule) is comparable to the dimensions of the system. On the other hand, in molecular statistics the space and time scales of the fluctuation field are small. In turbulence theory we are therefore interested in different time and space intervals; this leads to qualitatively different laws.

3. In molecular statistics extensive use is made of Markov processes, which correspond to infinite accelerations and higher time derivatives. Turbulence theory can derive very reliable results for the corresponding rms values, which approach infinity in the limiting case of vanishing viscosity. Some modification of the Onsager theory (Sec. 1) was thus required to take into account the internal time scale of fluctuations and to ensure the existence of accelerations; this can also be of interest for problems of molecular statistics.

4. In using the Langevin stochastic equations to describe the relative motion of fluid particles (Sec. 3) it is assumed that the random forces are localized in space; the time localization of the random forces follows from the fluctuation decay law. Through this hypothesis it became possible to relate the single-particle and two-particle descriptions of turbulence by means of exact numerical coefficients and to obtain new results. The proposed method of investigating turbulence is therefore called the random force method.

1. GENERALIZED DECAY LAW OF EQUILIBRIUM FLUCTUATIONS

Let us consider a stationary random function $\alpha(t)$ with zero mean value:

$$\langle \alpha(t) \rangle = 0, \tag{1.1}$$

and the correlation function

$$R(t) = \langle \alpha(t+s)\,\alpha(s) \rangle = \langle \alpha(t)\,\alpha_0 \rangle \quad (\alpha_0 \equiv \alpha(0)), \tag{1.2}$$

which, in virtue of the stationarity, is independent of s and is even in t.[1] Conditional averaging for a fixed value α_0 is denoted by the symbol $\langle\ \rangle_0$; we introduce the quantity

$$a(t) = \langle \alpha(t) \rangle_0, \tag{1.3}$$

$$a(0) = \alpha_0. \tag{1.4}$$

Averaging over the initial states will be denoted by a superior bar; complete averaging is therefore represented by

$$\langle\rangle = \overline{\langle\rangle_0}. \tag{1.5}$$

Obviously,

$$R(t) = \overline{a(t)\,\alpha_0}, \tag{1.6}$$

$$\overline{a(t)} = 0. \tag{1.7}$$

Unlike the randomly varying function $\alpha(t)$, the function $a(t)$ can be considered as decreasing monotonically in absolute value, with

$$\lim_{t\to\infty} a(t) = 0. \tag{1.8}$$

The latter condition is associated with the fact that after some time a fluctuating quantity "forgets" its initial value.

Onsager[4] assumed, on the basis of empirical laws, that the evolution of $a(t)$ is described by a linear law:

$$da(t)/dt = -\lambda a(t), \tag{1.9}$$

where λ is a positive constant. From (1.9) and (1.4) we have

$$a(t) = \alpha_0 e^{-\lambda t}. \tag{1.10}$$

This expression, which satisfies the requirements (1.7)—(1.8), is not in accord with evenness of the correlation function (1.6). In order to satisfy the condition of evenness, t is replaced formally by $|t|$; then from (1.10) and (1.6) we have

$$R(t) = \langle \alpha^2 \rangle\, e^{-\lambda |t|}, \tag{1.11}$$

since $\overline{\alpha_0^2} = \langle \alpha^2(t) \rangle \equiv \langle \alpha^2 \rangle$.

The time derivative of the correlation function (1.11) suffers a discontinuity at $t = 0$, whereas it should actually vanish. Indeed, for times that are small compared with the characteristic time scale τ_0, which will be determined below [see (1.13)] the function $\alpha(t)$ is smooth and the correlation function assumes the form

$$R(t) = \langle \alpha^2 \rangle - {}^1/_2 \langle [\alpha(t) - \alpha_0]^2 \rangle \approx \langle \alpha^2 \rangle - {}^1/_2 \langle (d\alpha/dt)^2 \rangle\, t^2 \tag{1.12}$$

$[(|t|) \ll \tau_0]$, Eqs. (1.11) and (1.12) are then equivalent for t of the order

$$\tau_0 = \lambda \langle \alpha^2 \rangle / \langle (d\alpha/dt)^2 \rangle. \tag{1.13}$$

The expression (1.13) can naturally be called the internal time scale of fluctuations,[2] in contrast to the external time scale $T = \lambda^{-1}$, which determines the fluctuation decay time. The ratio of these two scales, $m = \lambda\tau_0$, is a small parameter; as this parameter decreases in value the frequency range of real fluctuations is extended.

It is seen from the foregoing that the correlation function (1.11) and the initial equation (1.9) apply only to times larger than τ_0. The empirical equation (1.9) could be verified only for such (macroscopic) times. We shall attempt to generalize this equation so that it can also yield correct results for small times.

The time derivative of $a(t)$ must be a function of α_0 and t; assuming a one-to-one correspondence between $a(t)$ and α_0 for fixed t, we have

$$da(t)/dt = F(t, a).$$

Obviously, $F(t, 0) = 0$. Assuming small fluctuations [compared with the possible limits to the variation of the physical quantity $\alpha(t)$] and confining ourselves to the first term of $F(t, a)$ expanded in terms of a, we obtain

$$da(t)/dt = \partial F(t, a)/\partial a\,|_{a=0}\, a(t). \tag{1.14}$$

We find that the coefficient of $a(t)$ in the right-

[1] For the multivariate case the stationarity condition yields $R_{ik}(t) = \langle \alpha_i(t+s)\alpha_k(s) \rangle = R_{ki}(-t)$. Evenness and therefore symmetry follow from the principle of microscopic reversibility.

[2] We note that in the case of Brownian movement τ_0 signifies the collision time.

hand side of (1.14) is generally dependent on t.

Considering that (1.14) must go over into (1.9) for $t \gg \tau_0$, we finally obtain

$$da(t)/dt = -\lambda\varphi(t/\tau_0)\, a(t), \qquad (1.15)$$

$$\lim_{x\to\infty} \varphi(x) = 1. \qquad (1.16)$$

This is a natural generalization of (1.9) if it is remembered that a(t), unlike $\alpha(t)$, is not a stationary random function and that the equation for a(t) can contain variable coefficients.

From (1.15), (1.6), and (1.4) we have

$$R(t) = \langle\alpha^2\rangle \exp\left\{-m\int_0^{t/\tau_0}\varphi(x)\,dx\right\}, \qquad (1.17)$$

where the notation $m = \lambda\tau_0$ has been used. The evenness of R(t) leads to the oddness of $\varphi(x)$, while (1.12) gives

$$\varphi(x) \approx x \qquad (|x| \ll 1). \qquad (1.18)$$

Since $\alpha(t)$ is a stationary random function, the corresponding spectral function

$$\Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(t)\, e^{-i\omega t}\, dt \qquad (1.19)$$

must be positive for all ω. This imposes certain limitations on $\varphi(x)$. Finally, if we require the existence of mean squares and correlation functions for the derivatives of $\alpha(t)$ of all orders, $\varphi(x)$ must be infinitely differentiable on the real axis. It follows from this last condition that the spectrum $\Phi(\omega)$ decreases for $\omega\tau_0 \gg 1$ more rapidly than any power of the frequency.[3)] At low frequencies, with $m \ll 1$, the spectrum has the asymptotic form

$$\Phi(\omega) = \lambda\langle\alpha^2\rangle / \pi(\lambda^2 + \omega^2) \qquad (\omega\tau_0 \ll 1), \qquad (1.20)$$

which corresponds to the correlation function (1.11) and agrees with the classical fluctuation spectrum.

In the language of probability theory the correlation function (1.11) and the spectrum (1.20) correspond to a Markov random process $\alpha(t)$. The real physical process can be approximated by a Markov chain only for sufficiently large time intervals (or at low frequencies). In small time intervals we must take into account the microprocesses always existing in a physical system, which smooth out any sharp jumps of the fluctuating quantities.

The foregoing generalized decay law of equilibrium fluctuations (1.15) takes microprocesses into account phenomenologically. Moreover, this law eliminates the incompatibility between the principle of microscopic reversibility and Eq. (1.9), which is noninvariant under time reversal. This incompatibility was pointed out by Onsager [4] and remained in all subsequent publications on fluctuation theory (e.g. [5]).

2. VELOCITY FLUCTUATIONS OF A FLUID PARTICLE IN TURBULENT FLOW

Let us consider the homogeneous and statistically stationary flow of an incompressible fluid. A homogeneous flow is the simplest mathematical model, and can be used to describe the structure of real turbulent flow only on scales that are very small compared with the external turbulence scale L.

Let $v_i(t)$ be a vector representing the deviation of the velocity of a fixed fluid particle from the mean flow velocity. We have

$$\langle v_i(t)\rangle = 0, \qquad (2.1)$$

$$R_{ik}(t) = \langle v_i(t+s)\, v_k(s)\rangle = \langle v_i(t)\, v_{k0}\rangle \qquad (v_{k0} \equiv v_k(0)), \qquad (2.2)$$

where $R_{ik}(t)$ is the correlation tensor. In virtue of the stationarity we have

$$R_{ik}(t) = R_{ki}(-t). \qquad (2.3)$$

The dynamical equations of a viscous fluid are not invariant under time reversal; therefore we cannot in the general case require evenness of the tensor $R_{ik}(t)$. However, if this tensor is symmetric it is an even function because of (2.3).

$R_{ik}(t)$ is symmetric for isotropic flow and for axisymmetric flow, which is statistically invariant under reflections in any point. In the first case we have

$$R_{ik}(t) = {}^1/_3\, \delta_{ik} R_{ll}(t), \qquad (2.4)$$

where δ_{ik} is the Kronecker symbol; repeated indices are summed from 1 to 3. In the second case we have

$$R_{ik}(t) = R_{\parallel}(t)\, n_i n_k + R_{\perp}(t)\,(\delta_{ik} - n_i n_k), \qquad (2.5)$$

where n_i is the unit vector in a specified direction, $R_{\parallel}(t)$ is the correlation function of the velocity component parallel to n_i, and $R_{\perp}(t)$ is the correlation function of one of the velocity components perpendicular to n_i.

As in Sec. 1, we denote conditional averaging for a fixed value of v_{i0} by the symbol $\langle\ \ \rangle_0$ and introduce

$$u_i(t) = \langle v_i(t)\rangle_0, \qquad (2.6)$$

[3)]One can assume the interpolation formula $\varphi(x) = x(1+x^2)^{-1/2}$, which leads to a spectrum that decays exponentially at high frequencies.

$$u_i(\dot{0}) = v_{i0}. \quad (2.7)$$

Furthermore,

$$R_{ik}(t) = \overline{u_i(t)\, v_{k0}}, \quad (2.8)$$

$$\overline{u_i(t)} = 0, \quad (2.9)$$

$$\lim_{t\to\infty} u_i\,(t) = 0, \quad (2.10)$$

where the overbar denotes averaging over the parameters v_{i0}.

We shall now confine ourselves to a consideration of isotropic flow. The extension of the results to axisymmetric flow encounters no special difficulty. The generalized decay law of velocity fluctuations and the correlation tensor for isotropic flow are

$$du_i(t)/dt = -\lambda\varphi\,(t/\tau_0)u_i\,(t), \quad (2.11)$$

$$R_{ik}(t) = {}^1/_3\,\langle v_l^2\rangle\,\delta_{ik}\exp\left\{-m\int\limits_0^{t/\tau_0}\varphi\,(x)\,dx\right\}. \quad (2.12)$$

Here λ is the reciprocal relaxation time of fluid particle velocity; the function $\varphi(x)$ has the same properties that were enumerated in Sec. 1; $m = \lambda\tau_0$; the definition of the internal time scale is similar to (1.13):

$$\tau_0 = \varepsilon_*/\langle(dv_l/dt)^2\rangle,\ \ \varepsilon_* = \lambda\,\langle v_l^2\rangle. \quad (2.13)$$

In order to elucidate the meaning of ϵ_* we use the Langevin stochastic equation for the random velocity:

$$dv_i(t)/dt = -\lambda v_i\,(t) + f_i(t), \quad (2.14)$$

where $f_i(t)$ is the random force per unit fluid mass. Multiplying (2.14) by $v_i(t)$, averaging, and using the stationarity condition, we obtain

$$\varepsilon_* = \langle f_l\, v_l\rangle\,. \quad (2.15)$$

Thus ϵ_* is the energy afflux due to the work of random forces. Generally speaking, it differs by a factor of the order of unity from ϵ, the kinetic energy dissipation obtained in terms of the mean square of the Eulerian velocity gradient. The latter quantity is the basic parameter of the Kolmogorov-Obukhov theory.[1,2] The exact relation between these two parameters of turbulent flow requires special investigation.

By dimensional reasoning we have

$$\tau_0 \sim (\nu/\varepsilon)^{1/2}, \qquad \varepsilon \sim (\langle v_l^2\rangle)^{3/2}/L, \quad (2.16)$$

where ν is the kinematic viscosity and L is the external spatial scale of turbulence. Consequently, we have the small parameter

$$m = \lambda\tau_0 \sim \mathrm{Re}^{-1/2} \qquad (\mathrm{Re} = \langle v_l^2\rangle^{1/2} L/\nu), \quad (2.17)$$

where Re is the Reynolds number. We note that the same small parameter appears in the theory of the boundary layer.

For times greater than τ_0 Eqs. (2.12) and (1.16) lead to

$$R_{ik}\,(t) = {}^1/_3\,\langle v_l^2\rangle\,\delta_{ik}\,e^{-\lambda|t|} \qquad (|\,t\,| \gg \tau_0). \quad (2.18)$$

The corresponding asymptotic form of the turbulent-flow energy spectrum $E(\omega) = \frac{1}{2}\Phi_{ll}(\omega)$ [where $\Phi_{ik}(\omega)$ is the spectral tensor] is

$$E\,(\omega) = \varepsilon_*/2\pi\,(\lambda^2 + \omega^2) \qquad (\omega\tau_0 \ll 1) \quad (2.19)$$

When m is sufficiently small (the Reynolds number therefore being large), there exists the time interval

$$m \ll \lambda\,|\,t\,| \ll 1 \quad (2.20)$$

and the corresponding frequency interval

$$1 \gg \omega\tau_0 \gg m, \quad (2.21)$$

for which Eqs. (2.18) and (2.19) lead to

$$D(t) = \langle[v_i(t) - v_{i0}]^2\rangle = 2\varepsilon_*\,|\,t\,|, \quad (2.22)$$

$$E\,(\omega) = \varepsilon_*/2\pi\omega^2, \quad (2.23)$$

where D(t) is the structural function. We thus see that ϵ_* is the only parameter determining the behavior of fluid particles in the given time and frequency intervals.

Equation (2.22) corresponds to the Kolmogorov-Obukhov $\frac{2}{3}$-law[1] and to the Eulerian description of turbulence derived by dimensional reasoning for the inertial interval of the scales. Obukhov has derived a similar formula in [6], where he assumed a six-variate Markov process for the coordinates and velocity of a fluid particle under Galilean invariance.[4)] In Obukhov's equation the role of ϵ_* is assumed by the constant coefficient B in the Fokker-Planck equation (the diffusion coefficient in velocity space).

Returning to the Langevin equation (2.14), we can now determine the asymptotic form of the random force spectrum:

$$\mathcal{F}_{ll}\,(\omega) = 2\,(\lambda^2 + \omega^2)\,E\,(\omega) = \varepsilon_*/\pi \qquad (\omega\,\tau_0 \ll 1), \quad (2.24)$$

which corresponds to the classical "white noise." If we require that the mean square time derivative of velocity exist for all orders, the random force spectrum, like the energy spectrum, must decay in the region $\omega\tau_0 \gg 1$ more rapidly than any power of the frequency.

4)In actuality it is sufficient to require a Markovian velocity, since the statistical characteristics of the coordinates can be obtained from the equation $x_i(t) = x_{i0} + \int_0^t v_i(s)ds$.

From (2.12) and (2.14) we obtain the following expression for the correlation tensor of random forces:

$$F_{ik}(t) = \frac{\varepsilon_*}{3\tau_0}\delta_{ik}\Big[m - m\varphi^2\Big(\frac{t}{\tau_0}\Big) + \varphi'\Big(\frac{t}{\tau_0}\Big)\Big]\exp\Big\{-m\int_0^{t/\tau_0}\varphi(x)\,dx\Big\}, \qquad (2.25)$$

where the derivative with respect to the dimensionless argument is denoted by a prime. The properties of $\varphi(x)$ show that the random forces are essentially local in time and have the correlation time scale τ_0. The random forces thus differ from the accelerations, whose correlation also vanishes at $t \sim \tau_0$. On the other hand, in the interval $\tau_0 \ll t \ll \lambda^{-1}$ we have negative correlation, with

$$\int_0^\infty W_{ll}(t)\,dt = -\frac{dR_{ll}(t)}{dt}\Big|_0^\infty = 0, \qquad (2.26)$$

where $W_{ik}(t)$ is the acceleration correlation tensor. For the contraction of the random force correlation tensor we have, using (2.24),

$$\int_0^\infty F_{ll}(t)\,dt = \varepsilon_* \qquad (2.27)$$

3. RELATIVE MOTION OF FLUID PARTICLES

Combining two equations of a system of Langevin equations for different fluid particles, we obtain an analogous equation for the relative motion of two selected fluid particles:

$$d\delta v_i(t)/dt = -\lambda\delta v_i(t) + \delta f_i(t), \qquad (3.1)$$

where $\delta v_i(t)$ is the relative velocity of the particles, and $\delta f_i(t)$ is the difference between the forces acting on these particles. We have

$$\delta v_i(t) = \delta v_{i0}e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\delta f_i(s)\,ds, \qquad (3.2)$$

$$r_i(t) = r_{i0} + \int_0^t \delta v_i(s)\,ds. \qquad (3.3)$$

Here $r_i(t)$ is the distance between the particles; the zero subscripts denote initial values.

Let the initial distance between the particles belong to the so-called inertial interval of distances:

$$L \gg r_0 \gg l_0 \equiv \nu^{3/4}\varepsilon^{-1/4},$$

where l_0 is the internal spatial scale.[1] In this case we can determine the correlation of initial relative velocities using the $2/3$-law:[1]

$$\langle\delta v_{i0}\,\delta v_{k0}\rangle = A^2[\delta_{ik} - {}^1/_4\, r_0^{-2} r_{i0} r_{k0}](\varepsilon r_0)^{2/3}, \qquad (3.4)$$

where A^2 is a structural constant of the order of unity and the coefficient $1/4$ follows from the incompressibility condition.

We assume that the characteristic spatial scale of the mutual correlation of random forces acting on two different fluid particles is of the order l_0, i.e., that our random forces are localized in both time and space. Since $r_0 \geq l_0$ and the average separation of particles will subsequently increase, the random forces acting on the particles can be regarded as uncorrelated from the very beginning. The correlation tensor for $\delta f_i(t)$ is then equal to twice the correlation tensor (2.25).

We select two successive instants: $t_2 \geq t_1 \gg \tau_0$. It is easily seen that in calculating the correlations of relative velocities and distances between the particles for these times we can neglect the correlations between the random-force difference and the initial relative velocity; we then use (2.27) and the following expression for the correlation tensor of random-force differences:

$$\langle\delta f_i(t+s)\,\delta f_k(s)\rangle = {}^4/_3\,\varepsilon_*\delta_{ik}\,\delta(t), \qquad (3.5)$$

where $\delta(t)$ is the Dirac delta function.

From (3.2), (3.3), and (3.5) we obtain the following expressions for the correlations of relative velocities and distances between the particles:

$$\langle\delta v_i(t_1)\,\delta v_k(t_2)\rangle = \langle\delta v_{i0}\delta v_{k0}\rangle e^{-\lambda(t_1+t_2)} + {}^4/_3\,\varepsilon_*\lambda^{-1}\delta_{ik}e^{-\lambda t_2}\,\mathrm{sh}\,\lambda t_1, \qquad (3.6)^*$$

$$\langle r_i(t_1)\,r_k(t_2)\rangle = r_{i0}r_{k0} + \lambda^{-2}\langle\delta v_{i0}\delta v_{k0}\rangle(1-e^{-\lambda t_1})(1-e^{-\lambda t_2}) + {}^4/_3\varepsilon_*\lambda^{-3}\delta_{ik}[\lambda t_1 - (1-e^{-\lambda t_1}) - (\mathrm{ch}\,\lambda t_1 - 1)e^{-\lambda t_2}] \quad (t_2 \gg t_1 \gg \tau_0). \qquad (3.7)^\dagger$$

We shall now consider certain special cases. Putting $t_1 = t_2 \equiv t \ll \lambda^{-1}$ and equating individual terms in (3.6), (3.7), and (3.4), it is easily shown that the effect of the initial conditions is unimportant for times $t \gg r_0^{2/3}\epsilon^{-1/3}$. For such times we have

$$\langle[\delta v_l(t)]^2\rangle = (2\varepsilon_*/\lambda)(1-e^{-2\lambda t}), \qquad (3.8)$$

$$\langle r_l^2(t)\rangle = (2\varepsilon_*/\lambda^3)(2\lambda t - 3 - e^{-2\lambda t} + 4e^{-\lambda t}), \qquad (3.9)$$

$$K(t) = \frac{1}{2}\frac{d}{dt}\langle r_l^2(t)\rangle = \frac{2\varepsilon_*}{\lambda}(1 + e^{-2\lambda t} - 2e^{-\lambda t}), \qquad (3.10)$$

where $K(t)$ is the dispersion coefficient

Equations (3.8)—(3.10) can also be regarded as the parametric forms of the functions representing

*sh = sinh.

†ch = cosh.

the dependence of the velocity structural function and the dispersion coefficient on the separation of the particles. Specifically, for $\lambda t \ll 1$ (i.e., for $r \ll L$) we obtain from (3.8) and (3.9) the Kolmogorov-Obukhov $\frac{2}{3}$-law

$$\langle (\delta v_l)^2 \rangle = 2(6)^{1/3}(\varepsilon_* r)^{2/3}, \qquad (3.11)$$

while from (3.9) and (3.10) we obtain the Richardson $\frac{4}{3}$-law:[7]

$$K(r) = 2(3/4)^{2/3}\varepsilon_*^{1/3} r^{4/3}. \qquad (3.12)$$

With regard to the last equation it must be noted that Richardson determined the exponent $\frac{4}{3}$ empirically. On the similarity hypothesis the exponent $\frac{4}{3}$ is obtained from dimensional considerations if it is assumed that ϵ is the only parameter characterizing turbulent flow in the inertial interval of scales.[2] An equation similar to (3.12) has also been derived by Lin.[8] However, Lin's equation contains an additional constant that is not related to the velocity structural constant of a single selected fluid particle. In this sense Lin's equation supplies no new information compared with Richardson's law as derived by dimensional reasoning.

We note another special case of (3.6) and (3.7). For two different instants in the interval $r_0^{2/3}\epsilon^{-1/3} \ll t_1 \ll t_2 \ll \lambda^{-1}$, when the initial separation has been "forgotten," but the particles are still separated by a distance much smaller than L, we obtain

$$\langle \delta v_l(t_1)\, \delta v_l(t_2) \rangle = 4\varepsilon_* t_1, \qquad (3.13)$$

$$\langle r_l(t_1)\, r_l(t_2) \rangle = \tfrac{2}{3}\varepsilon_* t_1^2 (3t_2 - t_1). \qquad (3.14)$$

Equation (3.14) is perhaps most suitable for experimental checking. For example, one could observe the relative motion of small spheres suspended in a turbulent flow.

We note that if we consider the variability of dissipation, which was first pointed out by Landau[9] and which has been discussed by Obukhov[10] and by Kolmogorov,[11] all expressions for the correlation functions must also be averaged with respect to ϵ, or in our case with respect to ϵ_*. The corresponding averaging of the Eulerian second moments, in which ϵ has the power $\frac{2}{3}$, results in the appearance of a correction factor depending on the external scale of turbulent flow;[10,11] additional considerations are needed to determine this dependence. The Lagrangian second moments (2.22), (3.13), and (3.14) depend on ϵ_* linearly; therefore the secondary averaging does not change these but results simply in the replacement of ϵ_* by its mean value. This constitutes a decided advantage of the Lagrangian description of turbulence.

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