REGGE POLES IN THE SCATTERING FROM A δ POTENTIAL

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Submitted to JETP editor January 24, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 246-250 (August, 1963)

The Regge trajectories for scattering from a δ potential are investigated. Many of the results apply also to potentials without singularities at the origin.

1. INTRODUCTION

IN the present paper we investigate the motion of Regge poles [1] for the potential $U(r) = q\,\delta(r-a)$, where $q > 0$. In contrast to the case of the potential in the deuteron model (which is obtained in the limit $a \to 0$, $2qa \to 1$), the scattering matrix S_l is an analytic function of l for $a \neq 0$. In general, the nonanalyticity of the potential, with the exception of the neighborhood of $r = 0$, has evidently little effect on the analytic properties of the S matrix, and the qualitative picture of the motion of the poles in the l plane depends for large energies only on the behavior of the potential in the region of small r.

The simplicity of the δ potential allows us to study such details of the motion of the poles as the coincidence recession into the complex plane, etc. As will be clear from the following, bound states exist only for orbital momenta $l < qa + 1/2$ and are described by a single Regge trajectory.

2. EQUATION FOR THE POLE

The solution of the radial Schrödinger equation has the form:

$$\chi_l = \sqrt{r}\,[c_1 H_\nu^{(1)}(kr) + c_2 H_\nu^{(2)}(kr)], \tag{1}$$

where $c_1 = c_2 = 1$ for $r < a$ and $c_{1,2} = 1 \pm i\pi\nu_0 J_\nu(ka) H_\nu^{(2,1)}(ka)$ for $r > a$.

For the scattering matrix we obtain then

$$S_l = e^{2i\delta_l} = \frac{1 + i\pi\nu_0 J_\nu(x) H_\nu^{(2)}(x)}{1 - i\pi\nu_0 J_\nu(x) H_\nu^{(1)}(x)}. \tag{2}$$

Here J_ν and H_ν are the Bessel and Hankel functions, $\nu = l + 1/2$, $x = ka$, and $\nu_0 = qa$. The poles of the S matrix are determined by the zeros of the denominator:

$$1 - i\pi\nu_0 J_\nu(x) H_\nu^{(1)}(x) = 0. \tag{3}$$

In the analysis of Eq. (3) we shall use for large values of x or ν the known asymptotic representations of the Bessel functions which are obtained by the saddle point method [2] and correspond to the quasiclassical approximation. We introduce the complex variable $\tau = \cosh^{-1}(\nu/x)$. Then the equation for the pole for $|\arg x| < \pi/2$ has the following form:

$$\text{I.}\quad \frac{x}{\nu_0}\operatorname{sh}\tau + 1 = ie^{2x\varphi(\tau)},$$

$$\text{II.}\quad \frac{x}{\nu_0}\operatorname{sh}\tau + 1 = i(1 - e^{2i\pi\nu})\,e^{2x\varphi(\tau)},$$

$$\text{III.}\quad \frac{x}{\nu_0}\operatorname{sh}\tau - 1 = ie^{2x\varphi(\tau)}$$

$$\text{IV.}\quad \frac{x}{\nu_0}\operatorname{sh}\tau + 1 = ie^{-2x\varphi(i\pi-\tau)}, \tag{4}*$$

where $\varphi(\tau) = \sinh\tau - \tau\cosh\tau$, and the regions I to IV are shown in Fig. 1 (unmarked regions are of no interest). The condition for the validity of these equations is $|x\sinh\tau| \gg 1$.

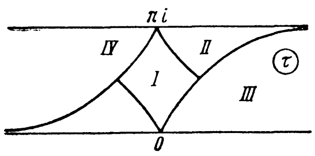

FIG. 1

3. SMALL AND MODERATE ENERGIES

For small energies we have a group of poles located near the real axis and another group near the imaginary axis. In the right half-plane there is one simple pole near the real axis corresponding to a bound state. In the region of small energies, positive as well as negative, the equation for this pole is

$$\nu = \nu_0 + \frac{\nu_0}{2(\nu_0^2-1)}x^2 + \ldots - \frac{\pi\exp(-i\pi\nu_0)}{\sin\pi\nu_0\,\Gamma(\nu_0)^2}\left(\frac{x}{2}\right)^{2\nu_0} + \ldots;$$

$$x^2 \ll \nu_0 + 1. \tag{5}$$

*sh = sinh.

This formula is obtained with the help of the power expansions of J_ν and $H_\nu^{(1)}$; the second and omitted terms correspond to the subtractions c_n in [3] and the last term describes the recession of the pole into the complex plane for positive energies.

For negative energies (x = iy) this pole moves along the real axis to the left according to the law

$$\nu = \sqrt{\nu_0^2 - y^2}. \tag{6}$$

Formula (6) has been obtained in the quasiclassical approximation ($e^{-\nu_0} \ll 1$). It is seen from this formula that the pole recedes into the complex plane for $y \approx \nu_0$ and ν close to zero. The energy of the bound states can be obtained from (6):

$$E_l = -[\nu_0^2 - (l + 1/2)^2]/2a^2,$$

and the number of levels is equal to the integer part of $\nu_0 + 1/2$.

In the left half-plane we have an infinite set of poles at small energies which are located near the negative integers:[1]

$$\nu_m = -m + \frac{\nu_0 - m}{m!\,(m-1)!\,\nu_0}\left(\frac{x}{2}\right)^{2m} - \frac{2(\nu_0 - m - 1)}{(m-1)!^2 (m^2-1)\nu_0}\left(\frac{x}{2}\right)^{2m+2} + \ldots + \frac{2\ln(2/x) + i\pi}{(m-1)!^4 \nu_0^2}\left(\frac{x}{2}\right)^{4m} + \ldots \quad (x^2 \ll m). \tag{7}$$

In the region of negative energies the even poles move first to the right and the odd poles, to the left, as long as $m < \nu_0$. The direction of the motion changes for $m > \nu_0$. The further motion of these poles along the real axis, the coincidence and the recession into the complex plane can be followed with the help of an equation which is obtained in the same approximation as Eq. (4) (more precisely, the condition for its validity is $\exp|\sqrt{\nu^2 + y^2}| \gg 1$):

$$\sqrt{\nu^2 + y^2}/\nu_0 - 1 = -2 \sin \pi\nu \, e^{f(y,\nu)} \qquad \operatorname{Re} \nu < 0;$$

$$f(y, \nu) = 2\nu \operatorname{Arsh}(\nu/y) - 2\sqrt{\nu^2 + y^2}. \tag{8*}$$

The function $f(y, \nu)$ changes sign at $y = -0.66\,\nu$. As long as $y < -0.66\,\nu$ the condition $\exp[f(y, \nu)] \gg 1$ is satisfied and the poles are located near the negative integers. In the opposite case, $\exp[f(y, \nu)] \ll 1$, Eq. (8) has no real solutions. Thus the recession into the complex plane occurs at $y \approx -0.66\,\nu$.

The exact as well as the approximate equations for the pole have, for $E < 0$, either real or complex conjugate roots. Therefore, the poles leave the real axis as a result of coincidence.[4]

In connection with the analysis of the motion of the poles we note that Eq. (3) has the following important property: it does not change under the interchange $\nu \to -\nu$ for integer ν.[2] Since there is a single pole for $\nu > 0$ which passes once through the integers from ν_0 to zero, not a single pole can pass through the integer points to the left of $-\nu_0$, and the integer points to the right of $-\nu_0$ can also be passed through only once. At the point of coincidence $\sin \pi\nu$ is negative if $\nu < -0.83 - \nu_0$ and positive if $-0.83\,\nu_0 < \nu < 0$. Therefore, the poles located between $-\nu_0$ and $-0.83\nu_0$ must turn around on the real axis and pass again through the point at which they were located at E = 0 before meeting again with their neighbors. The pole with the number equal to the closest odd integer to $0.83\,\nu_0$ will always remain unpaired and will, as y increases, move to the right along the real axis up to a point near $\nu = 0$, where it meets with the pole (6) coming from the right (see Fig. 2). The trajectory of these two poles after coincidence is given by the formula

$$\nu = -y^2 \frac{\operatorname{sh} \pi\sqrt{y^2 - \nu_0^2}}{\sqrt{y^2 - \nu_0^2}} e^{-2y} + i\sqrt{y^2 - \nu_0^2}, \tag{9}$$

which is valid for $|\nu|^2 \ll \nu_0$.

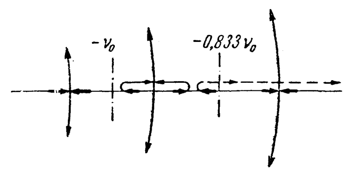

FIG. 2

According to (7) the poles recede for positive energies from negative integer points into the upper half-plane and move first to the right for $m > \nu_0$, and to the left for $m < \nu_0$. For large x these poles are determined by (4, II). The analysis of this equation shows that, beginning with $x \gtrsim m$, they all move to the right.

Let us now turn to the analysis of the other group of poles located near the imaginary axis for small energies. These poles were investigated by Gribov and Pomeranchuk[5] for an arbitrary short-range potential (threshold poles). For $|\nu| \ll 1$ and $x, y \ll 1$, their formula is valid with the parameters

[1]We note that the residues of the S matrix vanish for E = 0.
*Arsh = $\sinh^{-1}$.

[2]This property also holds in the case of an arbitrary potential, as can be seen from Eqs. (7) and (14) of the lectures of Gribov.[3]

$$\gamma = \frac{1}{2\nu_0^2}, \qquad \tau = \ln\frac{y^2}{4} + \frac{1}{\nu_0} + 2C, \tag{10}$$

where C is the Euler constant.

The far poles are described by the formula

$$\nu = \frac{i\pi k}{\Lambda} - \frac{1}{2\Lambda}\ln\left(\frac{2\pi|k|}{\nu_0\Lambda}\right),$$

$$\Lambda = \ln\frac{2\pi|k|}{y}, \qquad k = \pm 1, \ldots, \tag{11}$$

which is valid if the logarithms of the arguments are much larger than unity. This formula applies also to the region $E > 0$.

For positive energies the poles are located nearly symmetrically about $\nu = 0$, their trajectories are parabolas, and the front of the poles is close to a straight line slightly inclined toward the imaginary axis.

4. HIGH ENERGIES

Let us consider first the behavior of the pole (6) for large positive energies. Its trajectory lies entirely in the region III of Fig. 1, and we have, according to (4, III), for $x \ll \nu_0^{3/2}$

$$\nu = \sqrt{\nu_0^2 + x^2} + \frac{i\nu_0^2}{\sqrt{\nu_0^2 + x^2}}\exp\left\{-2\sqrt{\nu_0^2 + x^2}\,\mathrm{Arsh}\frac{\nu_0}{x} + 2\nu_0\right\}. \tag{12}$$

For larger values of x we can simplify (4, III), using the smallness of τ (as we shall see, $|\tau| \sim x^{-1/3}$). The equation then takes the form

$$x\tau/\nu_0 = ie^{-2/3 x\tau^3}. \tag{13}$$

This equation has the solution

$$\tau = (\lambda_k/x)^{1/3}e^{i\pi/3}, \tag{14}$$

where

$$\lambda_k = \ln(x/\nu_0^{3/2}) + 3i\pi\left(k - \tfrac{1}{12}\right) + \tfrac{1}{2}\ln\lambda_k, \qquad k = 0, \pm 1 \ldots$$

and $|\lambda_k| \ll x$.

The pole under consideration corresponds to $k = 0$. The poles corresponding to other values of k will be discussed below. Since $x \gg \nu_0^{3/2}$, the last term in the equation for λ_k is always small in comparison with the first, and it suffices for the determination of λ_k to iterate the equation once. Recalling that $\nu/x = \cosh\tau$, we obtain finally

$$\nu = x + \tfrac{1}{2}x^{1/3}e^{2i\pi/3}\lambda_k^{2/3}. \tag{15}$$

Let us now turn to the poles located at negative integer points for $E = 0$. Beginning with $x \gtrsim m$, they all move to the right. Near the imaginary axis the trajectory of these poles is given by the formula

$$\nu = \frac{2}{\pi}(x - m\pi) + \frac{1}{\pi}\,\mathrm{arc\,tan}^{-1}\frac{x}{\nu_0} + \frac{i}{2\pi}\ln(1 + x^2/\nu_0^2),$$

$$m = 1, 2, \ldots,$$

which is valid for $(x-m\pi)^2 \ll m\pi$. It is seen from this that the pole with the number m intersects the imaginary axis at $x \sim m\pi$. In the asymptotic region (large x) these poles move according to the law (15) within the region I of Fig. 1. But the equations in regions I and III differ only by the sign in front of unity, which is unimportant in the asymptotic region. Therefore, formula (15) applies to both of them, where $k = m$ runs through the values 1, 2,... The front of these poles makes an angle of 30° with the real axis for $3\pi m \ll \ln(x\nu_0^{-3/2})$ and becomes more and more inclined as m increases.

The poles corresponding to the values $k = -1, -2, \ldots$ in (14) are situated in region III of Fig. 1. Therefore, (15) describes the asymptotic behavior of the trajectory of the threshold poles of the series (10), which were located in the upper half-plane for small energies. If $-3\pi k \ll \ln(x\nu_0^{-3/2})$ the front of these poles has an inclination of 30° with respect to the real axis. As $|k|$ increases the angle becomes larger and reaches 60° when $x \gg -3\pi k \gg \ln(x\nu_0^{-3/2})$. For $-3\pi k \gg x$, (15) becomes invalid, and the poles are described by (11), which implies that the front of the threshold poles with very large numbers has an inclination of 90°.

Finally, we must investigate the motion of those threshold poles which were located in the lower half-plane for small x. First these poles were in the region $\mathrm{Re}\,\nu < 0$, $\mathrm{Im}\,\nu < 0$ (this corresponds to region IV in Fig. 1). It follows from (3) that no pole ever crosses the real axis for positive energies. Therefore, these poles will always remain in the third quarter of the ν plane. It can be shown that they never leave region IV. Let us introduce the new variable $\tau' = i\pi - \tau$ and note that $|\tau'|$ decreases as x increases. Then we obtain the approximate equation

$$x\tau'/\nu_0 = i\exp\{^2/_3\, x\tau'^3\}, \tag{16}$$

which has the solution

$$\tau' = (\lambda_k'/x)^{1/3}, \qquad \nu = -x - {}^1/_2\, x^{1/3}\lambda_k'^{2/3},$$
$$\lambda_k' = \ln(x/\nu_0^{3/2}) + 3\pi i(k - {}^1/_4) + {}^1/_2\ln\lambda_k' \quad (k > 0) \tag{17}$$

It is seen from (17) that the front of the threshold poles is vertical for small k. For $x \gg 3\pi k \gg \ln(x\nu_0^{-3/2})$ the inclination approaches 60° and tends again to 90° as k is further increased.

For sufficiently large negative energies all poles are located in the left half-plane symmetrically with respect to the real axis. If $\mathrm{Im}\,\nu > 0$ the poles with numbers $3\pi|k| \ll y$ are described by (14) and (15) with $x \to iy$. In the case $3\pi|k| \gg y$ the corresponding formulas for small energies are valid.

In conclusion we note that the quantity ν_0 enters in (10), (15), and (17) only in the argument of the logarithm, which implies that our results are evidently valid for an arbitrary potential without singularities at the origin.[6]

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[3] V. N. Gribov, Lectures in Voprosy fiziki elementarnykh chastits (Problems of Elementary Particle Physics), Erevan, 1962.

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[6] Patashinskiĭ, Pokrovskiĭ, and Khalatnikov, Preprint, Siberian Div. Acad. Sci.

Translated by R. Lipperheide
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