THE HYPOTHESIS OF REGGE POLES IN QUANTUM FIELD THEORY AND THE THRESHOLD SINGULARITIES OF INELASTIC PROCESSES

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A study is made of the manner in which the hypothesis of Regge poles in quantum field theory accords with the threshold singularities of inelastic processes in the s channel (but not in the t channel). Some additional new consequences of the Regge-pole hypothesis, bearing on the asymptotic behavior of inelastic processes, are derived.

In recent papers[1] there has been intensive study of the hypothesis of Regge poles (hereafter called RPH) in quantum field theory, which is based on the rigorous results of Regge[2] in quantum mechanics. A specific feature of the case of quantum field theory, which has no analog in the case of quantum mechanics, is that the region[1)] $t \le 0$, $s \ge 4\mu^2$, for which the basic results of the RPH are derived (for example, the asymptotic behavior for $s \to \infty$[1]), is, according to crossing symmetry,[4] the physical region of the s channel, and consequently in this region the scattering amplitude $A^{\pm}(s, t)$ must have the appropriate threshold behavior caused by inelastic processes in the s channel.

We shall here study precisely this specific feature of the RPH in quantum field theory, which is due to crossing symmetry. Some consequences of this feature have been considered earlier.[5]

1. According to the RPH the invariant amplitude $A^{\pm}(s, t)$ for the scattering of spinless particles can be represented in the form[1,3] [2)]

$$A^{\pm}(s,t) = -\frac{\pi}{2}\sum_k \frac{2l_k(t)+1}{\sin \pi l_k(t)}\left(1+e^{i\pi l_k(t)}\right) g^{\pm}_{l_k(t)} P_{l_k(t)}\left(1+\frac{2s}{t-4\mu^2}\right) + \frac{i}{4}\int_{l_c-i\infty}^{l_c+i\infty} \frac{2l+1}{\sin \pi l}\left(1+e^{i\pi l}\right) f_l^{\pm}(t)\, P_l\left(1+\frac{2s}{t-4\mu^2}\right) dl, \quad (1)$$

where $l_k(t)$ are the Regge poles, assumed in accordance with the basic hypothesis to exist in the analytic continuation of the amplitudes $f_l^{\pm}(t)$ into the half-plane $\mathrm{Re}\, l > l_c < 0$, $g^{\pm}_{l_k(t)}$ are the residues of $f_l^{\pm}(t)$ at these poles, and l_c is determined by the position of the nonpole singularities (for example, essential singularities) of $f_l^{\pm}(t)$ that lie farthest to the right in the l plane. In writing Eq. (1) we have assumed for definiteness that $0 > l_c > -1$, but this assumption does not in any way limit the generality of the conclusions of this paper.

For $\mathrm{Re}\, l > l_H > l_c$, where l_H is some finite number[3)] determined by the asymptotic behavior of $A^{\pm}(s, t)$ in the variable $s \to \infty$, we have[3] the relation

$$f_l^{\pm}(t) = \frac{4}{\pi}\int_{s_0}^{\infty} A_1(s,t)\, Q_l\left(1+\frac{2s}{t-4\mu^2}\right)\frac{ds}{t-4\mu^2}, \quad (2)$$

where $A_1(s, t) \equiv \mathrm{Im}\, A^{+}(s, t)$, and $s_0 = 4\mu^2$ is the threshold of the elastic process in the s channel.

According to the "optical" theorem,[4] [4)]

$$\mathrm{Im}\, A^{+}(s,0) = A_1(s,0) = \sqrt{s^2-4\mu^2}\,\sigma(s) = \sum_{m=0}^{\infty} A_1^{(m)}(s,0) = \sqrt{s^2-4\mu^2}\sum_{m=0}^{\infty}\sigma_m(s), \quad (3)$$

$$A_1^{(m)}(s,0) \equiv \begin{cases} A_1^{(m)}(s,0) \geqslant 0, & s > s_m \\ 0, & s \leqslant s_m \end{cases}. \quad (4)$$

$A_1^{(m)}$ is the contribution to the imaginary part of $A^{+}(s, 0)$ [i.e., to the total cross section $\sigma(s)$] of all possible processes in the s channel; $A_1^{(0)}(s, 0)$ is the contribution of the elastic scattering process, $A_1^{(1)}(s, 0)$ is that of the first inelastic process,

[1)]We adhere throughout to the usual notation, as adopted, for example, in a paper by Gribov;[3] in particular, s and t are the usual Mandelstam variables.

[2)]We here assume everywhere that the sum over k is a finite sum, i.e., that the number of Regge poles in the region $l_c < \mathrm{Re}\, l < l_H$ is finite.

[3)]The existence of a finite l_H for all t[3] is a consequence of the Mandelstam representation.[4]

[4)]By this we are confining ourselves to forward scattering only (t = 0), but the argument can easily be extended to the general case $t \neq 0$.

and so on; $s_0 = 4\mu^2 < s_1 < s_2 < \ldots < s_m < \ldots$ are the thresholds of the corresponding inelastic processes. It is important to emphasize that in the indicated definitions it is not assumed at all that only a finite number of inelastic processes in the intermediate states are taken into account. The threshold behavior of $A_1^{(m)}(s, 0)$—i.e., its behavior in the neighborhood $s \sim s_m + 0$—depends on the details of the process in question, and in any case it is characterized by a discontinuity of the derivative $\partial^p A_1^{(m)}(s, 0)/\partial s^p$ of some finite order p.

It is natural to require that for $t \le 0$, $s \ge 4\mu^2$ the expression for $A^{\pm}(s, t)$, Eq. (1), written in accordance with the RPH, should correspond to the threshold features which we have indicated for the inelastic processes (4).

2. According to the separation (3), we represent $f_l^{\pm}(t)$, on the basis of Eq. (2), in the form

$$f_l^{\pm}(t) \equiv \sum_{m=0}^{\infty} f_l^{\pm(m)}(t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \int_{s_0}^{\infty} A_1^{(m)}(s, t)\, Q_l\left(1 + \frac{2s}{t - 4\mu^2}\right) \frac{ds}{t - 4\mu^2}. \quad (5)$$

Then, according to Eq. (1), we also have

$$A^{\pm}(s, t) \equiv \sum_{m=0}^{\infty} A^{\pm(m)}(s, t) = -\frac{\pi}{2} \sum_{m=0}^{\infty} \sum_k \frac{2 l_k^{(m)}(t) + 1}{\sin \pi l_k^{(m)}(t)} (1 + \exp[i\pi l_k^{(m)}(t)]) \times g^{\pm}_{l_k^{(m)}(t)} P_{l_k^{(m)}(t)}\left(1 + \frac{2s}{t - 4\mu^2}\right) + \frac{i}{4} \sum_{m=0}^{\infty} \int_{l_c - i\infty}^{l_c + i\infty} \frac{2l + 1}{\sin \pi l} (1 + e^{i\pi l}) f_l^{\pm(m)}(t) \times P_l\left(1 + \frac{2s}{t - 4\mu^2}\right) dl. \quad (6)$$

In writing Eqs. (5) and (6) we have assumed that the RPH is valid for each partial amplitude $f_l^{\pm(m)}(t)$ and we do not have to deal with "collective" Regge poles caused by an infinite number of the $f_l^{\pm(m)}(t)$. This a priori possible case will also be considered in the following paragraphs.

According to Eq. (6), the contribution to $A^{\pm(m)}(s, t)$ and, consequently, also to $A_1^{(m)}(s, 0)$ from the Regge poles is a continuous function of s for $s \ge 4\mu^2$, and consequently threshold singularities at $s = s_m$ can be caused only by the integral term. It is not hard to realize that for this it is necessary to impose certain conditions on the asymptotic behavior of $f_l^{\pm(m)}(t)$ on the lines $l = l_c + il_2$ for $|l_2| \to \infty$. It follows from Eq. (6) that for $s_m \to \infty$ this all in fact reduces to problems in the theory of Fourier integrals in the complex region of semifinite functions $[A_1^{(m)}(s, 0)]$, which have been studied in detail earlier in papers on the quantum theory of decay.[6] Here we confine ourselves to the remark that the RPH in quantum field theory must be supplemented with hypotheses about the asymptotic behavior of $f_l^{\pm(m)}(t)$ on the lines $l = l_c + il_2$ for $|l_1| \to \infty$, and shall not write out in detail, as can be done on the basis of [6], the necessary and sufficient conditions that the asymptotic behavior of $f_l^{\pm(m)}(t)$ must satisfy. We remark that if we were to rewrite the basic formula (1) without the use of the RPH, i.e., by choosing the line $l_c = l_H$, where l_H is determined in accordance with Eq. (2), then it would not be necessary to impose any additional conditions on $f_l^{\pm(m)}(t)$, since then, as can be shown, Im $A^+(s, 0)$ would automatically coincide with $A_1(s, 0)$.

3. As can be seen from Eq. (6), for $s \le s_m$ the quantity Im $A^{+(m)}(s, 0)$ is made up of a nonzero part caused by Regge poles and the imaginary part of the integral term. According to Eq. (4), however, Im $A^{+(m)}(s, 0) = A_1^{(m)}(s, 0) \equiv 0$ for $s \le s_m$, and consequently the imaginary part of the integral term, which is also not zero, must cancel the contribution to the imaginary part of $A^{+(m)}(s, 0)$ from the Regge poles. It is quite obvious that the contribution from some (a finite number) of the Regge poles cannot exactly cancel that from the other Regge poles, as Eq. (4) requires, for $s < s_m$ and $s_m \to \infty$. From this it follows directly that it is actually only the integral term that can cancel the contribution from the Regge poles. In turn we can conclude from this that when considered in the entire l plane $f_l^{\pm(m)}(t)$ cannot be a function with a finite number of pole singularities (Regge poles). This is an indirect argument in favor of the existence of essential singularities of $f_l^{\pm}(t)$, which have been found by Gribov and Pomeranchuk.[5]

It follows from Eq. (6) that for Eq. (4), i.e., for the threshold behavior of the inelastic processes, to hold, it is at any rate necessary that for $s = s_m$ the contribution to the imaginary part of $A^{+(m)}(s, 0)$ from the Regge pole l_k be equal in absolute value to the contribution to the imaginary part from the integral term. For $s_m \to \infty$, however, we have

$$P_{l_k^{(m)}(0)}\left(1 - \frac{s_m}{2\mu^2}\right) \approx 2^{l_k^{(m)}(0)} \frac{1}{\sqrt{\pi}} \frac{\Gamma(l_k^{(m)}(0) + 1/2)}{\Gamma(l_k^{(m)}(0) + 1)} \left(1 - \frac{s_m}{2\mu^2}\right)^{l_k^{(m)}(0)}, \quad (7)$$

$$\left|\frac{i}{4}\int\limits_{l_c-i\infty}^{l_c+i\infty}\frac{2l+1}{\sin\pi l}(1+e^{i\pi l})f_l^{\pm(m)}(0)\,P_l\left(1-\frac{s_m}{2\mu^2}\right)dl\right|\leqslant Bs_m^{l_c}, \quad (8)$$

where B is a bounded constant which does not depend on m. We may say that the existence of such a constant independent of m is a direct consequence of the boundedness of all the $f_l^{\pm}(t)$ on the line $l = l_c + il_2$ and the one-signed nature of all the $g_{l_k^{(m)}}$. Consequently, in order for Eq. (4) to be satisfied it is necessary than the following restriction on the residues $g_{l_k^{(m)}} \equiv g_{l_k^{(m)}}(s_m)$ hold for $s_m \to \infty$:

$$g_{l_k^{(m)}}(s_m) \leqslant Cs_m^{l_c - l_k^{(m)}}, \quad (9)$$

where C is a bounded constant which does not depend on m.

4. Besides the case considered above, we logically admit also the case of "collective" Regge poles, i.e., the case in which each $f_l^{\pm(m)}$ has no singularities, and in particular no Regge poles, to the right of a certain line $\mathrm{Re}\, l > l_c < l_H$, but the infinite set of $f_l^{\pm(m)}(t)$,

$$f_l^{\pm}(s_{(M)}) \equiv \sum_{m=M}^{\infty} f_l^{\pm(m)}, \quad M < \infty, \quad (10)$$

has a Regge pole $l_k(M)$. We shall show, however, that this logical possibility is actually forbidden. In fact, when we carry through arguments analogous to those of the preceding section (actually the proof requires rather refined arguments, which we omit here), we arrive at the requirement that the residue at a "collective" Regge pole decrease as we increase M, since then $s_m \to \infty$. By hypothesis, however, the residue at a "collective" pole must not change with increasing $M(s_m)$, owing to the very definition of a "collective" Regge pole.

This contradiction shows that in the framework of the assumptions we have made the hypothesis that a "collective" Regge pole exists is illegitimate. From this it follows that we can discover the Regge asymptotic behavior by studying any allowed inelastic channel of the reaction. This underlines once more the correctness of the assumption made in [7] that in the framework of an exact theory of the spectral method of quantum field theory all particles are elementary, since the Regge behavior by no means involves effects from all possible elementary particles. An important result in this connection is that of papers by Arbuzov and others[8] and by Levy,[9] who obtained the Regge behavior in the framework of ordinary models of quantum field theory with a fixed number of elementary particles.

5. Let us consider the conclusion usually drawn that according to the RPH the total elastic cross section goes asymptotically ($s \to \infty$) to zero, and that for $s \to \infty$ the total cross section is due to all of the inelastic processes.[1]

Actually, as can be seen without difficulty from Eqs. (5), (6), this does not follow from the RPH at all, because we can imagine a case free from internal contradictions in which $f_l^{+(0)}(t)$ has in its analytic continuation a Regge pole (vacuum pole) at $l_0 = 1$, and all the amplitudes $f_l^{+(m)}(t)$ ($m = 1, 2, \ldots$) have their singularities to the left of $l = 1$. The only rigorous conclusion from the RPH is that as $s \to \infty$ the diffraction peak in the neighborhood of $t \sim 0$ indeed becomes narrower, so that for $s \to \infty$ the derivative of the elastic differential cross section with respect to angle becomes infinite in the forward direction (because this conclusion does not depend on the "source" of the Regge pole, i.e., on which of the $f_l^{\pm(m)}$ have it as a pole), but the conclusion that therefore $\sigma_0(s \to \infty) = 0$ is an assumption in addition to the RPH. If we assume, as is usually done,[1] that this assumption is correct, then the results we have obtained yield new consequences of the RPH.

First, as follows from Sec. 4, within the framework of the assumptions that have been made the vacuum pole, like Regge poles in general, cannot be a "collective" pole. Consequently, the cross section of any of the possible inelastic processes (because we do not understand how in this case one can single out just one inelastic process) must asymptotically approach a constant different from zero.

Second, as follows from Eq. (9), the asymptotic cross sections $\sigma_m(s \to \infty)$ of inelastic processes must satisfy the following restriction

$$\sigma_m(s \to \infty) \leqslant Cs_m^{l_c - 1}. \quad (11)$$

According to the result found by Gribov and Pomeranchuk[5] an extremely important singularity is $l_c = -1$, and therefore

$$\sigma_m(s \to \infty) \leqslant Cs_m^{-2}. \quad (12)$$

As for the results of Azimov,[11] it may be that the Mandelstam representation is actually valid only for the part of the scattering amplitude that does not have in its intermediate states particles with large values of the spin, and then all conclusions about the asymptotic behavior are not to be taken literally, but play the role of natural boundary conditions of a given approximation (cf. a preprint by Mandelstam[1]). If, in accordance with[11], the real singularities farthest to the right lie to the right of $l_c = -1$, then, according to Eq. (11), they can in principle be detected in the study of the asymptotic behavior of inelastic processes.

6. We have presented above some new consequences of the hypothesis of Regge poles in quantum field theory, which in principle admit of experimental test, if one believes at all in the possibility of testing by experiment asymptotic (infinitely high energy) theoretical predictions.[10] In this connection it would be useful to develop arrangements for the following experiments:
a) a study of the asymptotic behavior of inelastic processes, for example the process $p + p \rightarrow p + p + n\pi$ along with the study of the elastic process $p + p \rightarrow p + p$, for an additional test of the RPH;
b) a study of the asymptotic behavior of various inelastic processes for the purpose of determining, by Eq. (11), the positions of the farthest-right real singularities of the $f_l^{\pm(m)}(t)$.

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