# NONLINEAR THEORY OF THE DRIFT INSTABILITY OF AN INHOMOGENEOUS PLASMA IN A MAGNETIC FIELD

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We consider a weakly turbulent plasma. The turbulence arises by virtue of the characteristic instability of an inhomogeneous plasma.[1] The stabilizing effect due to the oscillations is analyzed and it is shown that this characteristic instability is important only in a collision-dominated plasma. The nonlinear growth rate is also estimated. The kinetic wave equation[3,4] is used to compute the steady-state oscillation spectrum, which is then used to estimate the turbulent flow of the plasma across the confining magnetic field. The turbulence diffusion coefficient is found to be greater than the classical coefficient but small compared with the Bohm coefficient.

1. In recent years a great number of authors have investigated stability at the inhomogeneous boundary of a plasma contained by a magnetic field.[1] We present a brief review of the results of the linear theory of stability obtained in the collisionless approximation.

A plasma layer, whose density and temperature depend only on x, is confined by a uniform magnetic field $H_z$. The electric field is zero everywhere. In a plasma of this kind oscillations occur at a frequency $\omega \sim k_y(cT/eH)(\nabla n)/n$ (drift waves). If the plasma pressure satisfies the inequality $1 > \beta > m_e/m_i$, where $\beta = 8\pi nT/H^2$, the most dangerous perturbations are those with phase velocities in the range $u_i \ll \omega/k_z \lesssim v_A$ ($u_\alpha = \sqrt{T/m_\alpha}$, $v_A = H/\sqrt{4\pi n m_i}$); we consider oscillations of the form $\varphi(x)\exp\{ik_z z + ik_y y - i\omega t\}$. In this case the electric field of the perturbations may be regarded as a potential field: $\mathbf{E} = -\nabla\varphi$. The drift-wave instability is caused by resonance electrons which move along the lines of force with velocities close to the phase velocity of the wave $\omega/k_z$.

The effect can be demonstrated if the electrons are described by the kinetic equation in the drift approximation

$$\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} - c\frac{[\nabla\varphi\mathbf{H}]}{H^2}\nabla f + \frac{e}{m}\frac{\partial\varphi}{\partial z}\frac{\partial f}{\partial v_z} = 0. \qquad (1.1)*$$

The rate of change of the wave energy $\dot{W}_{\mathbf{k}}$ is equal to the work transferred from the resonance electrons to the field per unit time

$$\dot{W}_{\mathbf{k}} = \tfrac{1}{4}(j_{z\mathbf{k}}E^*_{z\mathbf{k}} + \text{c.c.}) = -\tfrac{1}{4}e\left(E^*_{z\mathbf{k}}\int v_z f_{\mathbf{k}}dv_z + \text{c.c.}\right).$$

Substituting in this expression $f_{\mathbf{k}}$, the oscillatory correction to the zeroth order function $f_0$ found from the solution of the linearized equation (1.1), we have

$$\dot{W}_{\mathbf{k}} = \frac{\pi e^2}{m}\omega_{\mathbf{k}}\varphi_{\mathbf{k}}^2\int\left(k_z\frac{\partial f_0}{\partial v_z} - \frac{k_y}{\omega_{He}}\frac{\partial f_0}{\partial x}\right)\delta(\omega_{\mathbf{k}} - k_z v_z)\,dv_z. \qquad (1.2)$$

We can now easily formulate the stability criterion for drift waves. The wave energy increases if the distribution function for the resonance particles satisfies the condition

$$k_z\frac{\partial f}{\partial v_z} - \frac{k_y}{\omega_{He}}\frac{\partial f}{\partial x} > 0.$$

For a Maxwellian velocity distribution the frequency of the oscillations $\omega$ ($k_y r_i \ll 1$, where $r_i$ is the ion Larmor radius) is $k_y(cT/eH)(\nabla n)/n$. When $k_y r_i \gg 1$ the frequency becomes $u_i(\nabla n)/n$ and the growth rate vanishes. For all values of $\mathbf{k}$ the energy of the drift wave is of order $e^2\varphi_{\mathbf{k}}^2/T$. Hence, as follows from Eq. (1.2), the growth rate $\gamma_{\mathbf{k}} = \dot{W}_{\mathbf{k}}/2W_{\mathbf{k}}$ varies from $\gamma_{\mathbf{k}} \sim \omega_{\mathbf{k}}\sqrt{m_e/m_i\beta}$ at $k_y r_i \ll 1$ to $\gamma_{\mathbf{k}} \sim \omega_{\mathbf{k}} \sim u_i(\nabla n)/n$ for $k_y r_i \sim \sqrt{m_i\beta/m_e}$.[1)]

*$[\nabla\varphi\mathbf{H}] = \nabla\varphi \times \mathbf{H}$.

1)As in earlier papers,[1] we neglect the diamagnetic drift, that is, we assume

$$\omega_{\mathbf{k}} \gg k_y\frac{cT}{eH}\frac{H'(x)}{H(x)} \sim k_y\frac{cT}{eH}\frac{\nabla n}{n}\beta.$$

Hence, when $k_y r_i \sim \sqrt{m_i\beta/m_e}$ we obtain a limitation on the ratio of plasma pressure to magnetic pressure

$$(m_e/m_i)^{1/3} > \beta > m_e/m_i.$$

In an unstable plasma the fluctuation level of the electric field can be much greater than the thermal background and should cause increased plasma flow across the confining magnetic field. In order to compute the flow it is necessary to take account of nonlinear terms in the original equations.

We first consider the feedback effect on the electron distribution function due to oscillations associated with the instability. This process has been studied earlier for the case of a uniform plasma.[2] We use the method developed there for solving the problem, applying it to Eq. (1.1). The electron distribution function is divided into a slowly varying (in time) part $f_0$ and a small rapidly varying part $\delta f$. This procedure is valid if $\gamma_k \ll \omega_k$. The function $\delta f$ is determined from the linearized equation (1.1):

$$\delta f = \sum_{\mathbf{k}} (f_{\mathbf{k}} \exp\{-i\omega_{\mathbf{k}} t + ik_y y + ik_z z\} + \text{c.c.}),$$
$$f_{\mathbf{k}}(x) = \frac{e}{m}\varphi_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}} - k_z v_z}\left\{k_z \frac{\partial f_0}{\partial v_z} - \frac{k_y}{\omega_{He}}\frac{\partial f_0}{\partial x}\right\}. \qquad (1.3)$$

The expression for $\delta f$ is substituted in Eq. (1.1) and an average is taken over the rapid oscillations. As a result we obtain an equation that describes the change in $f_0$:[5,7]

$$\frac{\partial f_0}{\partial t} = \sum_{\mathbf{k}} \left(\frac{1}{v_z}\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_{\mathbf{k}}\omega_{He}}\frac{\partial}{\partial x}\right) D_{\mathbf{k}} \left(\frac{1}{v_z}\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_{\mathbf{k}}\omega_{He}}\frac{\partial}{\partial x}\right) f_0,$$
$$D_{\mathbf{k}}(x, t) = \pi \frac{e^2}{m^2}\varphi_{\mathbf{k}}^2 \omega_{\mathbf{k}}^2 \delta(\omega_{\mathbf{k}} - k_z v_z). \qquad (1.4)$$

The expression in (1.4) is a diffusion-like equation in $(x, v_z)$ space. An approximate solution can be found on the basis of the following considerations. The coefficient $D_k$ depends on the wave vector $k$ and reaches a peak for some value $k = \bar{k}$; $\bar{k}_z \sim \omega/v_A$, $\bar{k}_y r_i \sim \sqrt{m_i\beta/m_e}$.

To estimate the rate of equalization of the "plateau" in $(x, v_z)$ space given by Eq. (1.4) (cf. Vedenov, Velikhov, Sagdeev [2]) we replace $D_k$ by $D_{\bar{k}}$. It is also convenient, in the simplified equation (1.4), to transform from the variables $x$, $v_z$ to the new variables

$$\eta = v_z^2/2u, \qquad \xi = v_z^2/2u + \omega_{\bar{k}}\omega_{He}x/\bar{k}_y u,$$

where $u$ is the velocity of the resonance electrons $u \sim \omega_{\bar{k}}/k_z \sim v_A$. In this case Eq. (1.4) becomes

$$\frac{\partial f_0(\xi, \eta, t)}{\partial t} = \frac{\partial}{\partial \eta} D_{\bar{k}}(\xi, \eta, t) \frac{\partial f_0}{\partial \eta},$$

$$D_{\bar{k}}(\xi, \eta, t) = \pi(e^2/m^2 v_A^2)\,\omega_{\bar{k}}\varphi_A^2. \qquad (1.4')$$

This equation describes the feedback effect of the oscillations on the distribution function of the resonance particles.

For the resonance particles the values of the variable $\eta$ lie in the range $0 < |\eta| < v_A$. In this range of $\eta$ a plateau is formed on the distribution $f(\eta)$ in a time of order $\tau \sim v_A^2/D_{\bar{k}}$ and the oscillation growth is terminated, as is evident from Eq. (1.2). During the course of this process the coordinate $x$ and velocity $v_z$ of the resonance electrons are related by the expression $v_z^2/2 + \omega_{\bar{k}}\omega_{He}x/k_y = \text{const}$ and the velocity increment of these electrons in a time $\tau$ is $\delta v_z \sim v_z \sim v_A$, while the displacement $\delta x$ is approximately

$$\delta x = \frac{\delta v_z}{v_z}\frac{\bar{k}_y v_z^2}{\omega_{\bar{k}}\omega_{He}} \sim \frac{n}{\nabla n}\frac{v_A}{u_e}. \qquad (1.5)$$

The mean displacement of the nonresonant particles is zero in the approximation used here.

Thus, the instability at the boundary of a plasma contained by a magnetic field inhibits itself rapidly and the change in the initial density in the time required for the plasma to reach the stable state is small.

We have shown that in a low-density plasma (i.e., when collisions can be neglected) the instability of the boundary does not lead to an appreciable loss of plasma across the magnetic field. However, if the plasma lifetime is large compared with an electron mean free time $\tau_e$, the process by which a plateau is established in $(x, v_z)$ space must compete with the relaxation of the electron distribution function to an unstable local Maxwellian velocity distribution $f_M$.

We now analyze the steady-state problem taking account of collisions, assuming that the density distribution $n(x)$ and the temperature distribution $T(x)$ are given; we estimate the average (over time) plasma flux $\langle nv_x \rangle$ caused by the instability in this state. For this purpose, we add a collision term to the right side of Eq. (1.4) or (1.4′):

$$\frac{\partial f_0}{\partial t} = \sum_k \left(\underline{\frac{1}{v_z}\frac{\partial}{\partial v_z}} - \frac{k_y}{\omega_k\omega_{He}}\frac{\partial}{\partial x}\right) D_{\mathbf{k}} \left(\underline{\frac{1}{v_z}\frac{\partial}{\partial v_z}} - \frac{k_y}{\omega_k\omega_{He}}\frac{\partial}{\partial x}\right) f_0$$
$$+ \nu_e u_e^2 \underline{\frac{\partial}{\partial v_z^2}} (f_0 - f_M). \qquad (1.6)$$

Comparing the underscored terms in Eq. (1.6) we find that collisions are important when $\nu_e \gtrsim \nu_e^*$, where $\nu^* \sim \omega_{\bar{k}}(u_e/u_A)^2 e^2 \varphi_k^2/T^2$. If $\nu_e \ll \nu^*$ we have shown above that the oscillations have a strong stabilizing effect. If $\nu_e \gg \nu^*$ the collisions maintain the unstable Maxwellian distribution. The qualitative dependence of the nonlinear growth rate $\gamma'_{\bar{k}}$ on $\nu_e$ is given by the formula[2]

$$\gamma'_{\mathbf{k}} \sim \gamma_{\bar{\mathbf{k}}}/(1+\nu^*/\nu_e), \qquad (1.7)$$

where $\gamma_{\bar{\mathbf{k}}}$ is the growth rate obtained from the linear theory for a Maxwellian electron distribution function. [To determine the exact functional relation $\gamma'_{\bar{\mathbf{k}}}(\nu_e)$ we must solve the system of equations (1.2) and (1.6).]

We have shown that in the problem considered a consequence of the instability is a stationary oscillation level; it is natural to ask what the ultimate fate of these oscillations is, and to what amplitude they can grow. It follows from the linear theory that the growth rate $\gamma_{\mathbf{k}}$ is a maximum for waves with phase velocities $\omega/k_z \sim v_A \gg u_i$. These waves are weakly absorbed by the ions since the number of resonance ions (for which $v_z \sim v_A$) is exponentially small. These waves cannot be directly radiated from the plasma in the form of electromagnetic waves since their phase velocity is small compared with the velocity of light. Hence one expects that the wave energy will grow to a level such that different modes can interact, causing a net flux of modes to the region of the spectrum characterized by smaller phase velocities $\omega/k_z$. The waves with phase velocities $\omega/k_z \lesssim u_i$ produced as a result of this process will be absorbed by the resonance ions. The steady-state energy spectrum of the oscillations $W_{\mathbf{k}}$ is computed in Section 2. When $r_i k_y \gtrsim 1$

$$W_{\mathbf{k}} \sim e^2\varphi_{\mathbf{k}}^2/T \sim m_i\gamma'_{\mathbf{k}}\omega_{\mathbf{k}}/k^2. \qquad (1.8)$$

The mean particle flux $\langle nv_x \rangle$ can be found by averaging (over time) the y component of the equation of motion for the electrons and ions

$$\langle nv_x\rangle_j = (c/H)\,\langle E_y\delta n_j\rangle. \qquad (1.9)$$

In this expression we must use $\delta n_j = \int \delta f_j dv$ found from the solution of the linearized kinetic equation. For the electrons $\delta n_e$ is determined by Eq. (1.3):

$$\langle nv_x\rangle_e = 2\pi\frac{ec}{mH}\sum_{\mathbf{k}} k_y n_{\mathbf{k}}\varphi_{\mathbf{k}}^2 \int\left(k_z\frac{\partial f_0}{\partial v_z} - \frac{k_y}{\omega_{He}}\frac{\partial f_0}{\partial x}\right)$$

$$\times\delta(\omega_{\mathbf{k}} - k_z v_z)\,dv_z \sim \frac{ec}{TH}\sum_{\mathbf{k}} k_y\varphi_{\mathbf{k}}^2\frac{\gamma'_{\mathbf{k}}}{\omega_{\mathbf{k}}}. \qquad (1.10)$$

In making this estimate we have used Eq. (1.2).

The relation in (1.10) has a simple meaning. The resonance electrons lose a certain momentum per unit time in exciting the oscillations:

$$\sum \dot{p}_{ky}: \qquad \dot{p}_{ky} \sim -\gamma'_{\mathbf{k}}\frac{W_{\mathbf{k}}}{\omega_{\mathbf{k}}}k_y = -\frac{\gamma'_{\mathbf{k}}}{\omega_{\mathbf{k}}}k_y\frac{e^2\varphi_{\mathbf{k}}^2}{T}.$$

This momentum flux is absorbed by the ions. It can be shown that a frictional force acts between the electrons and ions. By virtue of this force a plasma flux is produced $\langle nv_x\rangle = -(cn/eH)\Sigma\dot{p}_{ky}$. It follows from these considerations that $\langle nv_x\rangle$ is the same for both kinds of particle so that plasma neutrality is not violated.

The "diffusion coefficient" D(x, t) is determined from the relation $\langle nv_x\rangle = -D\nabla n$:

$$D \sim -\frac{ecn}{TH\nabla n}\sum_{\mathbf{k}} k_y\frac{\gamma'_{\mathbf{k}}}{\omega_{\mathbf{k}}}\varphi_{\mathbf{k}}^2. \qquad (1.11)$$

We now use Eq. (1.8) and eliminate $\varphi_{\mathbf{k}}^2$ from Eq. (1.11), thus obtaining

$$D \sim \frac{n}{\omega_{Hi}\nabla n}\sum_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}'^2}{k}. \qquad (1.12)$$

The nonlinear growth rate $\gamma'_{\mathbf{k}}$ can be obtained from Eq. (1.7):

$$\gamma'_{\mathbf{k}} = \gamma_{\mathbf{k}}\left[1 + \frac{\gamma'_{\mathbf{k}}}{\nu_e}\frac{m_i}{m_e}\beta\frac{\omega_{\mathbf{k}}^2 m_i}{k^2 T}\right]^{-1}. \qquad (1.13)$$

The quantity $\gamma_{\mathbf{k}}'^2/k$ is a maximum when $kr_i \sim \sqrt{m_i\beta/m_e}$. For these values of k, $\omega_{\mathbf{k}} \sim \gamma_{\mathbf{k}} \sim u_i(\nabla n)/n$. Hence, when $\nu_e > \omega_{Hi}(r_i n^{-1}\nabla n)^3$

$$D \sim \sqrt{\frac{m_e}{m_i\beta}}\,r_i\frac{\nabla n}{n}\frac{cT}{eH}. \qquad (1.14)$$

When $\nu_e < \omega_{Hi}(r_i n^{-1}\nabla n)^3$

$$D \sim \sqrt{m_e/m_i\beta}\,\nu_e(n^{-1}\nabla n)^{-2} \sim \nu_i\beta^{-1/2}(n^{-1}\nabla n)^{-2}. \qquad (1.15)$$

We note that this result can be obtained from Eq. (1.11) for any energy spectrum[2)] $\varphi_{\mathbf{k}}^2$ so long as $\nu_e > \omega_{\mathbf{k}}(m_i/m_e)\beta e^2\varphi_{\mathbf{k}}^2/T^2$. The calculated diffusion coefficient is greater than the classical diffusion coefficient $D_c \sim (m_e/m_i)r_i^2\nu_e$ (it is only for this condition that the estimates we have given apply) but is relatively small. Thus, it is small compared with the Bohm diffusion coefficient $D_B \sim cT/eH$.

2. In this section we consider in detail the nonlinear mode interaction.[3,4] We shall also obtain the mode spectrum $\omega(\mathbf{k})$.

We write the potential $\varphi$ as a sum of potentials for the electric fields of the individual modes with amplitudes $C_{\mathbf{k}}(t, x)$ that vary slowly in time and space:

$$\varphi = \sum_{\mathbf{k}} C_{\mathbf{k}}(t, x)\,\bar{\varphi}_{\mathbf{k}}\exp\{i(\mathbf{kr} - \omega_{\mathbf{k}}t)\}$$

$$+\sum_{\mathbf{k}} C_{\mathbf{k}_-}(t, x)\,\bar{\varphi}_{\mathbf{k}_-}\exp\{-i(\mathbf{kr} - \omega_{\mathbf{k}}t)\},$$

$$C_{\mathbf{k}_-} = C_{\mathbf{k}}^*, \qquad \bar{\varphi}_{\mathbf{k}_-} = \bar{\varphi}_{\mathbf{k}}^*. \qquad (2.1)$$

In the same way we write expressions for the oscillating corrections to the $f_j$:

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[2)]This has been pointed out to us by R. Z. Sagdeev.

$$\delta f_j = \sum_{\mathbf{k}} C_{\mathbf{k}}(t, x)\, \bar{f}_{\mathbf{k}j} \exp\{i(\mathbf{kr} - \omega_{\mathbf{k}} t)\} + \sum_{\mathbf{k}} C_{\mathbf{k}-}(t, x)\, \bar{f}_{\mathbf{k}j-} \exp\{-i(\mathbf{kr} - \omega_{\mathbf{k}} t)\}.$$

As in [5], we do not use the rigorous quasiclassical (WKB) characteristic functions for the linear stability problem:

$$\varphi_{\mathbf{k}} = \bar{\varphi}_{\mathbf{k}} C_{\mathbf{k}} \exp\left\{i \int_{x_1}^{x} k_x(x, \omega)\, dx + i k_y y + i k_z z - i\omega t\right\},$$

but approximate them by plane waves $\exp\{i\mathbf{k}\cdot\mathbf{r} - i\omega t\}$ assuming that $k_x \sim k_y$.

The nonlinear mode interaction leads to a slow change in mode amplitude in time by virtue of the transfer of energy through the spectrum. A correct description of this process can be given only when the mode coupling (and the wave-particle interaction) is weak so that perturbation theory can be used; the small parameter is the ratio of the energy of interaction between modes to the total mode energy.[3] As a consequence of the weak coupling between modes the mode phases can be assumed to be random and averages over phase can be taken.

We first obtain the dynamic equation describing the time variation of the amplitudes $C_{\mathbf{k}}(t)$. For this purpose we write the Fourier component of the rapidly varying part of the distribution function $\bar{f}_{\mathbf{k}j}$ in the form of an integral over the unperturbed particle trajectories, taking account of nonlinear terms:[6]

$$\begin{aligned} C_{\mathbf{k}} \bar{f}_{\mathbf{k}j}(r, \mathbf{v}, t) &= \frac{e_j}{m_j} i\mathbf{k} \int_{-\infty}^{t} \varphi_k(r, t) \frac{\partial f_{0j}(\mathbf{v}, x)}{\partial \mathbf{v}} dt \\ &- \frac{\partial C_{\mathbf{k}}}{\partial t} \int_{-\infty}^{t} \bar{f}_{\mathbf{k}j}(r, \mathbf{v}, t)\, dt \\ &- \frac{dC_{\mathbf{k}}}{dx} \left\{ \int_{-\infty}^{t} v_x(t) \bar{f}_{\mathbf{k}j}(r, \mathbf{v}, t)\, dt - \frac{e_j}{m_j} \int_{-\infty}^{t} \varphi_{\mathbf{k}} \frac{\partial f_{0j}}{\partial v_x} dt \right\} \\ &+ \frac{e_j}{m_j} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} i\mathbf{k}' \int_{-\infty}^{t} \varphi_{\mathbf{k}'} \frac{\partial \delta f_{\mathbf{k}''j}}{\partial \mathbf{v}} dt \end{aligned} \quad (2.2)$$

making use of the neutrality condition

$$\sum_j e_j \int \bar{f}_{\mathbf{k}j}(r, \mathbf{v}, t)\, d\mathbf{v} = 0. \quad (2.3)$$

In this section we assume that collisions are successful in re-establishing a local Maxwellian velocity distribution for the electrons and ions. From Eq. (2.2), in the linear approximation we obtain an expression that relates the distribution function $\bar{f}_{\mathbf{k}j}$ and the potential $\bar{\varphi}_{\mathbf{k}}$:

$$\begin{aligned} \bar{f}_{\mathbf{k}j} = \xi_{\mathbf{k}j}(\mathbf{v})\, \bar{\varphi}_{\mathbf{k}} &= -\frac{e_j}{T_j}\Big[1 - \sum_{l=-\infty}^{+\infty} \frac{\omega_{\mathbf{k}} + k_y c T_j n_0'/e_j H n_0}{-\omega_{\mathbf{k}} - l\omega_{Hj} + k_z v_z} \\ &\times J_l\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right) \exp\left\{il(\theta - \omega_{Hj} t + \varphi) - il\frac{\pi}{2}\right\} \\ &\times \exp\left\{-i \frac{[\mathbf{k}\mathbf{v}(t)]_z}{\omega_{Hj}}\right\}, \\ f_{0j} &= n_0 \left[\frac{m_j}{2\pi T_j}\right]^{3/2} \left(1 + \frac{v_y}{\omega_{Hj}} \frac{1}{n_0} \frac{dn_0}{dx}\right) \exp\left\{-\frac{m_j v^2}{2T_j}\right\}, \end{aligned} \quad (2.4)$$

where $k_\perp^2 = k_x^2 + k_y^2$, $\mathbf{v} = \{v_\perp, \varphi, v_z\}$.

In Eq. (2.4) we retain the term with $l = 0$ thus obtaining an expression for the growth rate from Eq. (2.3)

$$\begin{aligned} \gamma_{\mathbf{k}} &= \mathrm{Im} \sum_j e_j \int \xi_{\mathbf{k}j}\, dv \Big/ \sum_j e_j \int \frac{\partial \xi_{\mathbf{k}j}}{\partial \omega_{\mathbf{k}}} dv \\ &= -\sqrt{\frac{\pi}{2}} \frac{\omega_{\mathbf{k}}}{|k_z| u_l} \left(\omega_{\mathbf{k}} + k_y \frac{cT}{eH} \frac{\nabla n}{n}\right) \end{aligned} \quad (2.5)$$

and the oscillation frequency

$$\omega_{\mathbf{k}} = -k_y \frac{cT}{eH} \frac{\nabla n}{n} F(k_\perp^2)/[2 - F(k_\perp^2)],$$
$$F(k_\perp^2) = e^{-k_\perp^2 r_i^2} I_0(k_\perp^2 r_i^2), \quad (2.6)$$

where $I_0$ is a Bessel function of zero order and imaginary argument. The details of the calculation for the linear approximation can be found in [1].

We now substitute the first approximation for $\bar{f}_{\mathbf{k}j}$ in the right side of Eq. (2.2). Keeping only the main terms in the nonlinear part we obtain an equation for the correction to the distribution function that takes account of the modes interaction:

$$\begin{aligned} C_{\mathbf{k}} \bar{f}_{\mathbf{k}j} = \Big(\xi_{\mathbf{k}j} C_{\mathbf{k}} &- \frac{\partial \xi_{\mathbf{k}j}}{\partial i\omega_{\mathbf{k}}} \frac{\partial C_{\mathbf{k}}}{\partial t} + \frac{\partial \xi_{\mathbf{k}j}}{\partial i k_x}\left(\frac{\partial C_{\mathbf{k}}}{\partial x} + \frac{\partial C_{\mathbf{k}}}{\partial k_x} \frac{dk_x}{dx}\right) \\ &+ \frac{\partial \xi_k}{\partial i\omega_k} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} V^j_{\mathbf{k}\mathbf{k}'\mathbf{k}''}(\mathbf{v}, t)\, C_{\mathbf{k}'} C_{\mathbf{k}''}\Big) \bar{\varphi}_{\mathbf{k}}, \end{aligned} \quad (2.7)$$

where $\xi_{\mathbf{k}} = \int \sum_j \xi_{\mathbf{k}j} (e_j/|e|)\, dv$;

$$\begin{aligned} \frac{\partial \xi_{\mathbf{k}}}{\partial \omega_{\mathbf{k}}} V^j_{\mathbf{k}\mathbf{k}'\mathbf{k}''} &= 2 \frac{e_j^2}{m_j T_j} \frac{[\mathbf{k}'\cdot\mathbf{k}'']_z}{\omega_{Hj}} \\ &\times \left(\frac{\omega_{\mathbf{k}} - k_y''(cT_j/e_j Hn)\nabla n}{\omega_{\mathbf{k}''} - k_z'' v_z} - \frac{\omega_{\mathbf{k}'} - k_y'(cT_j/e_j Hn)\nabla n}{\omega_{\mathbf{k}'} - k_z' v_z}\right) \\ &\times \frac{J_0(kv_\perp/\omega_{Hj}) J_0(k'v_\perp/\omega_{Hj}) J_0(k''v_\perp/\omega_{Hj})}{w_{k'} + \omega_{k''} + (k_z' + k_z'') v_z} \frac{\varphi_{\mathbf{k}'}\varphi_{\mathbf{k}''}}{\varphi_{\mathbf{k}}} \\ &\times \exp\{-i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_{\mathbf{k}})t\}. \end{aligned}$$

The dynamic equation for the amplitudes $C_{\mathbf{k}}(t)$ is

$$\frac{\partial C_{\mathbf{k}}}{\partial t} = \gamma_{\mathbf{k}} C_{\mathbf{k}} - \frac{d\omega_{\mathbf{k}}}{dk_x}\frac{\partial C_{\mathbf{k}}}{\partial x} + \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}(t)\, C_{\mathbf{k}'}C_{\mathbf{k}''},$$

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}(t) = \sum_j \int V^j_{\mathbf{k}\mathbf{k}'\mathbf{k}''}(\mathbf{v}, t)\frac{e_j}{|e|}\,d\mathbf{v}. \qquad (2.8)$$

If we normalize the state vector $\overline{\varphi}_{\mathbf{k}}$, in accordance with the relation

$$\widetilde{W}_{\mathbf{k}} = \frac{\partial(\omega_{\mathbf{k}}\varepsilon(\omega_{\mathbf{k}},\mathbf{k}))}{\partial\omega_{\mathbf{k}}}\frac{k^2\overline{\varphi}^2_{\mathbf{k}}}{8\pi} \equiv \frac{e^2\overline{\varphi}^2_{\mathbf{k}}}{2T} n_0(2 - F_{\mathbf{k}}) = \omega_{\mathbf{k}}, \qquad (2.9)$$

where

$$\varepsilon = \sum_j \int \frac{4\pi e_j \xi_{\mathbf{k}j}}{k^2}\,dv$$

is the dielectric constant of the plasma, the square of the modulus of the mode amplitude can be treated as the number of modes $\eta_{\mathbf{k}} = |C_{\mathbf{k}}(t)|^2$ with energy $\omega_{\mathbf{k}}$. In this normalization the matrix element $V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$, that characterized the interaction possesses the necessary symmetry properties (cf [4]); the change in the number of modes in time is described by the kinetic equation

$$\frac{\partial n_{\mathbf{k}}}{\partial t} + \frac{d\omega_{\mathbf{k}}}{dk_x}\frac{\partial n_{\mathbf{k}}}{\partial x} = 2\gamma'_{\mathbf{k}} n_{\mathbf{k}} + \mathrm{St}^{(0)}\{n_{\mathbf{k}'}n_{\mathbf{k}''}\};$$

$$\mathrm{St}^{(0)}\{n_{\mathbf{k}'}n_{\mathbf{k}''}\} = 2\pi \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \{|V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}|^2 (n_{\mathbf{k}'}n_{\mathbf{k}''} - n_{\mathbf{k}}n_{\mathbf{k}'} - n_{\mathbf{k}}n_{\mathbf{k}''})$$
$$\times\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}''} - \omega_{\mathbf{k}'}) + 2|V_{\mathbf{k}\mathbf{k}'_-\mathbf{k}''}|^2$$
$$\times\,(n_{\mathbf{k}'}n_{\mathbf{k}''} + n_{\mathbf{k}''}n_{\mathbf{k}} - n_{\mathbf{k}}n_{\mathbf{k}'})\,\delta(\omega_{\mathbf{k}''} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}})\}. \qquad (2.10)$$

The collision term in this equation takes account only of the nonlinear transfer of energy over the spectrum of characteristic oscillations, i.e., the conversion of two modes into one (and vice versa) resulting from elastic collisions. However, interference between modes can lead to driven oscillations as well. The amplitude of a driven oscillation is given by

$$C^{(1)}_{\mathbf{k}''} = e^{\gamma_{\mathbf{k}''}t}\int_0^t V_{\mathbf{k}''\mathbf{k}'_-\mathbf{k}} C_{\mathbf{k}'}C_{\mathbf{k}}$$
$$\times \exp\{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\,t - \gamma_{\mathbf{k}''}t\}\,dt$$
$$= V_{\mathbf{k}\mathbf{k}'_-\mathbf{k}} C_{\mathbf{k}'_-} C_{\mathbf{k}} \frac{\exp\{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\,t\} - \exp(\gamma_{\mathbf{k}''}t)}{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}) + \gamma_{\mathbf{k}''}}.$$

The absorption of these driven oscillations gives an additional channel for the outflow of energy from a given fluctuation scale and these processes must be included in the total collision term:

$$\mathrm{St}\{n_{\mathbf{k}}n_{\mathbf{k}'}\} = \mathrm{St}^{(0)}\{n_{\mathbf{k}}n_{\mathbf{k}'}\} + 2V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}V_{\mathbf{k}''\mathbf{k}'_-\mathbf{k}}$$
$$\times \frac{|\gamma_{\mathbf{k}''}|\,n_{\mathbf{k}}n_{\mathbf{k}'}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}''} - \omega_{\mathbf{k}'})^2 + \gamma^2_{\mathbf{k}''}}$$
$$+ 2V_{\mathbf{k}\mathbf{k}'_-\mathbf{k}''}V_{\mathbf{k}''\mathbf{k}'\mathbf{k}}\frac{|\gamma_{\mathbf{k}''}|\,n_{\mathbf{k}}n_{\mathbf{k}'}}{(\omega_{\mathbf{k}''} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}})^2 + \gamma^2_{\mathbf{k}''}}. \qquad (2.11)$$

This equation and the quasilinear equation (1.4) form a complete system for determining the kinetics of a turbulent plasma in the case of weak mode coupling ($\gamma_{\mathbf{k}}$, $C^{-1}_{\mathbf{k}}\partial C_{\mathbf{k}}/\partial t \ll \omega_{\mathbf{k}}$). However, the largest contribution to the diffusion coefficient is that of the short-wave oscillations $kr_i \sim \sqrt{m_i\beta/m_e}$ for which $\gamma_{\mathbf{k}} \sim \omega_{\mathbf{k}}$. Equation (1.4) applies in this case since the condition of weak coupling between the waves and electrons $\gamma_{\mathbf{k}}/k_z u_e \ll 1$ is satisfied; Eq. (2.10) is only qualitatively correct, but can be used for an order-of-magnitude estimate of the wave energy. (This estimate will be obtained as a limiting case for oscillations characterized by $kr_i \ll \sqrt{m_i\beta/m_e}$.)

Under conditions of quasi-stationary equilibrium in which the conversion of energy into turbulent fluctuations caused by the instability is compensated by absorption and the loss of energy because of the nonlinear transfer through the spectrum

$$2\gamma'_{\mathbf{k}} n_{\mathbf{k}} \sim \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} n_{\mathbf{k}} n_{\mathbf{k}'} \frac{1}{\omega_{\mathbf{k}}}.$$

When $kr_i \gtrsim 1$, an estimate of the matrix element gives

$$|V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}|^2 \sim k^2\omega_{\mathbf{k}}/nm_i.$$

Hence

$$W_{\mathbf{k}} \sim n_{\mathbf{k}}\omega_{\mathbf{k}} \sim e^2 n n_{\mathbf{k}} |\overline{\varphi}_{\mathbf{k}}|^2/T \sim nm_i\gamma'_{\mathbf{k}}\omega_{\mathbf{k}}/k^2. \qquad (2.12)$$

In principle, Eq. (2.10) can be used to estimate the wave energy numerically. We obtain the following value for $W_{\mathbf{k}}$:

$$W_{\mathbf{k}} \sim \tfrac{1}{200} nm_i\gamma'_{\mathbf{k}}\omega_{\mathbf{k}}/k^2.$$

If this numerical estimate is used in place of the symbolic expression (1.14) for D we find

$$\nu_e > \tfrac{1}{10}\,\omega_{Hi}\,(r_i n^{-1}\nabla n)^3, \quad D \sim \tfrac{1}{100}\sqrt{m_e/m_i\beta}\,(r_i n^{-1}\nabla n)\,cT/eH.$$

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