OPEN RESONATORS WITH SPHERICAL MIRRORS

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Electromagnetic oscillations in an open resonator formed by two identical circular or rectangular spherical mirrors located in vacuum opposite each other are studied theoretically. It is shown that natural oscillations with very small radiative losses can occur in such a resonator. Each of the oscillations may be interpreted as consisting of a set of rays which are alternately reflected by the mirrors and which are restricted by a caustic surface. Simple formulas are obtained for the oscillation frequency and field distribution. The evolution of the natural oscillations with change from plane to concentric spherical mirror surfaces is followed. It is found that minimum radiative losses occur when the resonator consists of confocal mirrors (the curvature radius of the mirrors equals the distance between them).

INTRODUCTION

WE have previously [1] presented a theory of natural oscillations in open resonators with plane mirrors. Open resonators with spherical mirrors also are of practical importance. The resonators most studied in the literature (see [2-4]) have confocal spherical mirrors with a radius of curvature equal to the maximum distance between mirrors. However, the use of confocal mirrors is in many cases inefficient (for example, in lasers), owing to the excessively strong competition between different oscillation modes and the excessively low radial extent of the oscillations with the smallest radiation losses.

We investigate below the natural oscillations of an open resonator with spherical mirrors, for an arbitrary ratio between the radius of curvature of the mirrors and the distance between them. The wave field is considered in a curvilinear (spheroidal) reference frame corresponding to the resonator geometry, with the problem reduced approximately to integration of a parabolic equation. This approach is called in diffraction theory the "parabolic equation method" or the "transverse diffusion method," since the longitudinal diffusion of the wave amplitude is not taken into account in the parabolic equation. This yields the simplest and clearest asymptotic solutions for diffraction problems if the wavelength is sufficiently small compared with the characteristic dimensions of the system.

1. SPHEROIDAL COORDINATE SYSTEM

We introduce spheroidal coordinates ρ, φ, and ζ, related with the cylindrical coordinates r, φ, and z by the equations

$$r = d\,\mathrm{ch}\,\zeta \sin\rho, \qquad z = d\,\mathrm{sh}\,\zeta \cos\rho \qquad (1)^*$$

and having a range of variation

$$0 < \rho < \pi/2, \qquad 0 \leqslant \varphi \leqslant 2\pi, \qquad -\infty < \zeta < \infty, \qquad (2)$$

so that the surface $\zeta = \pm$ const is an oblate ellipsoid, for which the z axis is the axis of revolution and the distance between foci is 2d, while the surface ρ = const is a single-sheet hyperboloid of revolution, which approaches asymptotically as $\zeta \to \pm\infty$ a cone whose generators pass through the origin and make an angle ρ with the z axis.

We specify the reflecting mirror surfaces with the aid of the relations

$$0 < \rho < \bar{\rho}, \qquad 0 \leqslant \varphi \leqslant 2\pi, \qquad \zeta = \pm\bar{\zeta}. \qquad (3)$$

If the condition

$$\sin^2\bar{\rho} \ll \mathrm{ch}^2\bar{\zeta} \qquad (4)$$

is satisfied, then the radius of curvature of such mirrors can be regarded as constant and equal to

$$r_0 = d\,\mathrm{ch}^2\bar{\zeta} / \mathrm{sh}\,\bar{\zeta}, \qquad (5)$$

so that relations (3) specify in practice spherical mirrors of circular form (along with such mirrors, we shall consider in Sec. 3 also spherical mirrors

*ch = cosh, sh = sinh

of rectangular form). The phase relations for spherical mirrors of radius r_0 and for the spheroidal mirrors (3) practically coincide, subject to the additional condition

$$\tfrac{1}{8} kd \sin^4 \bar{\rho} \operatorname{ch} \bar{\zeta} \ll 1, \qquad (6)$$

which we also assumed to be satisfied.

The maximum distance between mirrors is $2l$, where

$$l = d \operatorname{sh} \bar{\zeta}, \qquad (7)$$

and the mirror diameter is $2a$, where

$$a = d \operatorname{ch} \bar{\zeta} \sin \bar{\rho}. \qquad (8)$$

If we specify the geometrical parameters r_0, l, and a, then for $r_0 > l$ we can find d, $\bar{\zeta}$, and $\bar{\rho}$ such that the open resonator is "inscribed" in the corresponding spheroidal system of coordinates, which makes it possible to investigate in simplest fashion the field distribution in the resonator.

The Lamé coefficients are equal to

$$h_\rho = h_\zeta = h = d\sqrt{\operatorname{ch}^2 \zeta - \sin^2 \rho}, \quad h_\varphi = d \operatorname{ch} \zeta \sin \rho, \qquad (9)$$

and therefore the wave equation is of the form

$$\frac{1}{\sin\rho}\frac{\partial}{\partial\rho}\left(\sin\rho\frac{\partial\Phi}{\partial\rho}\right) + \left(\frac{1}{\sin^2\rho} - \frac{1}{\operatorname{ch}^2\zeta}\right)\frac{\partial^2\Phi}{\partial\varphi^2}$$
$$+ \frac{1}{\operatorname{ch}\zeta}\frac{\partial}{\partial\zeta}\left(\operatorname{ch}\zeta\frac{\partial\Phi}{\partial\zeta}\right) + \gamma^2(\operatorname{ch}^2\zeta - \sin^2\rho)\Phi = 0. \qquad (10)$$

Here $k = \omega/c$ is the wave number in free space, corresponding to the complex circular frequency ω of one of the natural oscillations of the resonator (time dependence $e^{-i\omega t}$), $\gamma = kd$ is a parameter, and Φ is a scalar function, the connection of which with the electromagnetic field will be considered in Sec. 5. For the time being we seek for (10) an asymptotic solution satisfying the boundary condition $\Phi = 0$ on the mirrors (3) in the form

$$\Phi = W(\rho, \varphi, \zeta)\, e^{i\gamma \operatorname{sh}\zeta} - (-1)^q\, W(\rho, \varphi, -\zeta)\, e^{-i\gamma\operatorname{sh}\zeta}, \qquad (11)$$

i.e., in the form of two waves propagating in the directions $\pm\zeta$ and which go over into each other when ζ is replaced by $-\zeta$ (apart from the factor $-(-1)^q$, where q is an integer). The function W determines the complex amplitude of these waves and satisfies the equation

$$\frac{1}{\sin\rho}\frac{\partial}{\partial\rho}\left(\sin\rho\frac{\partial W}{\partial\rho}\right) + \left(\frac{1}{\sin^2\rho} - \frac{1}{\operatorname{ch}^2\zeta}\right)\frac{\partial^2 W}{\partial\varphi^2}$$
$$+ \frac{1}{\operatorname{ch}\zeta}\frac{\partial}{\partial\zeta}\left(\operatorname{ch}\zeta\frac{\partial W}{\partial\zeta}\right) + 2i\gamma\operatorname{ch}\zeta\frac{\partial W}{\partial\zeta}$$
$$+ (2i\gamma\operatorname{sh}\zeta - \gamma^2\sin^2\rho)W = 0, \qquad (12)$$

which we shall solve approximately under the following conditions

$$\gamma \gg 1, \qquad \sin^2\bar{\rho} \ll 1, \qquad (13)$$

which are compatible with the condition (6).

The first condition of (13) is satisfied so long as all the dimensions of the open resonator are large compared with the wavelength (see, incidentally, Sec. 4). The second condition amplifies condition (4) and signifies that when $0 < \rho < \bar{\rho}$ the ζ axis makes between the mirrors small angles with the z axis, so that the factors $\exp(\pm i\gamma \sinh \zeta)$ in formula (11) correspond to propagation along the z axis with velocity close to c. The first condition of (13) enables us to neglect in (12) the term $\cosh^{-1}\zeta\partial(\cosh\zeta\,\partial W/\partial\zeta)\partial\zeta$, since it is small compared with the terms that follow, while the second condition of (13) makes it possible to replace in (12) $\sin\rho$ by ρ and neglect the term $-\cosh^{-2}\zeta\,\partial^2 W/\partial\varphi^2$. As a result we obtain a parabolic equation of rather simple form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial\varphi^2}$$
$$+ 2i\gamma\operatorname{ch}\zeta\frac{\partial W}{\partial\zeta} + (2i\gamma\operatorname{sh}\zeta - \gamma^2\rho^2)W = 0, \qquad (14)$$

whereas Eqs. (9), (10), and (12) are elliptic.

Making the substitution

$$W = \frac{1}{\operatorname{ch}\zeta}\Psi(\tau, \varphi, \sigma),$$
$$\tau = \sqrt{2\gamma}\,\rho, \qquad \sigma = \int_0^\zeta \frac{d\zeta}{\operatorname{ch}\zeta} = \arcsin(\operatorname{th}\zeta), \qquad (15)^*$$

we transform (14) to a still simpler form

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi}{\partial\tau}\right) + \frac{1}{\tau^2}\frac{\partial^2\Psi}{\partial\varphi^2} + i\frac{\partial\Psi}{\partial\sigma} - \frac{\tau^2}{4}\Psi = 0. \qquad (16)$$

The last equation coincides formally with the Schrödinger temporal equation for an isotropic two-dimensional harmonic oscillator. It makes it possible to analyze the natural oscillations between spherical mirrors of circular form.

2. SPHERICAL MIRRORS OF CIRCULAR FORM

For mirrors of circular form, the function Ψ should satisfy the boundary condition

$$\Psi(\tau, \varphi, -\alpha) = e^{i(2kl - \pi q)}\,\Psi(\tau, \varphi, \alpha) \text{ (for } 0 < \tau < \bar{\tau}), \qquad (17)$$

which follows from the condition $\Phi = 0$ on the mirrors. The parameters α and $\bar{\tau}$ in (17) are defined as

$$\alpha = \arcsin(\operatorname{th}\bar{\zeta}) = \arcsin\sqrt{l/r_0},$$
$$\bar{\tau} = \sqrt{2\gamma}\,\bar{\rho} = \sqrt{ka^2 l^{-1}}\sin 2\alpha, \qquad (18)$$

*th = tanh

with

$$\sin\alpha = \sqrt{l/r_0},\quad \cos\alpha = \sqrt{1 - l/r_0};$$
$$\sin 2\alpha = 2\sqrt{lr_0^{-1}(1 - l/r_0)}. \qquad (19)$$

The condition that must be satisfied on the geometrical continuation of the lower mirror is

$$\Psi(\tau, \varphi, -\alpha) = 0 \quad (\text{for } \tau > \bar{\tau}), \qquad (20)$$

signifying the absence of a wave propagating upward and excited by the currents on the lower mirror (the latter, according to Sec. 5, are proportional to $\Psi(\tau, \varphi, -\alpha)$ and are not present on the geometrical continuation of the mirror).

The presence of two boundary conditions (17) and (20) is connected with the diffraction and the edges of the mirrors when $\tau = \bar{\tau}$. It is relatively easy to take account of diffraction (see Sec. 4) only in the case of plane mirrors ($\alpha = 0$, $r_0 = \infty$) and concentric mirrors ($\alpha = \pi/2$, $r_0 = l$), for which it is of decisive significance, since oscillations with low radiation losses can exist between such mirrors only because of the influence of the edges. In order to calculate the diffraction phenomena in the general case, it becomes necessary to solve numerically the integral equation for the current density of the mirrors (see [2,3]). It turns out, however, that for oscillations with rather low radiation losses because of the formation of an external caustic surface (see below), the calculation of these losses is not of great practical significance since they are known to be exceeded by the Joule and other losses. We therefore investigate first the oscillations in an open resonator, disregarding diffraction and condition (20), i.e., assuming $\bar{\tau} = \infty$, and only then (in Sec. 4) will we consider the results of diffraction at finite $\bar{\tau}$.

By virtue of the symmetry of the problem we put

$$\Psi = \Psi_m(\tau, \sigma)\cos m\varphi \quad (m = 0, 1, 2, \ldots)$$
$$\text{for}\quad \Psi = \Psi_m(\tau, \sigma)\sin m\varphi \quad (m = 1, 2, \ldots). \qquad (21)$$

The function Ψ_m satisfies the equation

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi_m}{\partial\tau}\right) + i\frac{\partial\Psi_m}{\partial\sigma} + \left(-\frac{\tau^2}{4} - \frac{m^2}{\tau^2}\right)\Psi_m = 0, \qquad (22)$$

a particular solution of which has the form

$$\Psi_m = e^{-i\varkappa\sigma}\psi(\tau), \qquad (23)$$

where κ is a constant and ψ is the solution of the ordinary differential equation

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\psi}{d\tau}\right) + \left(\varkappa - \frac{\tau^2}{4} - \frac{m^2}{\tau^2}\right)\psi = 0, \qquad (24)$$

a solution finite as $\tau \to 0$ and vanishing as $\tau \to \infty$. The first requirement follows from the fact that the field must not have any singularities on the z axis, and the second from the fact that the function (23) should yield the approximate solution of the problem at finite albeit large values of $\bar{\tau}$. Indeed, we now solve the problem without taking condition (20) into account, but if the function ψ is sufficiently small when $\tau > \bar{\tau}$, where the parameter $\bar{\tau}$ is sufficiently large, then condition (20) is essentially satisfied approximately, and increased accuracy is only tantamount to a small perturbation of the function (23).

It is easy to show that (24) admits of a solution satisfying the foregoing requirements only if

$$\varkappa = \varkappa_{m,n} = m + 2n + 1 \quad (n = 0, 1, 2, \ldots). \qquad (25)$$

This solution has the form

$$\psi = \psi_{m,n}(\tau) = \frac{\tau^m L_n^{(m)}(\tau^2/2)}{2^{m/2}\sqrt{n!\,(m+n)!}}e^{-\tau^2/4}, \qquad (26)$$

where $L_n^{(m)}(x)$ are Laguerre polynomials, defined by the following formulas (see [5])

$$L_n^{(m)}(x) = (-1)^n x^{-m} e^x d^n (x^{m+n} e^{-x})/dx^n,$$
$$L_0^{(m)}(x) = 1, \qquad L_1^{(m)}(x) = x - (m+1) \qquad (27)$$

and satisfying the orthonormalization relation

$$\int_0^\infty \psi_{m,n}(\tau)\,\psi_{m,n'}(\tau)\,\tau d\tau = \delta_{nn'}. \qquad (28)$$

Substituting (23) in (17) and taking account of (21) and (26), we find that the function (23) corresponds to a natural oscillation, the frequency of which is given by the formula

$$2kl = \pi q + 2(m + 2n + 1)\alpha, \qquad (29)$$

where q is a large integer (since we assume that $kl \gg 1$).

An idea of the behavior of the function $\psi_{m,n}(\tau)$ can be obtained directly from (24). We put

$$\psi(\tau) = \tilde{\psi}(\tau)/\sqrt{\tau}, \qquad (30)$$

and then we obtain for $\tilde{\psi}$ the differential equation

$$d^2\tilde{\psi}/d\tau^2 + [\varkappa - U_m(\tau)]\tilde{\psi} = 0,$$
$$U_m(\tau) = \tau^2/4 + (m^2 - 1/4)/\tau^2. \qquad (31)$$

The function $U_0(\tau)$ is shown for symmetric oscillations in Fig. 1. The horizontal lines correspond to $\kappa_{0,n}$; their intersections with the curve $U_0(\tau)$ determines the values of $\tau_{0,n}$ for which the difference $\kappa_{0,n} - U_0(\tau)$ reverses sign. Therefore the function $\psi_{0,n}$ oscillates in the interval $0 < \tau < \tau_{0,n}$ and decreases monotonically when $\tau > \tau_{0,n}$. Since τ is essentially a dimensionless radius vector on the surface ζ = const, the function $\zeta_{0,n}(\tau)$ can be schematically represented on the τ, φ plane

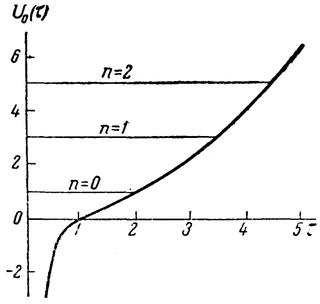

FIG. 1

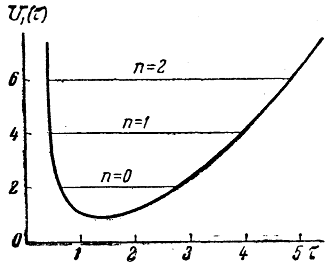

FIG. 2

in the form of a radial pencil of rays that experience total reflection from the circle $\tau = \tau_{0,n}$. At $\tau = 0$, these rays are focused. The field in the space between the mirrors can be represented in the form of rays that cross the z axis at a small angle and are then alternately reflected from each mirror. Owing to the concavity of the mirrors, the rays cannot penetrate beyond the hyperboloid $\tau = \tau_{0,n}$: this is the caustic surface, behind which the field decreases exponentially.

Figure 2 shows the function $U_1(\tau)$ and the values of $\kappa_{1,n}$ $(n = 0, 1, 2)$. Since $U_m(\tau) \to \infty$ as $\tau \to 0$ and $m = 1, 2, \ldots$, the equation $U_m(\tau) = \kappa_{m,n}$ has for $m > 0$ two roots: $\tau'_{m,n}$ and $\tau_{m,n} > \tau'_{m,n}$. The functions $\psi_{m,n}(\tau) \cos m\tau$ or $\psi_{m,n}(\tau) \sin m\tau$ correspond on the τ, φ plane to rays which fill the annular region $\tau'_{m,n} < \tau < \tau_{m,n}$. In the space between the mirrors, the field is represented in the form of rays between the hyperboloid $\tau = \tau'_{m,n}$ (the internal caustic surface that prevents the rays from coming too close to the axis of revolution) and the hyperboloid $\tau = \tau_{m,n}$ (the external caustic surface). The internal caustic surface is characteristic of asymmetrical wave fields in systems that have rotational symmetry (for example, for two dimensional fields of the form $J_m(kr) \cos m\varphi$, where J_m is a Bessel function), while the external caustic surface is due to the concavity of the mirrors.

If $\bar{\tau} > \tau_{m,n}$, then, owing to the external caustic surface, the field of a given mode is weak near the edge of the mirror, and the eigenfunction defined by (23), (25), and (26) is changed insignificantly by diffraction on the edge. On the other hand, if $\bar{\tau} < \tau_{m,n}$, then the function (26) "does not fit" in the given resonator and the corresponding oscillation is formed by diffraction on the edge, similar to oscillations in a resonator with flat mirrors.

It is seen from Figs. 1 and 2 that symmetrical oscillations $(m = 0)$ have smaller radiation losses than asymmetrical oscillations $(m = 1, 2, \ldots)$ with the same index n, because the latter have an internal caustic surface, owing to which the external caustic surface is situated farther away from the axis of revolution $(\tau_{0,n} < \tau_{1,n} < \tau_{2,n} < \ldots)$. If on the other hand we take an open resonator with spherical mirrors of annular form (the projection of the mirror on the plane $z = 0$ is a ring defined by the inequalities $b < r < a$), then the situation changes: the symmetrical oscillations begin to radiate strongly, and the smallest radiation losses will be for the asymmetrical oscillations satisfying the conditions $\sqrt{kb^2 l^{-1}} \sin 2\alpha < \tau'_{m,n}$ and $\tau_{m,n} < \sqrt{ka^2 l^{-1}} \sin 2\alpha$.

3. SPHERICAL MIRRORS OF RECTANGULAR FORM

Let us consider now spherical mirrors whose projections on the plane $z = 0$ are rectangles with sides 2a and 2b. The surface of such a mirror is defined by the relations

$$-a < d \operatorname{ch} \bar{\zeta} \sin \rho \cos \varphi < a, -b < d \operatorname{ch} \bar{\zeta} \sin \rho \sin \varphi < b, \quad \zeta = \pm \bar{\zeta}. \tag{32}$$

We denote by $\bar{\rho}$ the maximum value of the coordinate ρ on the mirrors and assume that conditions (13) are satisfied. Introducing "quasi-Cartesian" coordinates

$$\xi = \rho \cos \varphi, \qquad \eta = \rho \sin \varphi, \tag{33}$$

we can rewrite (14) in the form

$$\partial^2 W/\partial \xi^2 + \partial^2 W/\partial \eta^2 + 2i\gamma \operatorname{ch} \zeta \, \partial W/\partial \zeta + [2i\gamma \operatorname{sh} \zeta - \gamma^2 (\xi^2 + \eta^2)] W = 0, \tag{34}$$

assuming W to be a function of ξ, η, and ζ. The substitution

$$W = \Psi(\tau_x, \tau_y, \sigma)/\operatorname{ch} \zeta, \quad \tau = \sqrt{2\gamma}\xi, \quad \tau_y = \sqrt{2\gamma}\eta, \quad \sigma = \arcsin(\operatorname{th} \zeta) \tag{35}$$

yields the equation

$$\partial^2 \Psi/\partial \tau_x^2 + \partial^2 \Psi/\partial \tau_y^2 + i\partial \Psi/\partial \sigma - \tfrac{1}{4}(\tau_x^2 + \tau_y^2) \Psi = 0, \tag{36}$$

which must be solved with account of the boundary conditions

$$\Psi(\tau_x, \tau_y, -\alpha) = e^{i(2kl - \pi q)} \Psi(\tau_x, \tau_y, \alpha) \quad \text{for } |\tau_x| < \tau_a, \ |\tau_y| < \tau_b; \tag{37}$$

$$\Psi(\tau_x, \tau_y, -\alpha) = 0 \quad \text{for} \quad |\tau_x| > \tau_a, \ |\tau_y| > \tau_b. \tag{38}$$

We denote by $2l$, as before, the distance between the mirrors along the z axis, while r_0 is the radius of curvature of the mirrors; the parameter α is determined from (18) and (19), while the parameters τ_a and τ_b are equal to, respectively,

$$\tau_a = \sqrt{ka^2 l^{-1} \sin 2\alpha}, \qquad \tau_b = \sqrt{kb^2 l^{-1} \sin 2\alpha} \tag{39}$$

Equation (36) has a solution

$$\Psi = \Psi_a(\tau_x, \sigma)\,\Psi_b(\tau_y, \sigma), \tag{40}$$

where the functions Ψ_a and Ψ_b satisfy the equations

$$\partial^2\Psi_a/\partial\tau_x^2 + i\partial\Psi_a/\partial\sigma - \tfrac{1}{4}\tau_x^2\Psi_a = 0,$$

$$\partial^2\Psi_b/\partial\tau_y^2 + i\partial\Psi_b/\partial\sigma - \tfrac{1}{4}\tau_y^2\Psi_b = 0. \tag{41}$$

Conditions (37) and (38) are satisfied if the functions Ψ_a and Ψ_b are subjected to the conditions

$$\Psi_a(\tau\ , -\alpha) = e^{i\chi_a}\Psi_a(\tau_x, \alpha) \quad \text{for} \quad |\tau_x| < \tau_a,$$

$$\Psi_b(\tau_y, -\alpha) = e^{i\chi_b}\Psi_b(\tau_y, \alpha) \quad \text{for} \quad |\tau_y| < \tau_b; \tag{42}$$

$$\Psi_a(\tau_x, -\alpha) = 0 \quad \text{for} \quad |\tau_x| > \tau_a,$$

$$\Psi_b(\tau_y, -\alpha) = 0 \quad \text{for} \quad |\tau_y| > \tau_b. \tag{43}$$

The natural frequency is determined by the formula

$$2kl = \pi q + \chi_a + \chi_b. \tag{44}$$

The functions Ψ_a and Ψ_b correspond to two-dimensional natural oscillations of an open resonator with cylindrical mirrors (of infinite length and of finite width 2a and 2b, see [6]). If we neglect diffraction on the edges, we have

$$\Psi_a = e^{-i(m+1/2)\sigma}\psi_m(\tau_x), \qquad \Psi_b = e^{-i(n+1/2)\sigma}\psi_n(\tau_y),$$

$$\chi_a = (2m+1)\,\alpha, \qquad \chi_b = (2n+1)\,\alpha,$$

$$\chi_a + \chi_b = 2(m+n+1)\,\alpha, \quad m, n = 0, 1, 2, \ldots \tag{45}$$

where

$$\psi_m(\tau) = (H_m(\tau)/(2\pi)^{1/4}\sqrt{m!})\,e^{-\tau^2/4}, \tag{46}$$

and H_m is the Hermite polynomial (see, for example, [5]):

$$H_m(\tau) = (-1)^m e^{\tau^2/2}\, d^m e^{-\tau^2/2}/d\tau^m,$$

$$H_0(\tau) = 1, \quad H_1(\tau) = \tau. \tag{47}$$

The functions (46) satisfy the differential equation

$$d^2\psi_m/d\tau^2 + (m + 1/2 - \tau^2/4)\,\psi_m = 0 \tag{48}$$

and the orthonormality condition

$$\int_{-\infty}^{\infty} \psi_m(\tau)\,\psi_n(\tau)\,d\tau = \delta_{mn}. \tag{49}$$

The oscillation defined by formulas (40) and (45) can be interpreted as an assembly of rays filling the region

$$-2\sqrt{m+1/2} < \tau_x < 2\sqrt{m+1/2},$$

$$-2\sqrt{n+1/2} < \tau_y < 2\sqrt{n+1/2}, \tag{50}$$

the boundaries of which are caustic surfaces. If this region is completely contained between the mirrors without projecting beyond their edges, then the eigenfunctions and the eigenvalues of (45) are only weakly perturbed by the edges, otherwise the perturbation is strong.

We note that

$$\psi_{0,0}(\tau) = \sqrt{2\pi}\,\psi_0(\tau_x)\,\psi_0(\tau_y). \tag{51}$$

In the general case the function $\psi_{m,n}(\tau)\cos m\varphi$ or $\psi_{m,n}(\tau)\sin m\varphi$ is a linear combination of the products $\psi_{m'}(\tau_x)\,\psi_{n'}(\tau_y)$, corresponding to the same frequency (29). The point is that the natural frequencies defined by (29), (44), and (45) are degenerate when $m > 0$ and $n > 0$, so that the eigenfunctions, which do not take into account the influence of the edges, can be chosen in different ways. The perturbing action of the edges eliminates the arbitrariness in the choice of the unperturbed system of functions and makes it necessary to use the functions (26) for mirrors of round form and functions (46) for mirrors of rectangular form.

4. DIFFRACTION PHENOMENA

We have taken diffraction into consideration only to the extent that the behavior of the wave field in the presence of caustic surfaces is essentially a diffraction phenomenon. Diffraction on the edges of the mirrors becomes important in open resonators, and diffraction at the focus in the case of concentric mirrors (see below).

To take account of diffraction on the edges, we rewrite (29) in the form

$$2kl = \pi q + 2(m+2n-1)\,\alpha + 2\pi p$$

$$(m = 0, 1, 2, \ldots;\ n = 1, 2, \ldots), \tag{52}$$

and (44) and (45) in the form

$$2kl = \pi q + 2(m+n-1)\,\alpha + 2\pi p \quad (m, n = 1, 2, \ldots), \tag{53}$$

where $p = p' - ip''$ is the diffraction correction, which depends on the indices m and n and which takes into account the influence of the edges on the oscillation frequency and on the attenuation of the oscillations with time.

The effect of diffraction on oscillations with caustic surfaces can be understood as follows: the unperturbed field at the edge of round spherical mirrors is proportional, in accordance with (26), to $\exp(-\bar{\tau}^2/4)$ (we disregard the pre-exponential factor which depends on m and n). If we calculate the deviation from the resonator by using the unperturbed field (as is frequently done for approximate estimates), then the complex parameter p will obviously be proportional to $\exp(-\bar{\tau}^2/2)$, so that oscillations with small indices m and n will have quite small radiation losses at moderate values of $\bar{\tau}$. This conclusion is confirmed by the numerical results of Fox and Li [2] for confocal mirrors ($r_0 = 2l$), but it turns out that p decreases more rapidly, almost like $\exp(-\bar{\tau}^2)$. Therefore diffraction on the edges leads to a decrease in the radiation losses. In the absence of caustic surfaces, diffraction on the edges ensures in itself small radiation losses (see [1]), although these losses exceed greatly the losses for the formation of caustics in open resonators with the same dimensions.

The curvature of the mirrors is taken into account by the term $\tau^2\Psi/4$ in (22) and by the term $(\tau_x^2 + \tau_y^2)\Psi/4$ in (36). If the parameter $\bar{\tau}$ is small, this term can be neglected and the influence of the curvature is slight; in particular, the caustics cannot be formed so that the diffraction on the edges occurs in the same way as on the edges of plane mirrors, which is considered in [1]. It is seen from (18) and (19) that small $\bar{\tau}$ can be obtained when $\sin 2\alpha \approx 0$, and $\sin 2\alpha$ vanishes either when $r_0/l = \infty$ or when $r_0/l = 1$. The first case corresponds to plane mirrors and the second to concentric spherical mirrors.

A simple connection exists between the distribution of the currents on plane and concentric mirrors (see [4]). This connection does not extend to the field distribution between the mirrors, and can be readily established with the aid of an integral equation for the current on the mirrors. We introduce, for $\sigma > \sigma'$, the function

$$G(\tau_x, \tau_x', \tau_y, \tau_y', \sigma - \sigma') = \frac{-i}{4\pi \sin(\sigma - \sigma')} \times \exp\left\{\frac{i}{4}\left[\frac{\tau_x^2 + \tau_x'^2 + \tau_y^2 + \tau_y'^2}{\mathrm{tg}(\sigma - \sigma')} - \frac{2(\tau_x\tau_x' + \tau_y\tau_y')}{\sin(\sigma - \sigma')}\right]\right\}, \quad (54)^*$$

satisfying Eq. (36) and the limiting relation

$$\lim_{\sigma \to \sigma' + 0} G(\tau_x, \tau_x', \tau_y, \tau_y', \sigma - \sigma') = \delta(\tau_x - \tau_x')\,\delta(\tau_y - \tau_y'). \quad (55)$$

This is the Green's function for the parabolic equation (36). It allows us to write for the function Ψ with $\sigma > -\alpha$ the expression

$$\Psi(\tau_x, \tau_y, \sigma) = \iint_S G(\tau_x, \tau_x', \tau_y, \tau_y', \sigma + \alpha)\,\Psi(\tau_x', \tau_y', -\alpha)\, d\tau_x' d\tau_y', \quad (56)$$

which satisfies automatically Eq. (36) and condition (20) or (38); S is chosen to be a region in the τ_x, τ_y plane corresponding to the surface of the mirror (i.e., a circle or rectangle). The boundary conditions (17) or (37) lead to the integral equation

$$\Psi(\tau_x, \tau_y, -\alpha) = e^{i\chi} \iint_S G(\tau_x, \tau_x', \tau_y, \tau_y', 2\alpha)\,\Psi(\tau_x', \tau_y', -\alpha)\, d\tau_x' d\tau_y', \quad (57)$$

from which we obtain the eigenfunctions $\Psi(\tau_x, \tau_y, -\alpha)$, the eigenvalues $e^{i\chi}$, and the natural frequencies of the oscillations defined by the formula

$$2kl = \pi q + \chi. \quad (58)$$

We now consider two "conjugate" resonators, with curvature radii r_0 and $\hat{r}_0$ satisfying the relation

$$1/r_0 + 1/\hat{r}_0 = 1/l, \quad (59)$$

while the dimensions of the mirrors (a or a and b) and the distance between mirrors $2l$ are the same in both. It is seen from (19) that the parameters α and $\hat{\alpha}$ of the conjugate resonators are related by the equations

$$\hat{\alpha} = \pi/2 - \alpha, \quad \mathrm{tg}\, 2\hat{\alpha} = -\mathrm{tg}\, 2\alpha, \quad \sin 2\hat{\alpha} = \sin 2\alpha, \quad (60)$$

from which we have (the asterisk denotes the complex conjugate)

$$G(\tau_x, \tau_x', \tau_y, \tau_y', 2\hat{\alpha}) = -G^*(-\tau_x, \tau_x', -\tau_y, \tau_y', 2\alpha). \quad (61)$$

By virtue of the symmetry of the region S, the eigenfunctions satisfy the relation

$$\Psi(\tau_x, \tau_y, -\alpha) = \pm\Psi(-\tau_x, -\tau_y, -\alpha), \quad (62)$$

so that the function Ψ^* is the solution of the integral equation

$$\Psi^*(\tau_x, \tau_y, -\alpha) = \mp e^{-i\chi^*} \iint_S G(\tau_x, \tau_x', \tau_y, \tau_y', 2\hat{\alpha})\,\Psi^*(\tau_x', \tau_y', -\alpha)\, d\tau_x' d\tau_y', \quad (63)$$

which must be satisfied by the function $\Psi(\tau_x, \tau_y, -\hat{\alpha})$ for the conjugate resonator. It follows therefore that

$$\Psi(\tau_x, \tau_y - \hat{\alpha}) = \Psi^*(\tau_x, \tau_y, -\alpha), \quad e^{i\hat{\chi}} = \mp e^{-i\chi^*}. \quad (64)$$

The latter relations together with formula (52) enables us to find the connection between $\hat{\chi}$ and χ for mirrors of circular form

$$\hat{\chi} = (m + 2n - 1)\pi - \chi^*. \quad (65)$$

For rectangular mirrors we have

$$\hat{\chi} = (m + n - 1)\pi - \chi^* \quad (m, n = 1, 2, \ldots). \quad (66)$$

For the conjugate resonators the diffraction corrections in formulas (52) and (53) are related by

$$\hat{p} = -p^*. \quad (67)$$

Thus, a resonator with confocal mirrors ($r_0 = 2l$, $\alpha = \pi/4$) is self-conjugate: its eigenfunctions

*tg = tan

$\Psi(\tau_x, \tau_y, -\alpha)$ are real and the diffraction corrections $p = -i|p|$ are pure imaginary. A resonator with concentric spherical mirrors is conjugate with respect to a resonator with plane parallel mirrors of the same form, so that the expressions given in Secs. 4 and 5 of [1] for the current density on the surface of plane mirrors (rectangular and round) enable us to obtain without difficulty the current density on concentric mirrors. The diffraction correction for round concentric mirrors is

$$p = -\nu_{mn}^2/\pi\,(M + \beta - i\beta)^2,$$
$$M = \sqrt{2ka^2/l}, \qquad \beta = 0.824, \tag{68}$$

where ν_{mn} is the n-th zero of the Bessel function J_m. For rectangular concentric mirrors

$$p = -\frac{\pi m^2}{4(M_a + \beta - i\beta)^2} - \frac{\pi n^2}{4(M_b + \beta - i\beta)^2},$$
$$M_a = \sqrt{\frac{2ka^2}{l}}, \qquad M_b = \sqrt{\frac{2kb^2}{l}}. \tag{69}$$

It must be noted for plane and concentric mirrors the variables τ, τ_x, and τ_y are not suitable; in addition, $d = 0$ and $\gamma = 0$ for concentric mirrors. Since we have assumed $\gamma \gg 1$ in Sec. 1, the concentric mirrors must be considered separately. We introduce spherical coordinates R, ρ, and φ such that the angle ρ varies in the range $0 < \rho < \pi/2$, but the radius vector R assumes both positive and negative values; as will be seen below, such coordinates are convenient in the analysis of diffraction at the focus.

We seek the solution of the wave equation in the form

$$\Phi = W(\rho, \varphi, R)\, e^{ikR} - (-1)^q\, W(\rho, \varphi, -R)\, e^{-ikR} \tag{70}$$

and obtain from W the equation

$$\frac{1}{\sin\rho}\frac{\partial}{\partial\rho}\left(\sin\rho\,\frac{\partial W}{\partial\rho}\right) + \frac{1}{\sin^2\rho}\frac{\partial^2 W}{\partial\varphi^2}$$
$$+ \frac{\partial}{\partial R}\left(R^2\frac{\partial W}{\partial R}\right) + 2ikR\,\frac{\partial}{\partial R}(RW) = 0. \tag{71}$$

Specifying the surface of the mirrors by means of the relations

$$0 < \rho < \bar{\rho}, \qquad 0 \leqslant \varphi \leqslant 2\pi, \qquad R = \pm l \tag{72}$$

and assmuing that $\sin^2\bar{\rho} \ll 1$, we can replace $\sin\rho$ in (71) by ρ. Neglecting, in addition, the term $\partial(R^2\partial W/\partial R)\,\partial R$, we obtain the parabolic equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial W}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial\varphi^2} + 2ikR\,\frac{\partial}{\partial R}(RW) = 0, \tag{73}$$

which assumes in the coordinates (33) the form

$$\frac{\partial^2 W}{\partial\xi^2} + \frac{\partial^2 W}{\partial\eta^2} + 2ikR\,\frac{\partial}{\partial R}(RW) = 0 \tag{74}$$

The Green's function for this equation has the form $(R > R')$

$$\Gamma(\xi - \xi', \eta - \eta', R, R')$$
$$= -\frac{ikR'}{2\pi(R - R')}\exp\left\{\frac{ikRR'}{2}\,\frac{(\xi - \xi')^2 + (\eta - \eta')^2}{R - R'}\right\}, \tag{75}$$

with

$$\lim_{R \to R'+0} \Gamma(\xi - \xi', \eta - \eta', R, R') = \frac{1}{R'}\,\delta(\xi - \xi')\,\delta(\eta - \eta'). \tag{76}$$

It is easy to see that for $k|R'| \gg 1$ and arbitrary kR the discarded term $\partial(R^2\partial\Gamma/\partial R)/\partial R$ is actually small compared with the remaining terms.

In place of (54) we now have

$$W(\xi, \eta, R) = -l \iint\limits_S \Gamma(\xi - \xi', \eta - \eta', R, -l)$$
$$W(\xi', \eta', -l)\, d\xi' d\eta', \tag{77}$$

where S is the region in the ξ, η plane corresponding to the surface of the mirror (circle for the mirror (72), or rectangle). From (77) we usually obtain an integral equation for the function W $(\xi, \eta, -l)$. It has the same form as for plane mirrors, confirming the already formulated connection between resonators with concentric and plane mirrors.

At the same time, expression (77) shows that the field is concentrated near the focus $R = 0$. If, for example, put $W = 1$ on the lower mirror (72), then we get the formula

$$W = -\frac{ikl^2}{R + l}\int\limits_0^{\bar{\rho}} e^{i\nu(\rho^2 + \rho'^2)/2}\, J_0(\nu\rho\rho')\,\rho' d\rho', \quad \nu = -\frac{kRl}{R + l}, \tag{78}$$

which agrees with the classical expression for the field at the focus (see [7]). The asymptotic jump in the phase at the focus follows from the identity

$$\iint\limits_{-\infty}^{\infty} \Gamma(\xi - \xi', \eta - \eta', R, -l)\, d\xi' d\eta' = \frac{1}{R}, \tag{79}$$

since $R < 0$ prior to the passage of the wave We^{ikR} through the focus and $R > 0$ following passage through the focus. The field distribution of the natural oscillations near the focus is more complicated than that given by (78), since the function W is not constant in amplitude or phase on the mirror.

If we start with plane mirrors and gradually increase the curvature, leaving their dimensions and the distance between them constant, then the most favorable conditions for the formation of the caustic surfaces will occur when $r_0 = 2l$ — in the case of confocal mirrors. This is seen from (18) and (39), since $\sin 2\alpha$ reaches a maximum when $\alpha = \pi/4$; for confocal mirrors the radiative damping of the oscillations with caustics is minimal, since the field turns out to be weakest at the edge of the resonator.

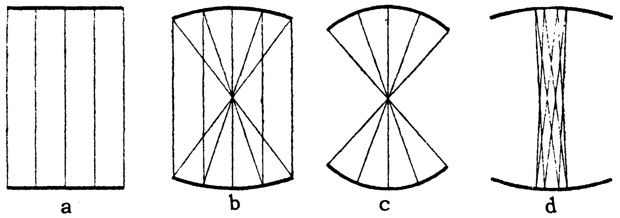

FIG. 3

With further increase in the curvature, the caustic surfaces near the mirrors move towards the edges, so that the radiative damping of the oscillations increases. At the same time, the concentration of the field near the center of the resonator increases; this is seen from the fact that in the plane $z = 0$ we have

$$r = \tau \sqrt{(l/2k)\operatorname{ctg}\alpha}, \tag{80}*$$

so that as α increases the caustic surfaces move here towards the z axis and enclose an ever decreasing area. Finally, as $\alpha \to \pi/2$, the caustic surfaces disappear and a focus is formed at the place where they contract. Further increase in the curvature is not rational, for we arrive then at resonators that are conjugates to the resonators with convex mirrors (see [6]) and have therefore large radiative losses.

If we consider open resonators from the point of view of geometrical optics, then we can construct for plane and concentric mirrors sets of parallel (Fig. 3a) and radial rays (Fig. 3c), which go over into each other upon reflection and which correspond to a certain degree to the wave fields in the resonators. For confocal mirrors we can construct (Fig. 3b) parallel rays which pass through the center of the resonator after one reflection, and which become parallel again after the second reflection. Such rays, however, have no relation whatever with the natural oscillations. If the oscillation has an external caustic surface, then it can be represented (see Secs. 2 and 3), as shown in Fig. 3d, by rays that are alternately reflected from the mirrors and do not reach the edges.

5. ELECTROMAGNETIC FIELDS

So far we have solved the scalar wave equation with boundary condition $\Phi = 0$ on the mirrors. We are interested, however, in the electromagnetic oscillations, so that we must solve Maxwell's equations

*ctg = cot

$$\operatorname{rot}\mathbf{H} = -ik\mathbf{E}, \qquad \operatorname{rot}\mathbf{E} = ik\mathbf{H} \tag{81}*$$

together with the auxiliary equations

$$\operatorname{div}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{H} = 0 \tag{82}$$

and the boundary conditions on the perfectly conducting mirrors

$$E_\xi = E_\eta = H_\zeta = 0 \tag{83}$$

Since we are seeking an approximate (asymptotic) solution, we can express the electromagnetic field of the linearly polarized natural oscillations in terms of the scalar function Φ in the following manner. We put

$$\mathbf{E} = ik^{-1}\operatorname{rot}\operatorname{rot}\mathbf{A}, \qquad \mathbf{H} = \operatorname{rot}\mathbf{A}, \tag{84}$$

and then the first equation of (81) and both equations of (82) are satisfied. If the vector potential A satisfies a vector wave equation, then the second equation of (81) is also satisfied and the electric field is represented in the form

$$\mathbf{E} = ik^{-1}(\operatorname{grad}\operatorname{div}\mathbf{A} + k^2\mathbf{A}). \tag{85}$$

We now put $\Phi = A_\xi$ or $\Phi = A_\eta$ and assume that the other two components of the vector potential are equal to zero. We stipulate here, for example, that the component A_ξ satisfy the equation obtained if the values obtained for the components E_ξ from (84) and (85) are equated. By calculating the components E_η and E_ξ from (84) and (85) we readily verify that these formulas give practically equivalent result—the terms which do not coincide are smaller than those which do by a factor γ.

The frame ξ, η, and ζ, defined by (1), (9), and (33), is orthogonal only approximately, to the extent to which we can put $h_\varphi = h\rho$. We shall use such an approximation, assuming that $h = d\sqrt{\cosh^2\zeta - \rho^2}$, and then $h_\xi = h_\eta = h_\zeta = h$. For the functions A_ξ and A_η we obtain equations that differ from the scalar wave equation. However, if we solve them in the form (11) and neglect factors analogous to those neglected in Sec. 1, then we obtain for the function W the previous parabolic equations (14) and (34). If furthermore $A_\xi = 0$ and $A_\eta = 0$ on the mirrors, then the fields (84) will satisfy the boundary conditions (83) exactly.

For mirrors of rectangular form we can put, in accordance with Sec. 3,

$$\Psi = e^{-i(m+n-1)\sigma}\,\psi_{m-1}(\tau_x)\,\psi_{n-1}(\tau_y)\ (m, n = 1, 2, \ldots), \tag{86}$$

and then we can easily calculate from (84) the electromagnetic fields of the oscillations, which can be naturally called $E^{(x)}_{mnq}$ and $E^{(y)}_{mnq}$ modes in a res-

*rot = curl

onator with spherical mirrors of rectangular form; the frequency of these oscillations is determined by (53).

For mirrors of circular form we assume in accordance with Sec. 2

$$\Psi_m = e^{-i(m+2n-1)\sigma}\psi_{m,n}(\tau)$$
$$(m = 0, 1, 2, \ldots; n = 1, 2, \ldots) \quad (87)$$

and we obtain the $E^{(x)}_{mnq}$ and $E^{(y)}_{mnq}$ modes in such a resonator, with the frequency given by (52).

The current surface density on the mirrors has for $E^{(x)}_{mnq}$ modes a single component along the ξ axis, and for the $E^{(y)}_{mnq}$ modes a single component along the η axis. This component is equal to

$$f = \frac{i\omega e^{ikl}}{2\pi \operatorname{ch} \bar{\zeta}} \Psi(\xi, \eta, \bar{\zeta}) = \frac{i\omega e^{-i(kl-\pi q)}}{2\pi \operatorname{ch} \bar{\zeta}} \Psi(\xi, \eta, -\bar{\zeta}) \quad (88)$$

on the upper mirror and to $-(-1)^q f$ on the lower mirror. In the $E^{(x)}_{mnq}$ and $E^{(y)}_{mnq}$ modes the current distribution on the mirror surfaces is qualitatively the same as for the like oscillations in a resonator with plane mirrors of rectangular and circular form ([1], Figs. 10 and 11), but then the points are limited not by the dimensions of the mirror but by the dimensions of the caustic surface, beyond which the currents are insignificant.

CONCLUSION

We have developed a theory for natural oscillations in open resonators with spherical mirrors. The most interesting from the theoretical and practical points of view are oscillations with very small radiation losses. These oscillations can be represented in the form of two waves or two beams propagating towards each other and protected by the caustic surfaces against losses due to lateral radiation. In some cases these beams may be hollow because of the presence of an internal caustic surface.

Such waves are of interest not only for open resonators, but also for systems which transmit or guide electromagnetic radiation energy.

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