STABILITY OF THE LAMINAR FLOW OF THIN LIQUID LAYERS

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The stability, with respect to infinitesimal disturbances, of the laminar flow of a viscuous incompressible liquid film along a vertical wall is investigated. It is shown that the laminar flow possesses convective instability for disturbances with a sufficiently small frequency. Absolute instability is absent.

LIQUID films (10^{-2} cm) flowing under the influence of gravity along a hard vertical wall exhibit several specific peculiarities.[1-3] The usual laminar flow of such films has been observed only for Reynolds numbers Re not exceeding 20—30. When the Re numbers are larger, there occurs a so-called wave flow and wave-like oscillations with comparatively large wave lengths are observed on the surface of the film. Finally, when Re $\gtrsim$ 1500, turbulence begins.

The wave process has been investigated theoretically by P. L. Kapitza [1] and experimentally by P. L. Kapitza and S. P. Kapitza.[2] In [1] it was shown that in the flow of experimental films, in addition to viscous forces and the force of gravity, surface tension plays an important role. It was also established that when the Re numbers are precisely on the order of 20—30, the wave motion is more stable than the laminar motion. The fundamental characteristics of the wave motion were calculated and found to be in good agreement with the experimental data.[2]

Until now, however, no one had investigated the stability of simple laminar flow with respect to infinitesimal disturbances.[1)] The fact of the matter is that in the present case, just as in the case, for example, of the flow of a liquid in a tube, the question of stability involves specific peculiarities connected with the uniformity of the problem along the flow direction. As was first indicated by Landau and Lifshitz,[4] one should also differentiate between convective and absolute instabilities.

The present article deals with an investigation of the convective and absolute instabilities of the laminar flow of a film.

1. Let the surface of the wall over which the liquid flows coincide with xy plane and let the x axis be directed vertically downward. Further, let the equation for the free liquid surface be $z = h(x)$. We consider $h(x)$ to vary slightly within distances on the order of the thickness of the film, i.e., large wavelengths play an important role. The condition for this actually being so will be given presently. We merely note now that the condition is known to be fulfilled when the film is thin enough.

If $h(x)$ is a slowly varying function, it can be assumed that for a given x the z dependence of vx, the velocity component along x, is the same as for a case of laminar flow with a constant film thickness

$$v_x = \frac{3}{2} U(2 - z/h)z/h, \tag{1}$$

where, however, U (the cross-sectional average velocity) and h are slowly varying functions of x. The z component of the velocity, v_z, can also easily be expressed by the functions $U(x)$ and $h(x)$, if one uses the continuity equation, div v = 0; hence

$$v_z = -\int \frac{\partial v_x}{\partial x} dz. \tag{2}$$

Substituting (1) and (2) in the Navier-Stokes equation

$$\partial v_x/\partial t + v_x \partial v_x/\partial x + v_z\, \partial v_x/\partial z = -\rho^{-1}\, \partial p/\partial x + \nu \Delta v_x + g, \tag{3}$$

where g is the acceleration of gravity, and averaging over z for a given x, we obtain[1,3]

$$\partial U/\partial t + \frac{9}{10} U\, \partial U/\partial x = \sigma \rho^{-1}\, d^3h/dx^3 - 3\nu h^{-2} U + g, \tag{4}$$

where allowance is made for the fact that $\partial p/\partial x = -\sigma d^3h/dx^3$ (σ being the surface tension).

The second equation, which connects the two

[1)]The question of convective instability was studied by Bushmanov.[5] However, his results are unreliable, since they are based on an expansion in the parameter z/h ($0 \leq z/h \leq 1$), which is by no means small. Furthermore, his free surface boundary conditions are false. In the present investigation the expansion is in the parameter kh, which is small for the essential wave numbers k.

unknown functions U(x) and h(x), is the continuity equation

$$\partial h/\partial t = -\partial(Uh)/\partial x. \tag{5}$$

2. We shall seek solutions to (4) and (5) in the form of $U = U_0 + U'$, $h = a + h'$, where $U_0 = ga^3/3\nu$, and U' and h' are small perturbations proportional to $\exp i(kx - \omega t)$. Then we obtain for U' and h' a system of two equations

$$U'\left(\omega - \tfrac{9}{10} kU_0 + 3i\nu/a^2\right) + h'\left(-\sigma k^3/\rho - 6i\nu U_0/a^3\right) = 0,$$

$$kaU' + (kU_0 - \omega)h' = 0, \tag{6}$$

which has non-trivial solutions only when the corresponding determinant is equal to zero, i.e.,

$$(kU_0 - \omega)\left(\omega - \tfrac{9}{10} kU_0 + 3i\nu/a^2\right) + ka\left(\sigma k^3/\rho + 6i\nu U_0/a^3\right) = 0. \tag{7}$$

Equation (7) determines the relation between the disturbance "frequency" ω and the "wave vector" k. If k is real and the imaginary part of ω can have positive values, then the disturbance will be intensified in proportion to its "drift" down the flow. We shall call this kind of instability "displacement instability." Solving Eq. (7) for ω, we obtain

$$\omega = \frac{19\,U_0}{20}k - \frac{3i}{2}\frac{\nu}{a^2} \pm \left[\frac{\sigma a}{\rho}k^4 + \left(\frac{U_0}{20}\right)^2 k^2 + \frac{123}{20} i \frac{U_0\nu}{a^2} k - \left(\frac{3\nu}{2a^2}\right)^2\right]^{1/2}. \tag{8}$$

It is easy to see that one of the branches of (8) does not in general have values that lie in the upper half-plane and can therefore be disregarded. The condition that the imaginary part of the other branch be positive is provided for by the inequality

$$k < \sqrt{21/5}\, a^{-1}\Lambda^{-1/2}, \tag{9}$$

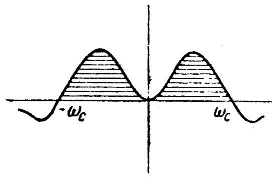
where $\Lambda = \sigma/\rho U_0^2 a$. When $k = a^{-1}\sqrt{21/5\Lambda}$, ω is real and equal to

$$\omega = \omega_c = 3\sqrt{21/5\Lambda}\, U_0/a. \tag{10}$$

Now we formulate the condition for the applicability of Eq. (4), namely that the main role must be played by disturbances with wavelengths that are large in comparison to the thickness of the film or with small values for k ($ka \ll 1$). It directly follows from (9) that this condition is fulfilled if

$$\Lambda \gg 1. \tag{11}$$

The question of convective instability is posed experimentally in the following manner. For some



fixed value of x, a disturbance with an assigned (real) frequency is applied to the film. Depending on the values of the imaginary part of the "wave vector" k, which corresponds to the assigned frequency, the disturbance will increase or decrease down the flow, with an increase occurring if Im k < 0 and a decrease if Im k > 0.

We have seen that the frequency determined by Eq. (10) corresponds to a real k. It is easy to see that Im k < 0 when $\omega < \omega_c$ and Im k > 0 when $\omega > \omega_c$. We note that ω_c coincides with the frequency of the wave motion investigated in [1] (see also [3]). Thus the laminar flow of a film always possesses displacement instability with respect to disturbances with sufficiently small frequencies.

In the investigation conducted by P. L. Kapitza and S. P. Kapitza convective instability was observed only when the Reynolds numbers exceeded a certain critical value. This appears to be due to the fact that the authors used only a limited interval of disturbance frequencies.

3. The presence of displacement instability in the laminar flow of a film does not in the least imply that the flow cannot be produced. When there is displacement instability, the disturbance increases with time, and in addition "drifts" down the flow. And since actual films always have finite dimensions, a disturbance at any point in it can be however small, if the amplitude of the disturbance is sufficiently small at the point of its origin. A flow cannot be produced if it possesses absolute instability, i.e., an instability causing an increase in the disturbance at a given point in space.

Landau and Lifshitz [4] have shown that if the integral along the real axis

$$\int a(k)\, e^{-i\omega(k)t}\, dk, \tag{12}$$

where a(k) is the amplitude determined by the initial conditions, tends to infinity as $t \to \infty$, then the flow possesses absolute instability. Otherwise absolute instability is absent. To investigate this question it is helpful, as Sturrock [6] has demonstrated, to transform from an integration over k to an integration over ω, i.e., to write instead of (12)

$$\int_L a[k_1(\omega)]\, e^{-i\omega t}\, \frac{dk_1(\omega)}{d\omega}\, d\omega. \tag{13}$$

The integration in (13) is over a contour L in the ω-plane, which is the image of the real axis of the k-plane and is given by the function $\omega(k)$, and $k_1(\omega)$ is that branch of $k(\omega)$ which yields real values on L. The L contour for the example investigated by us is shown in the figure above. We invariably deal with only that branch of $\omega(k)$ which has a positive imaginary part, since (12) obviously becomes zero for the other branch when $t \to \infty$. If this positive valued branch of $k(\omega)$ has no singularities in region G, which is shaded in the figure, then the integration contour in (13) can be shifted so that it lies wholly in the lower half-plane. Then (13) tends to zero as $t \to \infty$, and absolute instability is absent. We shall show that this situation occurs in our case.

For future convenience we transform ω and k to the dimensionless quantities ε and κ, to that

$$\omega = \varepsilon U_0/a\sqrt{\Lambda}, \qquad k = \varkappa/a\sqrt{\Lambda}. \tag{14}$$

We easily obtain from (7) the relation that determines $\kappa(\varepsilon)$,

$$\varkappa^4 - \tfrac{9}{10}\varkappa^2 + \varkappa\left(\tfrac{19}{10}\varepsilon + 9iA\right) - \varepsilon^2 - 3iA\varepsilon = 0, \tag{15}$$

where $A = \sqrt{\Lambda/\mathrm{Re}}$ and $\mathrm{Re} = U_0a/\nu$ is the Reynolds number.

The algebraic function $\kappa(\varepsilon)$, determined by Eq. (15), has singularities (branch points) for those values of ε for which Eq. (15) yields multiple roots for κ. It is easy to see that the equation $\kappa^4 - \alpha\kappa^2 + \beta\kappa - \gamma = 0$ has at least one multiple root if, and only if, there exists among the coefficients α, β, and γ the relationship

$$\tfrac{27}{16}\beta^4 + 8\alpha^2\gamma^2 - \tfrac{1}{4}\beta^2\alpha^3 - 9\alpha\beta^2\gamma + \alpha^4\gamma + 16\gamma^3 = 0, \tag{16}$$

with the value of the multiple root as determined by one of the branches being

$$\varkappa = \left[\tfrac{1}{6}\left(\alpha + \sqrt{\alpha^2 - 12\gamma}\right)\right]^{1/2}. \tag{17}$$

Applying this procedure to Eq. (15), we determine the existence of singular points of $\kappa(\varepsilon)$ from Eq. (16) with

$$\alpha = \tfrac{9}{10}, \qquad \beta = \tfrac{19}{10}\varepsilon + 9iA, \qquad \gamma = \varepsilon^2 + 3iA\varepsilon.$$

First let us examine the case where $A \gg 1$. In this case Eq. (16), which is a sixth-degree equation in ε, is easily solved

$$\varepsilon_{1,2,3} = -3iA, \qquad \varepsilon_4 = \tfrac{1}{2}\left(\tfrac{9}{4}\right)^{1/3}\sqrt[3]{A}\left(\sqrt{3} - i\right),$$

$$\varepsilon_5 = -\tfrac{1}{2}\left(\tfrac{9}{4}\right)^{1/3}\sqrt[3]{A}\left(\sqrt{3} + i\right), \qquad \varepsilon_6 = \left(\tfrac{9}{4}\right)^{1/3} i\sqrt[3]{A}. \tag{18}$$

The equation for L, as is apparent from (8), will be

$$\varepsilon'' = A^{-1}\left[{}^{7}/_{45}\varepsilon'^2 - {}^{1}/_{243}\varepsilon'^4\right],$$

where ε' and ε'' are respectively the real and imaginary parts of ε. Obviously, none of the points of (18) falls into the region G.

In order to demonstrate that this holds true for any A it is sufficient to prove that for no A do the singularities of $\kappa(\varepsilon)$ lie on the boundary of G. Not all the singular points concern us, but only those that are branch points for $\kappa_1(\varepsilon)$, which is that branch of $\kappa(\varepsilon)$ which becomes real valued on L. If such a point lies on L, the corresponding value for the multiple root in (17) is real, which is possible only where $\gamma = \varepsilon^2 + 3iA\varepsilon$ is real. Here either $\varepsilon'' = -3A$ or $\varepsilon' = 0$. The first case is of no interest, since then $\mathrm{Im}\,\varepsilon < 0$, and the second case is not possible, because L intersects the imaginary axis when $\varepsilon = 0$, and $\varepsilon = 0$ is not a solution for Eq. (16). Thus, the singularities that concern us cannot lie in the part of L that forms a boundary of G.

Nor can they lie on any remaining part of the boundary of G, since Eq. (16) does not generally have real solutions for ε. Actually, assuming ε to be real and separating the real from the imaginary parts in (16), we obtain two equations:

$$\varepsilon^4 + \varepsilon^2(0.63 - 3A^2) - 51.3A^2 - 0.03 = 0, \tag{19}$$

$$\varepsilon^6 + \varepsilon^4(0.05 - 27A^2) - \varepsilon^2(118.6A^2 + 0.0001) + 692.0A^4 + 0.92A^2 = 0,$$

which when $A > 0$ do not have real roots in common, as one can easily see.

We have proved that when $\Lambda \gg 1$ the laminar flow of a film does not possess absolute instability. Consequently, such instability can appear only when $\Lambda \lesssim 1$. In which case, however, there is nothing to isolate small values of k, and therefore, a disruption of the laminar process causes turbulence to begin at once.

Thus, by a careful elimination of disturbances it is possible, in principle, to transfer a film from a laminar process directly into turbulence, skipping the wave-flow process.

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