REGGE POLES IN THE NONRELATIVISTIC PROBLEM WITH NONLOCAL AND SINGULAR INTERACTION

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The singularities of the amplitude in the l plane are studied with the aid of the Schrödinger equation in the momentum representation. The relation between the analytic properties of the amplitude and the behavior of the potential at small distances is elucidated. It is shown that the nonlocality of the potential leads to pole condensation or branching at $l = -1$.

1. INTRODUCTION

It is known that the study of the singularities of the relativistic amplitude as a function of the orbital momentum leads to information on the behavior of amplitudes at large energies.[1,2] It has been shown that in the nonrelativistic Schrödinger problem the only singularities in the complex l plane are simple poles. In the relativistic theory this has not only not been proved, but it is indeed known that more complicated singularities exist; thus, condensation of poles may occur at $l = -1$.[3]

The investigation of the singularities of the amplitude directly on the basis of unitarity and analyticity is rather complicated, and it is very useful to consider nonrelativistic models in which the amplitude is more similar to the relativistic amplitude than in the usual Schrödinger theory. A characteristic feature of the relativistic interaction is its nonlocality and, possibly, singularity. It is therefore interesting to investigate the singularities in the l plane of an amplitude obtained from a Schrödinger equation with a nonlocal and singular interaction.

It is convenient to consider the nonrelativistic equation for the amplitude in the momentum representation. In this form it corresponds to the relativistic Bethe-Salpeter equation, where the potential plays the role of the irreducible four-point vertex. Single-meson exchange corresponds to the usual local potential, and the amplitude has in this case only poles, which, for small couplings or large negative energies, tend to values determined by the behavior of the potential at small distances.[4,1]

The graphs giving rise to the Mandelstam spectral density $\rho(s, u)$ correspond to a nonlocal potential. In this case a condensation of poles occurs at $l = -1$.[3] If one assumes[5] that the interaction between the elementary particles contains a repulsive "core" corresponding to a singular potential which grows more rapidly than r^{-2} at small distances, the poles connected with the behavior of the potential at small distances disappear but the condensation of the poles caused by the nonlocality of the potential remains. If the nonlocal potential is singular, the amplitude can have fixed (i.e., energy independent) singular points in the l plane.

2. EQUATION FOR THE AMPLITUDE

The Schrödinger equation with a nonlocal interaction has the form

$$\frac{1}{2m}\Delta\psi(\mathbf{r}) + E\psi(\mathbf{r}) = \int V(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\,d\mathbf{r}',$$
$$V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r}', \mathbf{r}). \tag{1}$$

In the usual case

$$V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r})\,\delta(\mathbf{r} - \mathbf{r}').$$

Instead of considering this integro-differential equation it is convenient to go over to the momentum representation:

$$\psi_{\mathbf{pp}'} = \delta(\mathbf{p} - \mathbf{p}') - \frac{1}{\varepsilon - E - i\delta}\int\frac{d\mathbf{q}}{(2\pi)^3}V(\mathbf{p}, \mathbf{q})\,\psi_{\mathbf{qp}'};$$

$$V(\mathbf{p}, \mathbf{q}) = \int d\mathbf{r}\,d\mathbf{r}'\,e^{i(\mathbf{pr} - \mathbf{qr}')}V(\mathbf{r}, \mathbf{r}'), \quad \varepsilon_p = \frac{p^2}{2m}. \tag{2}$$

From the equation for the wave function we obtain an equation for the amplitude $\Gamma_{\mathbf{pp}'}$ which is connected with the wave function by

$$\psi_{\mathbf{pp}'} = \delta(\mathbf{p} - \mathbf{p}') - \frac{\Gamma_{\mathbf{pp}'}}{\varepsilon_p - E - i\delta}, \quad \Gamma_{\mathbf{pp}'} = \int d\mathbf{q}\,V(\mathbf{p}, \mathbf{q})\,\psi_{\mathbf{qp}'}. \tag{3}$$

For $\epsilon'_p = E$ Eqs. (2) and (3) coincide with (1), but it is convenient to consider p′ and E as independent parameters. The amplitude $\Gamma_{\mathbf{pp}'}$ is a function of the energy E, the angle between the momenta, and the squares of momenta. On the energy shell, where $\epsilon_p = \epsilon'_p = E$, Γ differs from the usual amplitude only by the factor $-2\pi/m$.

It is easily seen with the help of (3) that $\Gamma_{pp'}$ satisfies an equation which is a generalization of the unitarity condition off the energy shell. To this end we form the expression

$$\int dq(\psi^+_{pq}\Gamma_{qp'} - \Gamma^+_{pq}\psi_{qp}).$$

Expressing Γ through ψ and ψ through Γ with the help of (3), we obtain

$$\int \frac{dq\,dq'}{(2\pi)^6}\psi^+_{pq}(V_{qq'} - V^+_{qq'})\psi_{q'p'}$$
$$= \Gamma_{pp'} - \Gamma^+_{pp'} - \int \frac{dq}{(2\pi)^3}\Gamma^+_{pq}\Gamma_{qp'}\left(\frac{1}{\varepsilon_q - E + i\delta} - \frac{1}{\varepsilon_q - E - i\delta}\right).$$

for a Hermitian Hamiltonian, $V = V^+$, the left-hand side of this equation vanishes. As a result

$$\frac{1}{2i}(\Gamma_{pp'} - \Gamma^+_{pp'}) = -\pi\int\frac{dq}{(2\pi)^3}\Gamma^+_{pq}\Gamma_{qp}\delta(\varepsilon_q - E). \quad (4)$$

Formula (4) goes over into the usual unitarity condition if $\epsilon_p = \epsilon'_p = E$.

Formula (4) can be generalized to the case when V acts on spinor indices and even can create new particles. In this case the indices p, q, and p′ in (2) denote not only momenta but also all other quantum numbers characterizing the system. Accordingly, one must replace in (4) the integral over the momenta by a sum over all intermediate states with different spins and particle numbers:

$$\frac{1}{2i}(\Gamma^+_{ab} - \Gamma_{ab}) = \pi\sum_c\int d\tau_c\,\Gamma^+_{ac}\Gamma_{cb}\delta(\varepsilon_c - E). \quad (4')$$

Eliminating ψ from (3), we obtain an equation for the amplitude:

$$\Gamma_{pp'} = V(\mathbf{p}, \mathbf{p}') - \int\frac{dq}{(2\pi)^3}V(\mathbf{p}, \mathbf{q})\frac{1}{\varepsilon_q - E - i\delta}\Gamma_{qp'}. \quad (5)$$

This equation can be obtained by summing the Feynman graphs for a particle in an external field.

In the case of a local potential, V depends only on the square of the momentum transfer s $= (\mathbf{p} - \mathbf{p}')^2 = p^2 + p'^2 - 2pp'z$. If the potential is nonlocal, V depends also on the total momentum $u = (\mathbf{p} + \mathbf{p}')^2 = p^2 + p'^2 + 2pp'z$ and on $p^2 - p'^2$. The exchange terms of a local potential depend only on u. It will be shown below that the condensation of poles occurs only when the potential is nonlocal, i.e., depends on s and on u.

As an example of a nonlocal potential we shall consider a product of Yukawa potentials:

$$V(\mathbf{r}, \mathbf{r}') = \int_{\mu_0}^{\infty}\frac{\rho(\mu^2)\,d\mu^2}{(2\pi)^2}\exp\left(-\frac{1}{2}\mu|\mathbf{r} + \mathbf{r}'|\right)\exp\left(-\frac{1}{2}\mu|\mathbf{r} - \mathbf{r}'|\right),$$

$$V(\mathbf{p}, \mathbf{p}') = \int_{\mu_0}^{\infty}\frac{\rho(\mu^2)\,d\mu^2}{(\mathbf{p} - \mathbf{p}')^2 + \mu^2}\frac{1}{(\mathbf{p} + \mathbf{p}')^2 + M^2}. \quad (6)$$

For $M \to \infty$ this interaction goes over into a superposition of Yukawa potentials.

Equation (5) can be written for the partial amplitudes:

$$\Gamma_l(p, p') = V_l(p, p') - \frac{1}{2\pi^2}\int_0^{\infty}V_l(p, q)\frac{q^2dq}{\varepsilon_q - E}\Gamma_l(q, p'), \quad (7)$$

$$V_l(p, p') = \frac{1}{2}\int_{-1}^{1}dz\,P_l(z)\,V(\mathbf{p}, \mathbf{p}')$$
$$= \int_0^{\infty}V_l(r, r')J_{l+1/2}(p\,r)\quad J_{l+1/2}(p'r')\frac{(2\pi)^3r^2\,dr\,r'^2\,dr'}{\sqrt{p'\,rp'r}}. \quad (8)$$

For a local potential

$$V_l(p, p') = \int_0^{\infty}V(r)\,J_{l+1/2}\quad(pr)\quad J_{l+1/2}(p'r)\frac{2\pi^2rdr}{\sqrt{pp'}}. \quad (8')$$

The quantity l in (7) can be regarded as a parameter which may also take on values other than integers. In the region where the kernel of the equation V_l is an analytic function of l the singularities of the amplitude can only be simple poles in the points where the Fredholm determinant vanishes, i.e., for such values of l where the associated homogeneous equation has a solution. In the points where the kernel has a pole, the solution may contain a condensation of poles. This theorem applies if the interaction does not increase too rapidly at small distances, i.e., if it decreases sufficiently rapidly with increasing momenta, so that we have

$$\int V_l^2(p, q)\frac{q^2dq}{\varepsilon_q - E}\frac{p^2dp}{\varepsilon_p - E} < \infty. \quad (9)$$

Condition (9) is not satisfied for any value of l if the quantity $V_l(p, p')$ falls off like p^{-1} or slower for large $p \sim p'$. This corresponds to an increase $|V(\mathbf{r}, \mathbf{r}')| \geq r^{-5}$ for small $r \sim r'$. For a local potential we have $|V(r)| \geq r^{-2}$. Such singular potentials which lead to an equation of non-Fredholm type for the amplitude will be considered in Secs. 5 and 6. For nonsingular potentials the kernel may be non-Fredholm for large negative l. The amplitude in this region may be found by analytic continuation from the region of positive l.

3. LOCAL POTENTIAL

In the case of the usual local potential V(r) the kernel of the equation may have singularities in the l plane only in such points where the integral (8′) diverges at small distances. Expanding the potential in a series for small r, we obtain for $V(r) = gr^{\nu}$

$$V_l(p, q) = 2\pi^2 g \frac{2^{\nu+1} p^l q^l \Gamma(l + (3+\nu)/2)}{(p+q)^{2l+\nu+3}\Gamma(l + 3/2)\Gamma(-\nu/2)}$$
$$\times F(l + (3+\nu)/2,\ l+1,\ 2l+2,\ 4pq/(p+q)^2). \quad (10)$$

Near the pole with $l = -\frac{1}{2}(3+\nu)$ the potential V_l has the form

$$V_l(p, q) = \frac{2\pi^2 g 2^{\nu+2}}{2l+\nu+3}\frac{p^l q^l}{\Gamma^2(-\nu/2)}. \quad (11)$$

There will be no poles if ν is an even number, owing to the poles of the Γ function. In all other cases V_l has a pole. However, this pole does not lead to a condensation of poles in the amplitude, since the kernel of the equation is degenerate. As is seen from (10), $V_l(p, q)$ has near the pole the form of a product of a function depending only on p and a function depending only on q. In this case the integral equation reduces to an algebraic equation, and the solution has only one pole, which is, in general, far from the pole of the potential.

However, if the interaction constant g goes to zero or if the energy E goes to infinity, the integral in (8) differs appreciably from zero only in the neighborhood of the poles of the potential. Substituting (11) in (8), we find

$$\Gamma_l(p, p') = \frac{2\pi^2 g 2^{\nu+1} p^l q^l}{\Gamma^2(-\nu/2)}\left[l + \frac{\nu+3}{2} - \frac{\pi m g 2^{\nu+1}(-2mE)^{l+1/2}}{\cos l\pi\ \Gamma^2(-\nu/2)}\right]^{-1}. \quad (12)$$

For a potential of the Yukawa type, $\nu = -1$, there is a pole at $l = -1 - mg/\sqrt{-2mE}$. This behavior of the poles has been described in the lectures of Gribov [1] and the paper of Azimov, Ansel'm, and Shekhter. [4] Here the same results have been obtained with the aid of the equation for the amplitude in the momentum representation. This representation is convenient for the generalization to the relativistic case. Thus, if in the Bethe-Salpeter equation we take account only of single meson exchange, corresponding to a local potential, we obtain a formula of the type (12) for a weak interaction.

For a nonsingular potential ($\nu > -2$) the equation can become of non-Fredholm type for such values of l for which the integral defining the iterated kernel diverges:

$$V_l^{(2)}(p, p') = \frac{1}{2\pi^2}\int V_l(p, q)\frac{q^2 dq}{\varepsilon_q - E}V_l(q, p'). \quad (13)$$

With the potential (10) the integral (13) begins to diverge for large q if $l \le -\frac{5}{2} - \nu$. In this region we can use for $V_l^{(2)}$ the analytic continuation of (13) which has a pole at $l = -\frac{5}{2} - \nu$ for noninteger ν. [For integer ν the pole arising from the divergence of the integral is compensated by the pole of $\Gamma(l + \frac{3}{2})$ in the denominator of (10).] In order to determine the poles of the amplitude one can replace the kernel of the homogeneous equation by the iterated kernel. Near the pole the iterated kernel will be degenerate and, for noninteger ν the amplitude will have a pole near $l = -\frac{5}{2} - \nu$ for small coupling or large energies.

The integral (13) diverges for small q if $l \le -\frac{3}{2}$. The solutions in this region may be found by analytic continuation. For example, in the case of a Yukawa potential we find for l close to negative integer $-n-1$

$$\Gamma_l(p, q) = \frac{2\pi g}{pp'}\frac{1}{l+n+1 - mgP_n(z_{kk})/ik}\left\{P_n(z_{pq}) + \frac{mg}{l+n+1}\left[P_n(z_{pk})P_n(z_{kq}) - P_n(z_{kk})P_n(z_{pq})\right]\right\},$$

where

$$z_{pq} = (p^2 + q^2 + \mu^2)/2pq, \qquad k^2/2m = E.$$

4. NONLOCAL POTENTIAL

For the analytic continuation of a nonlocal potential into the complex l plane we write $V(p, p')$ in the form of a dispersion integral over the angle z:

$$V(z) = \int \frac{V_1(z')\,dz'}{z - z'}.$$

According to (8) we obtain then for integer l

$$V_l(p, p') = \int Q_l(z)V_1(z)\,dz. \quad (14)$$

The local potential depended on z only through $s = p^2 + p'^2 - 2pp'z$ and had no singularities for positive s, i.e., negative z, so that the integration in (14) went only over positive z. Formula (14) could therefore be regarded as defining an analytic function of l which vanishes when $l \to \infty$ in the right half-plane.

A nonlocal potential has singularities also for negative z, since it depends also on $u = p^2 + p'^2 + 2pp'z$. Therefore the integral (14) contains Q_l with a negative argument, which does not vanish for $l \to \infty$. In this case the analytic continuation must be carried out separately for even and odd l. [2] We write

$$V_l = \tfrac{1}{2}(1 + (-1)^l)V_l^+ + \tfrac{1}{2}(1 - (-1)^l)V_l^-,$$
$$\Gamma_l = \tfrac{1}{2}(1 + (-1)^l)\Gamma_l^+ + \tfrac{1}{2}(1 - (-1)^l)\Gamma_l^-;$$
$$V_l^{\pm} = \int_0^{\infty} Q_l(z)\,[V_1(z) \mp V_1(-z)]\,dz.$$

Equation (7) for integer l can be written separately for $\Gamma_l^{\mp}$:

$$\Gamma_l^{\pm}(p, p') = V_l^{\pm}(p, p') - \frac{1}{2\pi^2}\int V_l^{\pm}(p, q)\frac{q^2 dq}{\varepsilon_q - E}\Gamma_l^{\pm}(q, p'). \quad (15)$$

These equations can be continued to noninteger l. Since the potential $V_l^{\mp}$ is defined as an integral over positive z, it vanishes for $l \to \infty$ in the right half-plane, and so does $\Gamma_l^{\mp}$.

Below we shall consider only the even amplitude and leave out the plus sign in the superfix. We have for the potential (6)

$$V_l(p, q) = \int_{\mu_0}^{\infty}\frac{\rho(\mu^2)\,d\mu^2}{pq(2p^2 + 2q^2 + M^2 + \mu^2)}\left[Q_l\left(\frac{p^2 + q^2 + \mu^2}{2pq}\right) + Q_l\left(\frac{p^2 + q^2 + M^2}{2pq}\right)\right].$$

Near $l = -1$ the equation for the amplitude has the form

$$\Gamma_l(p, p') = \frac{1}{(l+1)pp'}\int_{\mu_0}^{\infty}\frac{2\rho(\mu^2)\,d\mu^2}{2p^2 + 2p'^2 + M^2 + \mu^2} - \frac{1}{l+1}\int_{\mu_0}^{\infty}\frac{\rho(\mu^2)\,d\mu^2}{2p^2 + 2q^2 + M^2 + \mu^2}\frac{q\,dq}{p(\varepsilon_q - E)}\Gamma_l(q, p'). \quad (16)$$

Let us consider the analytic properties of the amplitude as a function of the energy. The kernel of the equation depends analytically on the energy E, except on the real positive axis. Therefore the amplitude has, as a function of the energy, only poles and a cut on the positive real axis. The discontinuity across this cut is determined by the unitarity condition. Off the energy shell it is given by (4):

$$\operatorname{Im}\Gamma_l(p, p') = -(m/2\pi)\sqrt{2mE}\,\Gamma_l^*(p, \sqrt{2mE})\,\Gamma_l(\sqrt{2mE}, p'). \quad (17)$$

On the energy shell $p^2 = p'^2 = 2mE$ the amplitude has singularities in the points where $\Gamma_l(p, p')$ has singularities as a function of p^2 and p'^2, i.e., for negative E. The closest singularity is given by the singularity of the potential. For the potential (6) we see from (16) that the closest singularity is the cut with the jump $(l+1)^{-1}\rho(-8mE - M^2)$ for $8mE < -(\mu_0^2 + M^2)$. For $4mE = -(\mu_0^2 + M^2)$ the amplitude corresponding to (13) branches out because of the integral term, and for $-4mE > \mu_0^2 + M^2$ the jump of the amplitude contains a condensation of poles at $l = -1$.

In the relativistic theory this branching does not occur on the energy shell, since the jump on the left cut has a simple pole at $l = -1$.[2] This is due to the fact that the irreducible four-point vertices, which play the role of a potential, depend on the energy.[6] If we introduce an interaction which depends explicitly on the energy, by replacing M^2 by $M^2 - 4mE$ in (16), then the singularities of the amplitude for negative E coincide with the singularities of the potential for physical values $p^2 = p'^2 = 2mE$. However, branching then occurs for positive $E = (4m)^{-1}(M^2 + \mu_0^2)$. A new term appears in the unitarity relation, corresponding to inelastic processes. Thus even with a nonlocal and energy dependent potential it is apparently impossible to construct a model with a jump proportional to $(l+1)^{-1}$ on the left cut and a jump corresponding to a two-particle unitarity condition on the right cut.

Independently of the explicit form of the nonlocal potential, the solution of (16) contains a condensation of poles at $l = 1$. Indeed, the kernel of the equation is nondegenerate, and the corresponding homogeneous equation has an infinite number of eigenvalues $\lambda_n = (l_n + 1)^{-1}$. As n increases the eigenvalues go to infinity, i.e., $l_n = -1$. Expanding the kernel in eigenfunctions $\varphi_n(p)$ of the homogeneous equation, we obtain

$$\Gamma_l(p, p') = \sum_n \frac{(l_n + 1)\varphi_n(p)\varphi_n(p')}{l - l_n}. \quad (18)$$

It can be shown[6] that for $\rho(\mu^2) > 0$ the kernel of (13) is positive definite and hence all poles satisfy $l_n > -1$. If $\rho(\mu^2) < 0$ all poles satisfy $l_n < -1$.

For $E \to -\infty$ all poles tend to -1. If the interaction is weak, the poles lie near -1 for all energies and only for $E \to 0$ does one pole move far away. If the interaction is not weak, some poles move away from the region near -1 already for not very large E, and their location cannot be determined with the help of (16), where, only the term largest in $(l+1)$ is retained in the kernel.

One may consider a third possibility which gives an intermediate picture for the motion of the poles. Let us assume that the interaction is weak but that we have an additional strong interaction without singularities at $l = -1$ which in the absence of a nonlocal potential would lead to a pole $l(E)$ intersecting the line $l = -1$ in some point E_0. Then, for values of l not close to -1, the pole moves in the same manner as in the absence of the nonlocal interaction.

In order to see how this pole intersects the condensation line of poles, let us consider the case where we have a potential of the form ge^{-Mr} in addition to the nonlocal interaction. For simplicity we shall assume that this potential is short ranged,

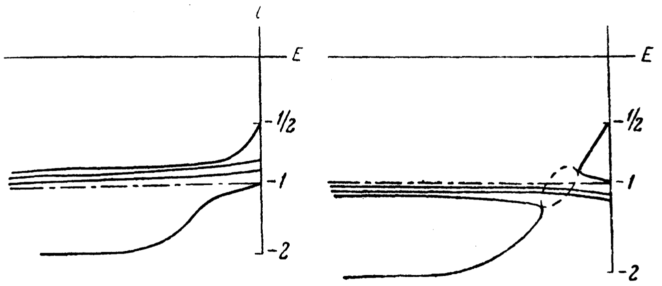
$M^2 \gg 2mE$. In momentum space we obtain then the degenerate kernel

$$V_l(pq) = \frac{4\pi^{3/2}g}{Mpq}\frac{\Gamma(l+2)}{\Gamma(l+3/2)}\left(\frac{pq}{M^2}\right)^{l+1}.$$

The solution corresponding to this kernel can be found if the eigenfunctions $\varphi_n(p)$ and eigenvalues l_n in the absence of the strong local potential are known. After some simple transformations we obtain an equation for the position of the poles:

$$\left[\frac{2}{\sqrt{\pi}}\frac{mg}{M^2}\Gamma(l+2)\Gamma\left(-l-\frac{1}{2}\right)\left(\frac{-2mE}{M^2}\right)^{l+1/2}-1\right]$$
$$=\frac{4\pi g}{M}\sum_n\frac{(l_n+1)}{l-l_n}\left[\frac{1}{2\pi^2}\int\frac{\varphi_n(p)\,pdp}{\varepsilon_p-E}\right]^2.$$

The function $l(E)$ defined by this equation is shown in the figure. Case a corresponds to the condensation of poles for $l > -1$ and case b, for $l < -1$.



a b

In case b, the poles collide: the two poles indicated by the dotted lines go off into the complex l plane and circumvent the dangerous region there. There is no collision of poles near -1, since (16) has only real eigenvalues. In the case shown in the figure it was assumed that the functions $\varphi_n(p)$ form a complete set. If this is not so, the sum on the right-hand side of (18) is not equal to -1 for $l = -1$, and the pole takes the value $l = -1$ at some value of $E \neq 0$.

A detailed investigation of the effect of the poles on the asymptotic behavior of the amplitude has been carried out in another paper of the authors, [6] where the analog of (16) was considered in a relativistic theory.

5. SINGULARITY IN A LOCAL POTENTIAL

Up to now we have assumed that the equation for the amplitude satisfies the Fredholm condition (9). This condition is not satisfied if the potential grows like r^ν at small distances, where $\nu \leq -2$. The motion of the poles for a local repulsive potential is well explored.[1,5] For $\nu = -2$ (the singular potential may be added to the centrifugal term) the amplitude has only branch points in the l plane.

For $\nu < -2$ the amplitude is an even function of $l + 1/2$. As the interaction constant decreases or $E \to -\infty$ the poles stay on the line $\operatorname{Re} l = -1/2$ and do not tend to real values as in the case of a nonsingular potential. If the interaction contains a nonlocal term besides the local singular potential, the amplitude will contain a condensation of poles near $l = -1$. However, the proof given above is not applicable in this case, since Eq. (7) for the amplitude in the momentum representation is not of the Fredholm type.

It is convenient to separate out the singular part of the potential as a zero-order Hamiltonian and to go over to a representation defined by the functions $\varphi_{pl}(r)$ satisfying the equation

$$\frac{1}{2mr}[r\varphi_{pl}(r)]'' + \left[\frac{\gamma}{r^{2(1+\varepsilon)}} + \frac{l(l+1)}{r^2} + \varepsilon_p\right]\varphi_{pl}(r) = 0. \tag{19}$$

Denoting the nonsingular part of the potential by $V_l(r, r')$, we obtain from the Schrödinger equation

$$\psi_{pl}(r) = \varphi_{pl}^{(1)}(r) + \int G(r, r')V_l(r', r_1)\psi(r_1)\,dr'dr_1, \tag{20}$$

where G is the Green's function of (19),

$$G_E(r, r') = \frac{1}{2\pi^2}\int p^2dp\frac{\varphi_{pl}^{(1)}(r)\varphi_{pl}^{(1)}(r')}{\varepsilon_p - E}$$
$$= \varphi^{(2)}(r)\varphi^{(1)}(r')\theta(r-r') + \varphi^{(1)}(r)\varphi^{(2)}(r')\theta(r'-r), \tag{21}$$

and $\varphi^{(1)}$ and $\varphi^{(2)}$ are the solutions of (16) with the following properties:

for $r \to 0$

$$\varphi^{(1)} \to r^{(\varepsilon-1)/2}\exp(-r^{-\varepsilon}/\varepsilon), \quad \varphi^{(2)} \to r^{(\varepsilon-1)/2}\exp(r^{-\varepsilon}/\varepsilon);$$

for $r \to \infty$

$$\varphi^{(1)} \to r^{-1}\sin(pr + \pi l/2 + \delta_l),$$
$$\varphi^{(2)} \to r^{-1}\exp[i(pr + \pi l/2 + \delta_l)],$$

the phase δ_l is found from (19) and has no singularities for real l, as noted above.

Finding the asymptotic form of the wave function at large distances with the aid of (20), we obtain for the amplitude

$$f_l = (e^{2i\delta_l} - 1)/2i\sqrt{2mE} - (m/2\pi)e^{2i\delta_l}\Gamma_l(\sqrt{2mE}, \sqrt{2mE}),$$

where $\Gamma_l(p, p')$ satisfies the equation

$$\Gamma_l(p, p') = V_l(p, p') - \frac{1}{2\pi^2}\int V_l(p, q)\frac{q^2dq}{\varepsilon_q - E}\Gamma_l(q, p'). \tag{7'}$$

This equation has the same form as (7) with the only difference that $V_l(p, q)$ is defined by

$$V_l(p, q) = \int_0^\infty V^l(r, r') \varphi_{lp}^{(1)}(r) \varphi_{lq}^{(1)}(r') r^2 dr r'^2 dr' \qquad (22)$$

instead of (8) and (8′).

In the case of a local potential $V^l(r, r') = V(r)\delta(r - r')$ and the kernel of the equation has no poles connected with the behavior of the potential at small distances, since the integral (19) converges for all l in virtue of the rapid vanishing of $\varphi^{(1)}(r)$ for small r. For a nonlocal potential the arguments of Sec. 4 can be repeated without change and we find that Γ_l and hence also f_l contain a condensation of poles for $l = -1$, if only the equation is of the Fredholm type.

The Fredholm condition (9) can be rewritten using (22) and (21):

$$\int V_l(r_1, r_2) G(r_2, r_3) V(r_3, r_4) \times G(r_4, r_1) r_1^2 dr_1 r_2^2 dr_2 r_3^2 dr_3 r_4^2 dr_4 < \infty.$$

It follows from (21) that $G(r, r) \to r^{\epsilon-1}$ for $r \to 0$, and thus the integral will converge for small r if the function $V_l(r, r')$ increases less rapidly than $r^{-5-\epsilon}$ for $r \sim r' \to 0$. Thus the equation will be of the Fredholm type if only it is of this type in the absence of the singular potential, and the amplitude will contain a condensation of poles for $l = -1$.

6. SINGULAR NONLOCAL POTENTIAL AND BRANCH POINTS

Let us now consider the case when the nonlocal potential has a strong increase at small distances. If $V(r, r') \geq r^{-5}$ for $r \sim r' \to 0$ the kernel is of non-Fredholm type for all l. In the preceding section we have considered the case of a local singular potential which does not depend on l in coordinate space. In the case of a nonlocal potential the kernel does depend on l and the eigenvalues of the equation are proportional to $(l + 1)^{-1}$ near $l = -1$. One might think that the homogeneous equation with non-Fredholm kernel has a solution for any eigenvalue in some region (continuous spectrum). The amplitude, as the solution of the inhomogeneous equation, will then have branch points at the ends of the continuous spectrum.

As an illustration of these assertions let us consider the equation

$$\Gamma(p, p') = \frac{g}{(l+1)\pi(p^2 + p'^2)} + \frac{g}{(l+1)\pi} \int \frac{dq^2}{p^2 + q^2} \Gamma(q, p'). \qquad (23)$$

Although this equation is not obtainable by the choice of a definite form of a nonlocal potential, it nevertheless has the form of Eq. (16) and may serve to illustrate the singularities occurring in such a case. Using the fact that the corresponding homogeneous equation has a solution of the form p^ν, we obtain the solution of the inhomogeneous equation in the form

$$\Gamma_l(p, p') = \frac{1}{\pi p p'} \int_0^\infty \frac{d\nu\, [l(\nu) + 1]}{l(\nu) - l} \cos \nu \ln \frac{p^2}{p'^2},$$

$$l(\nu) = -1 + \frac{g}{\cosh \nu\pi}. \qquad (24)$$

The amplitude, as a function of l, has a cut from $l = -1$ to $l = -1 + g$. Starting from $l = -1$, the cut goes to the right for attractive ($g > 0$) and to the left for repulsive potentials.

The equation with the kernel $\sim (p^2 + q^2 + \mu^2)^{-1}$ has similar properties. Its solution differs from (24) by the replacement of $\ln(p^2/p'^2)$ by $\nu P_{-1/2+i\nu}(p^2/\mu^2) Q_{-1/2+i\nu}(p'^2/\mu^2)$ and has the same singularities in the l plane.

The amplitude (24) has the same form as in the Fredholm case (18), except that the sum is replaced by an integral.

The simplicity of the example (24) allows us to find the contribution to the asymptotic form of the amplitude from the region of l close to -1. This contribution is given by [7]

$$\Gamma(z) = \frac{i}{4\pi} \int dl \Gamma_l [Q_{-l-1}(z) + Q_{-l-1}(-z)].$$

For a small coupling constant $g \ll 1$, when the branch point is located near -1, and large $z \gg 1$ we obtain

$$\Gamma(z) = \frac{-i\pi}{2p^2} \int_0^\infty d\nu\, [l(\nu) + 1]^2 z^{l(\nu)}. \qquad (25)$$

For simplicity we take $p = p'$.

Let us first consider the case of an attractive interaction, $g > 0$. For very large z such that $g \ln z \gg 1$, the most important region in the integral is the extreme right end of the cut, $l = -1 + g$. We obtain in this case

$$\Gamma(z) = -i\pi z^{-1+g} / 4p^2 \sqrt{\pi g \ln z}. \qquad (26)$$

This corresponds to the contribution from the pole farthest to the right. If $z \gg 1$ but $g \ln z \ll 1$, we obtain from (25)

$$\Gamma(z) = \frac{-i\pi}{2p^2 z} \int_0^\infty d\nu\, [l(\nu) + 1]^2 = \frac{-ig^2}{2zp^2}. \qquad (27)$$

The last result (27) does not depend on the sign of g and is also valid for the case when the cut goes to the left (repulsion). If $g < 0$ and $-g \ln z \gg 1$ the integral (25) is evaluated by the method of steepest descent and becomes

$$\Gamma(z) = -i/2zp^2 \ln^2 z. \qquad (28)$$

The result (28) is also valid for large g when the cut goes from -1 to the left.

Formulas (26) to (28) are rather insensitive to the specific form (24) of the amplitude and one might hope that the results will not be changed much if the integrals in (24) and (25) are replaced by a sum of an infinite number of pole terms, i.e., if one goes over to formula (18) of the Fredholm case, which has been discussed in detail in another paper.[6]

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